Tensors

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1 Introduction

These notes are the product of an effort to conciliate the concept of a tensor between the classical and modern differential geometry approaches. Another motivation for writing this document is a need to have simple geometrical interpretation for the concepts covariant and contravariant.

The references used for this work were:

- McConnell [4]. This is a classical text on Tensor Analysis with applications in dynamics, electricity, elasticity, hydrodynamics and special theory of relativity.
- Kreyszig [3]. This is a classical book in Differential geometry.
- Rektorys [5]. This is an encyclopedic book in mathematics. Here the treatment of tensors suggest a natural introduction of the metric tensor g_{ij}.
- Guggenheimer [1]. This book describes a modern approach to Differential Geometry.

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- Munkres [2]. An exellent balance between being formal and being clear. I use Munkres for the definition of the determinant and the brief notions of differential form (in the appendix).

2 Tensors

3 Contravariant components, covariant bases

V is a finite dimensional vector space over the real numbers. Given a basis $\boldsymbol{e}_i$ of V, any vector $\boldsymbol{x} \in$ V can be written as

$$\boldsymbol{x} = x^i \boldsymbol{e}_i \tag{3.1}$$

where the summation convention is used. Here and in what follows a repeated index indicates summation over its range of values. I use capital letters $I, J, K, ...$ when no summation should be performed along these variables. We call the components x^i as *contravariant* coordinates and the base vectors $\boldsymbol{e}_i$ as *covariant* vectors.

3.1 example

Given that

$$\boldsymbol{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.2}$$

then

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.3}$$

with $\alpha = 1$ and $\beta = 2$. Here $x^1 = \alpha = 1$ and $x^2 = \beta = 2$ are the contravariant coordinates of $\boldsymbol{x}$ in the covariant basis $\boldsymbol{e}_1, \boldsymbol{e}_2$ of the vector space V. In words: to get to $\boldsymbol{x}$ go 1 unit of $\boldsymbol{e}_1$ in the direction of $\boldsymbol{e}_1$ and 2 units of $\boldsymbol{e}_2$ in the direction of $\boldsymbol{e}_2$. The units of

$\boldsymbol{e}_1$ are given by its size 1, and the units of $\boldsymbol{e}_2$ are given by its size $\sqrt{2}$. Figure 1 shows an illustration of the representation of the vector $\boldsymbol{x}$ in terms of its contravariant coordinates along the base vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. This means that *the contravariant coordinates x^i represent the parallel projections of the vector $\boldsymbol{x}$ along the covariant direction $\boldsymbol{e}_i$*. We

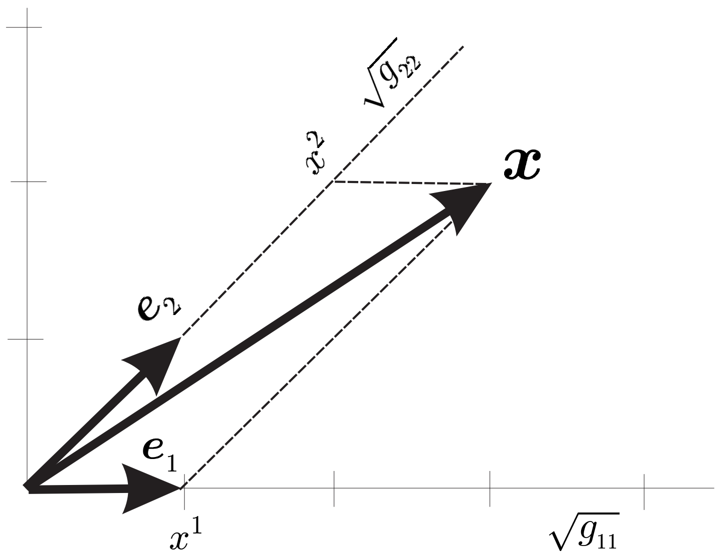

Figure 1: Illustration for the contravariant coordinates of the vector $\boldsymbol{x} = (3, 2)$ along the covariant basis lines. The base (covariant) vectors are $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. The contravariant coordinates are x^1 and x^2 respectively and the metric (units) that weight x^1 are $\sqrt{g_{11}} = 1$ and those that weight x^2 are $\sqrt{g_{22}} = \sqrt{2}$. It is clear from the figure that the contravariant coordinates are parallel projections of the vector $\boldsymbol{x}$ along the base directions $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.

make use of the following definition

$$g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j, \tag{3.4}$$

where the usual dot product of the Euclidean space is assumed. This is known as the metric * tensor, or the first fundamental form of the differential geometry. In our example

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \tag{3.5}$$

With this notation, the units that go with x^1 are $\sqrt{g_{11}} = 1$ and the units that goes with x^2 are $\sqrt{g_{22}} = \sqrt{2}$. This make sense, since the diagonal of a square grows $\sqrt{2}$ as fast as its side. I will also show below that the length of a vector is independent of the coordinate system of reference. For this particular example

$$r = |\boldsymbol{x}| = \sqrt{3^2 + 2^2} = \sqrt{13} \tag{3.6}$$

where the canonical basis defined by $\hat{\boldsymbol{i}}_j = \delta^i_j$ is assumed. The Kronecker delta δ^i_j is defined by the formula

$$\delta^i_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

After using the basis $\boldsymbol{e}_1, \boldsymbol{e}_2$ we find

$$r = \sqrt{1^2 + (2\sqrt{2})^2 - 2 \times 2\sqrt{2}\cos 3\pi/4} = \sqrt{13}. \tag{3.8}$$

We used the trigonometric cosine law. The angle between $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ is interpreted as $3\pi/4$ from the geometry of the problem. The reader can easily verify that (following indicators in the footnote $*$)

$$r = \sqrt{g_{ij}\, x^i\, x^j} = \sqrt{13}. \tag{3.9}$$

*because the length element is given by

$$ds ds = d\boldsymbol{x}.d\boldsymbol{x} = dx^i \boldsymbol{e}_i \cdot dx^j \boldsymbol{e}_j = dx^i dx^j g_{ij}.$$

.

4 Covariant components, contravariant bases

Let V^* be the dual space of V. This is the space of functionals over the real numbers. That is, any element ϕ of V^* is a linear function

$$\phi : V \longrightarrow \mathbb{R}. \tag{4.10}$$

We first introduce the contravariant base vectors. Given that $\boldsymbol{e}_i$ is a covariant basis for the space V, we define $\boldsymbol{e}^i$ as

$$\begin{aligned} \boldsymbol{e}^i \;\; : \;\; & V \longrightarrow \mathbb{R} \\ & \boldsymbol{e}_j \longmapsto \delta^i_j, \end{aligned} \tag{4.11}$$

that is $\boldsymbol{e}^i(\boldsymbol{e}_i) = \delta^i_j$. The vectors $\boldsymbol{e}^i \in V^*$ are called contravariant base vectors corresponding to the covariant base vectors $\boldsymbol{e}_i \in V$. It is left to the reader to prove that $\boldsymbol{e}^i$ is a basis for V^*. Let us assume that x_i are the components of a functional $\phi_{\boldsymbol{x}} \in V^*$ under the basis $\boldsymbol{e}^i$, then

$$\begin{aligned} \phi_{\boldsymbol{x}}(\boldsymbol{y}) &= x_i\, e^i(\boldsymbol{y}) \\ &= x_i\, y^j\, e^i(e_j) \\ &= x_i\, y^j\, \delta^i_j \\ &= x_i\, y^i. \end{aligned} \tag{4.12}$$

This means that any functional in V^* can be interpreted as a dot product –in the regular sense– of the components of it, in terms of its contravariant basis $\boldsymbol{e}^i$, and the components of the vector $\boldsymbol{y}$ where it is applied to; these components of y seen in its covariant basis

$\boldsymbol{e}_j$. Let us now pick two vectors $\boldsymbol{x} \in \mathrm{V}$, $\boldsymbol{y} \in \mathrm{V}$ and define

$$\begin{aligned} \gamma_{\boldsymbol{x}}(\boldsymbol{y}) &= \boldsymbol{x} \cdot \boldsymbol{y} \\ &= x^i \boldsymbol{e}_i \cdot y^j \boldsymbol{e}_j \\ &= (x^i \boldsymbol{e}_i \cdot \boldsymbol{e}_j)\, y^j \\ &= x_j\, y^j, \end{aligned} \tag{4.13}$$

where

$$x_j = x^i \boldsymbol{e}_i \cdot \boldsymbol{e}_j = x^i g_{ij}. \tag{4.14}$$

Then

$$\begin{aligned} \gamma_{\boldsymbol{x}}(\boldsymbol{y}) &= x_i \delta^i_j y^j \\ &= x_i \boldsymbol{e}^i(\boldsymbol{e}_j) y^j \\ &= x_i \boldsymbol{e}^i(y^j \boldsymbol{e}_j) \\ &= x_i \boldsymbol{e}^i(\boldsymbol{y}). \end{aligned} \tag{4.15}$$

So, in fact $x_j = x^i g_{ij}$ are the coordinates of $\gamma_{\boldsymbol{x}}$ in terms of its contravariant basis $\boldsymbol{e}^j$ given that $\boldsymbol{y}$ is expressed in the covariant basis $\boldsymbol{e}_i$. The coordinates x_j are called the covariant coordinates of the function $\gamma_{\boldsymbol{x}}$.

There is an isomorphism between the space V and the space V^*. Given any vector $\boldsymbol{x} \in V$ the functional $\gamma_{\boldsymbol{x}}$ in equation (4.13) defines the isomorphic image of $\boldsymbol{x}$ in the space of functionals V^*. Therefore we will identify $\gamma_{\boldsymbol{x}}$ with $\boldsymbol{x}$ as the "same" object —although this is in different spaces.

Now we interpret the meaning of covariant coordinates. Given that

$$\begin{aligned} x_j &= x^i(\boldsymbol{e}_i \cdot \boldsymbol{e}_j) \\ &= (x^i \boldsymbol{e}_i) \cdot \boldsymbol{e}_j \\ &= \boldsymbol{x} \cdot \boldsymbol{e}_j, \end{aligned} \tag{4.16}$$

then *the covariant coordinates* x_j *represent the orthogonal projections of the vector* $\boldsymbol{x}$ *along the covariant direction* $\boldsymbol{e}_j$. As in the interpretation of the contravariant coordinates we have to take into consideration the fact that the covariant vectors $\boldsymbol{e}_i$ are not unit vectors. For example the covariant coordinate x_1 is given by

$$\begin{aligned} x_1 &= \boldsymbol{x} \cdot \boldsymbol{e}_1 \\ &= |\boldsymbol{x}||\boldsymbol{e}_1| \cos\theta \\ &= r\sqrt{g_{11}} \cos\theta, \end{aligned} \tag{4.17}$$

where θ is the angle between $\boldsymbol{x}$ and $\boldsymbol{e}_1$. The length of the projection of $\boldsymbol{x}$ along the coordinate axis in the direction of $\boldsymbol{e}_1$ is given by

$$r\cos\theta = \frac{x_1}{\sqrt{g_{11}}}. \tag{4.18}$$

In general the length of the projection of the vector $\boldsymbol{x}$ into the coordinate axis in the direction of the $\boldsymbol{e}_J$ vector is $x_J/\sqrt{g_{JJ}}$ (recall that no summation is performed along capital letter indexes.) Figure 2 shows an illustration of the representation of the vector $\boldsymbol{x}$ in terms of its covariant coordinates along the base vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.

5 Change of basis

Let us call A the matrix of change from the covariant basis $\boldsymbol{e}_i$ in V to the covariant basis $\bar{\boldsymbol{e}}_i$ in V, so

$$\bar{\boldsymbol{e}}_i = a_i^j\, \boldsymbol{e}_j \tag{5.19}$$

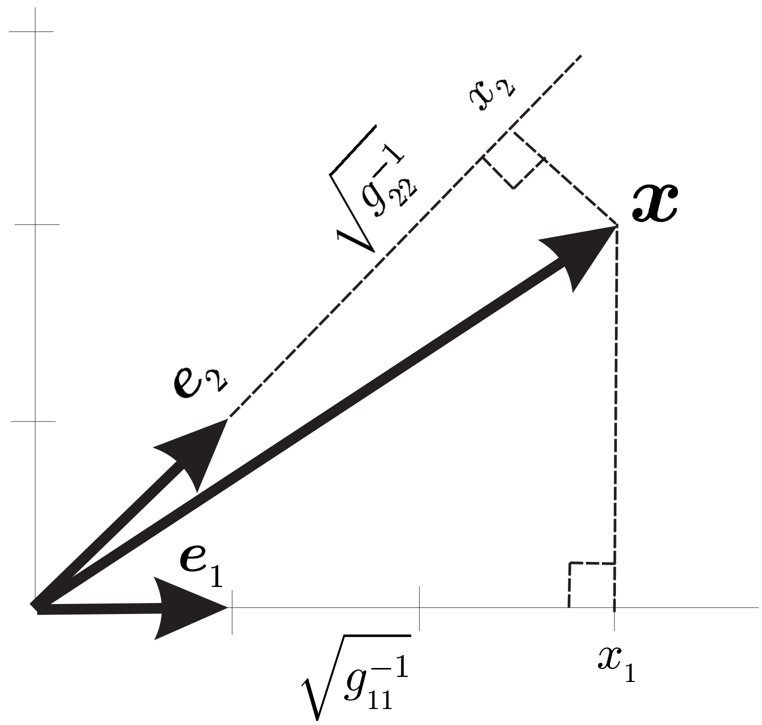

Figure 2: Illustration for the covariant coordinates of the vector $\boldsymbol{x} = (3, 2)$ along the covariant basis lines. The base (covariant) vectors are $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. The covariant coordinates are x_1 and x_2 respectively and the metric (units) that weight x_1 are $\sqrt{g_{11}^{-1}}$ and those that weight x_2 are $\sqrt{g_{22}^{-1}}$. It is clear from the figure that the contravariant coordinates are orthogonal projections of the vector $\boldsymbol{x}$ to the base directions $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.

A vector $\boldsymbol{x} \in V$ should be independent of any coordinate system, so

$$\boldsymbol{x} = x^i\, \boldsymbol{e}_i = \bar{x}^j\, \bar{\boldsymbol{e}}_j = \bar{x}^j\, a^i_j\, \boldsymbol{e}_i \tag{5.20}$$

so

$$x^i = a^i_j\, \bar{x}^j. \tag{5.21}$$

If $B = A^{-1}$ then

$$x^i\, b^k_i = \bar{x}^j\, a^i_j\, b^k_i = \bar{x}^j\, \delta^k_j = \bar{x}^k \tag{5.22}$$

that is

$$\bar{x}^i = b^i_j \, x^j \tag{5.23}$$

Let's now call C the matrix of change from the contravariant basis $\boldsymbol{e}^i$ in V* to the contravariant basis $\bar{\boldsymbol{e}}^i$ in V*. That is

$$\bar{\boldsymbol{e}^i} = c^i_j \, \boldsymbol{e}^j. \tag{5.24}$$

Pick a vector $\boldsymbol{x} \in$ V*, so

$$\boldsymbol{x} = x_i \, \boldsymbol{e}^i = \bar{x}_j \, \bar{\boldsymbol{e}}^j = \bar{x}_j \, c^j_i \, \boldsymbol{e}^i \tag{5.25}$$

therefore

$$x_i = c^j_i \, \bar{x}_j. \tag{5.26}$$

now if $D = C^{-1}$ then

$$d^i_k \, x_i = d^i_k \, c^j_i \, \bar{x}_j = \delta^j_k \, \bar{x}_j = \bar{x}_k \tag{5.27}$$

so

$$\bar{x}_i = d^j_i \, x_j \tag{5.28}$$

A more interesting relation between C and A is shown next. This relation links the space V with its dual V*. Given that $\bar{\boldsymbol{e}^i}(\bar{\boldsymbol{e}}_j) = \delta^i_j$ then

$$\begin{aligned} \delta^i_j &= \bar{\boldsymbol{e}^i} \, (\bar{\boldsymbol{e}}_j) \\ &= c^i_k \, \boldsymbol{e}^k (a^l_j \, \boldsymbol{e}_l) \\ &= c^i_k \, a^l_j \, \delta^k_l \\ &= c^i_k \, a^k_j, \end{aligned} \tag{5.29}$$

so $C = A^{-1}$.

In summary we have the four fundamental relations

$$\begin{aligned} \bar{\boldsymbol{e}}_i &= a_i^j\, \boldsymbol{e}_j & \bar{x}^i = (a^{-1})^i_j\, x^j, \\ \bar{\boldsymbol{e}}^i &= (a^{-1})^i_j\, \boldsymbol{e}^j & \bar{x}_i = a_i^j\, x_j, \end{aligned} \tag{5.30}$$

where (a^{-1}) means that we are taking the components of the inverse matrix A^{-1}. These relations show the duality between the contravariant and covariant base vectors and their corresponding coordinates in the spaces V and its dual V*.

The first consequence of the dualities in equation (5.30) is the invariance of the inner product under a change of coordinates. Let us see

$$\bar{x}^i\, \bar{y}_i = (a^{-1})^i_j\, a_i^k\, x^j\, y_k = \delta^k_j\, x^j\, y_k = x^j\, y_j. \tag{5.31}$$

in the same manner we see that $\bar{\boldsymbol{e}}_i\, \bar{\boldsymbol{e}}^i = \boldsymbol{e}_i\, \boldsymbol{e}^i$. Note here that we are fixing the basis in both coordinate systems while the vectors on each one of them are arbitrary. That is why I used y's in the inner product and only $\boldsymbol{e}$'s in the product of base vectors.

If the basis $\boldsymbol{e}_j$ is written in terms of the canonical basis $\hat{\boldsymbol{\imath}}_j$ of the space V we find

$$(\boldsymbol{e}_i)_k = e_i^k = a_i^j\, (\hat{\boldsymbol{\imath}}_j)^k = a_i^j\, \delta_j^k = a_i^k \tag{5.32}$$

by interpreting the subindex as a column we say that the base vectors $\boldsymbol{e}_i$ are the columns of the matrix A.

Let us take the covariant coordinates of the contravariant base vector $\boldsymbol{e}^i$ as e^i_j. Then by using equation (4.14) on the canonical basis $\hat{\boldsymbol{\imath}}_j$ we find

$$e^i_j = e^{ik}\, (\hat{\boldsymbol{\imath}}_j \cdot \hat{\boldsymbol{\imath}}_k) = e^{ik}\, \delta_{jk} = e^{ij}. \tag{5.33}$$

This means that if the covariant basis, is the canonical bases then there is not distinction between covariant and contravariant vectors and coordinates therefore for any arbitrary contravariant base vector $\boldsymbol{e}^j$ we find

$$(\boldsymbol{e}^j)_i = (a^{-1})^j_k\, (\hat{\boldsymbol{\imath}}_k)_i = (a^{-1})^j_k\, \delta_{ki} = (a^{-1})^j_i. \tag{5.34}$$

by interpreting the super-index as a row we say that the base vectors $\boldsymbol{e}^i$ are the rows of the matrix A^{-1}. In our example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{5.35}$$

Therefore

$$\boldsymbol{e}^1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{e}^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.36}$$

In appendix A equation (A.102) shows that $\boldsymbol{e}^i$ would be the generalized cross "×" product of all but the i-th covariant base vectors (preserving the order) "normalized" by the determinant of A. In particular in the 3D space

$$\boldsymbol{e}^1 = \frac{\boldsymbol{e}_2 \times \boldsymbol{e}_3}{\det A}, \qquad \boldsymbol{e}^2 = \frac{\boldsymbol{e}_1 \times \boldsymbol{e}_3}{\det A} \qquad \boldsymbol{e}^3 = \frac{\boldsymbol{e}_1 \times \boldsymbol{e}_2}{\det A}. \tag{5.37}$$

In 2D, we define we define

$$\times \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} x^2 \\ -x^1, \end{pmatrix} \qquad \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1, \end{pmatrix}, \tag{5.38}$$

or in compressed form $\times x^i = \epsilon_{ij} x^j$, and $\times x_i = \epsilon^{ij} x_j$, Then $\boldsymbol{e}^1 = \times \boldsymbol{e}_2 / \det(A)$, $\boldsymbol{e}^2 = -\times \boldsymbol{e}_1 / \det(A)$. This is in agreement with equation 5.36. The "−" sign in the previous definition is needed for consistency. That is, when the dimension is larger than two, the vector order determines the sign (according to the order of the permutation). In the case of two dimensions there is only one vector and the sign has to be forced into the second basis vector.

Given that the space V* is isomorphic to V, we can identify V* with V, and so find a geometrical interpretation of the contravariant base vectors $\boldsymbol{e}^i$ and the covariant coordinates x_i as embedded in V.

We first consider any vector $\boldsymbol{x} \in \mathrm{V}^*$ as spanned by the covariant basis $\boldsymbol{e}^i$

$$\boldsymbol{x} = x_i \, \boldsymbol{e}^i. \tag{5.39}$$

Let us now consider the inner product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in V^*, that is

$$\begin{aligned}\boldsymbol{x}\cdot\boldsymbol{y} &= x_i\,\boldsymbol{e}^i\cdot y_j\,\boldsymbol{e}^j\\ &= (x_i\,\boldsymbol{e}^i\cdot\boldsymbol{e}^j)\,y_j\\ &= x^j\,y_j\end{aligned}\tag{5.40}$$

where

$$x^j = x_i\,\boldsymbol{e}^i\cdot\boldsymbol{e}^j = x_i\,g^{ij}\tag{5.41}$$

Here we define $g^{ij} = \boldsymbol{e}^i\cdot\boldsymbol{e}^j$. We previously (see equation (4.14)) defined $x_j = x^i\,g_{ij}$ that definition will be consistent with formula (5.41) if and only if the symmetric matrices g_{ij} and g^{ij} are inverses of each other. Let us prove this.

From equation (5.32) we have that $(\boldsymbol{e}_i)_k = a_i^k$ and from equation (5.34) we have that $(\boldsymbol{e}^j)_i = (a^{-1})_i^j$ for any given covariant and contravariant vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}^j$ so

$$\begin{aligned}g_{ij}\,g^{jk} &= (\boldsymbol{e}_i\cdot\boldsymbol{e}_j)\,(\boldsymbol{e}^j\cdot\boldsymbol{e}^k)\\ &= (a_i^n\,a_j^n)(a^{-1})_m^j\,(a^{-1})_m^k\\ &= a_i^n\,\delta_m^n\,(a^{-1})_m^k\\ &= a_i^m\,(a^{-1})_m^k\\ &= \delta_i^k.\end{aligned}\tag{5.42}$$

So, indeed g_{ij} and g^{ij} are inverses of each other and equation (5.41) is well defined. Furthermore

$$\begin{aligned}x^j &= x_i\,\boldsymbol{e}^i\cdot\boldsymbol{e}^j\\ &= (x_i\,\boldsymbol{e}^i)\cdot\boldsymbol{e}^j\\ &= \boldsymbol{x}\cdot\boldsymbol{e}^j\end{aligned}\tag{5.43}$$

so we can interpret x^j as the orthogonal projection of $\boldsymbol{x}$ along the $\boldsymbol{e}^j$ contravariant base vector.

6 Geometrical interpretation of contravariant and covariant coordinates and contravariant and covariant base vectors

With all the machinery so far developed we are ready to give a geometrical interpretation of all the elements so far developed. We have two ways to represent base vectors (covariant in V and contravariant in its dual V*) and two ways to represent coordinates (contravariant as parallel components of the vector along the base vectors, and covariant as orthogonal projections with respect to the base vectors). These give us four combinations that we now study.

6.0.1 space V

The space V is spanned by the covariant base vectors $\boldsymbol{e}_i$. When these base vectors are written in terms of the canonical basis $\hat{\boldsymbol{i}}_j$ they can be seen as columns of certain matrix A.

- For a given vector $\boldsymbol{x} \in$ V, its i–th component with respect to $\boldsymbol{e}_i$ is the contravariant component x^i. Equation (3.1) $\boldsymbol{x} = x^i\,\boldsymbol{e}_i$ represents this statement. As indicated above, in our particular example

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.44}$$

  and $x^1 = 1$, and $x^2 = 2$.

- For a given vector $\boldsymbol{x} \in$ V, its projection on the I–th covariant base vector $\boldsymbol{e}_I$ is $\sqrt{g_{II}}$ times the covariant component x_i. Equation (4.16) $x_j = \boldsymbol{x} \cdot \boldsymbol{e}_j$ represents this statement. In our particular example: $x_1 = 3$, and $x_2 = 4$

6.0.2 dual space V*

The dual space V* is spanned by the covariant base vectors $\boldsymbol{e}_i$. When these base vectors are written in terms of the canonical basis $\hat{\boldsymbol{\imath}}_j$ they can be seen as rows of A^{-1}. The rows of A^{-1} are orthonormal to the columns of A, therefore the covariant vectors $\boldsymbol{e}_i$ are orthonormal to the contravariant base vectors $\boldsymbol{e}_i$.

- For a given vector $\boldsymbol{x} \in \mathrm{V}^*$, its i–th component with respect to $\boldsymbol{e}^i$ is the covariant component x_i. Equation (5.39) $\boldsymbol{x} = x_i\, \boldsymbol{e}^i$ represents this statement.

  In our example:

  $$\boldsymbol{e}^1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{e}^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6.45}$$

  with $x_1 = 3$ and $x_2 = 4$.

- For a given vector $\boldsymbol{x} \in \mathrm{V}^*$, its projection on the I–th contravariant base vector $\boldsymbol{e}^I$ is $\sqrt{g_{II}}$ times the covariant component x^i. Equation (5.43) $x^i = \boldsymbol{x} \cdot \boldsymbol{e}^i$ represents this statement.

Figure (3) shows the representation of the vector $(3, 2)^T$ (our example) in both spaces V and V^* and using both parallel and orthogonal projections. Both bases and components are labeled and weighted with the appropriate metric tensor component so as to respect the length of the elements.

7 Tensor products

We now consider two finite dimensional vector spaces V and W of dimensions n and m respectively. We construct their product † space

$$V \times W = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \in V, \boldsymbol{y} \in W\} \tag{7.46}$$

†the notation of product "×" of vector spaces is the same as that of vectors in the cross product. The context will tell which one is which.

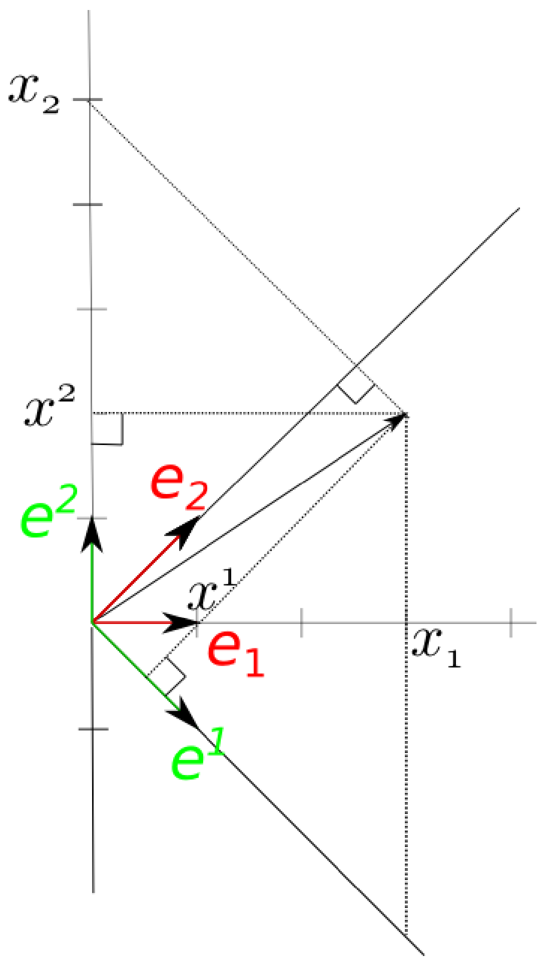

Figure 3: Illustration for the contravariant (x^1, x^2) and covariant (x_1, x_2) coordinates along the covariant $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ and contravariant $(\boldsymbol{e}^1, \boldsymbol{e}^2)$ base vectors. The coordinates are labeled at their distances from the origin. Those distances are measured by multiplying the contravariant coordinates x^I along the the covariant basis by $\sqrt{g_{II}}$ and by dividing them by the same amount along the contravariant basis. The covariant coordinates x_I should be multiplied by $\sqrt{g_{II}}$ along the contravariant basis and divided by the same amount along the covariant basis.

The dimension of $V \times W$ is $n \times m$. The functionals that we are considering for the space $(V \times W)^*$ are bilinear functions. That is they are linear in the "big" coordinates $\boldsymbol{x}$ and $\boldsymbol{y}$ of the compound vector $(\boldsymbol{x}, \boldsymbol{y})$.

I first show that $(V \times W)^* = V^* \times W^*$. The product vector space "$\times$" in the right hand side term is the normal product of functionals, and it is usually noted as

$$V^* \times W^* = V^* \otimes W^*. \tag{7.47}$$

that is any vector in $V^* \otimes W^*$ is seen as

$$\phi(\boldsymbol{x}) \otimes \phi(\boldsymbol{y}) = \phi(\boldsymbol{x})\phi(\boldsymbol{y}). \tag{7.48}$$

Given that the dimensions of V^* and W^* are n and m respectively, the dimension of $V^* \times W^*$ is also $n \times m$.

Let $\boldsymbol{e}_{ij}$ be a basis for $V \times W$ with $i = 1, ..., n$, $j = 1, ..., m$. Let us show that the set $\boldsymbol{e}^{ij}$ defined by the relations

$$\boldsymbol{e}^{ij}(\boldsymbol{e}_{kl}) = \delta^i_k \, \delta^j_l \tag{7.49}$$

form a basis for the dual space $(V \times W)^*$.

We first show that any $\phi \in (V \times W)^*$ is of the form

$$\phi(\boldsymbol{x}, \boldsymbol{y}) = w_{ij} e^{ij}(\boldsymbol{x}, \boldsymbol{y}) \tag{7.50}$$

for some w_{ij}.

We expand $\boldsymbol{x}$ and $\boldsymbol{y}$ along the basis $\boldsymbol{e}_i$ and $\boldsymbol{f}_i$ of the spaces V and W respectively.

$$
\begin{aligned}
\phi(\boldsymbol{x}, \boldsymbol{y}) &= \phi(x^i\, \boldsymbol{e}_i, y^j\, \boldsymbol{f}_j) \\
&= x^i\, y^j \phi(\boldsymbol{e}_i, \boldsymbol{f}_j) \\
&= \delta^i_k\, x^k\, \delta^j_l\, y^l w_{ij} \quad [\text{where} \quad w_{ij} = \phi(\boldsymbol{e}_i, \boldsymbol{f}_j)] \\
&= \boldsymbol{e}^{ij}(\boldsymbol{e}_k, \boldsymbol{f}_l)\, x^k\, y^l\, w_{ij} \\
&= \boldsymbol{e}^{ij}(\boldsymbol{x}, \boldsymbol{y}) w_{ij}.
\end{aligned} \tag{7.51}
$$

This shows that $\boldsymbol{e}^{ij}$ spans $(V \times W)^*$.

Now, if

$$
c_{ij}\, \boldsymbol{e}^{ij} = 0, \tag{7.52}
$$

then for each pair $\boldsymbol{x}$, $\boldsymbol{y}$

$$
c_{ij}\, \boldsymbol{e}^{ij}(\boldsymbol{x}, \boldsymbol{y}) = x^k y^l \boldsymbol{e}^{ij} \boldsymbol{e}_{kl} = c_{ij}\, \delta^i_k \delta^j_l x^k y^l = c_{kl}\, x^k\, y^l = 0 \tag{7.53}
$$

Since we are free to chose any real numbers x^k or y^l then $c_{kl} = 0$. We then found that $\boldsymbol{e}^{ij}$ forms a basis for $(V \times W)^*$. It can be shown that the basis for $V \times W$ is

$$
\boldsymbol{e}_{ij} = (\boldsymbol{e}_i, \boldsymbol{f}_j) \tag{7.54}
$$

so

$$
\begin{aligned}
\boldsymbol{e}^{ij}(\boldsymbol{e}_{ij}) &= \delta^i_k\, \delta^j_l \\
&= \boldsymbol{e}^i(\boldsymbol{e}_k) \boldsymbol{e}^j(\boldsymbol{f}_l)
\end{aligned} \tag{7.55}
$$

but $\boldsymbol{e}^i\, \boldsymbol{e}^j$ form the basis for the products of functionals in the spaces V^* and W^*. In other words, both $(V \times W)^*$ and $V^* \times W^*$ share the same basis and are of the same dimension. They are the same spaces.

We found that the covariant coordinates of a functional ϕ in $(V \times W)^*$ are the product of covariant coordinates in the spaces V^* and W^* respectively. We can then inherit what we know from these spaces. If

$$\begin{aligned} \bar{\boldsymbol{e}}_i &= a_i^j\, \boldsymbol{e}_j \\ \bar{\boldsymbol{f}}_i &= b_i^j\, \boldsymbol{f}_j \end{aligned} \tag{7.56}$$

then from equation (5.30) we obtain that a *second order contravariant component* $\bar{c}^{ij}$ in $V \times W$ transforms as

$$\bar{c}^{ij} = \bar{x}^i\, \bar{y}^j = (a^{-1})^i_k\, x^k\, (b^{-1})^j_l\, y^l = (a^{-1})^i_k\, (b^{-1})^j_l\, x^k y^l = (a^{-1})^i_k\, (b^{-1})^j_l\, c^{kl}. \tag{7.57}$$

and a *second order covariant component* $\bar{c}_{ij}$ in $(V \times W)^* = V \otimes W$ transforms as

$$\bar{c}_{ij} = \bar{x}_i\, \bar{y}_j = a_i^k\, x_k b_j^l\, y_l = a_i^k\, b_j^l\, x_k\, y_l = a_i^k\, b_j^l\, c_{kl} \tag{7.58}$$

Let us now consider the space $V \times W^*$ with elements $(\boldsymbol{x}, \boldsymbol{y})$ with $\boldsymbol{x} \in V$ and $\boldsymbol{y} \in W^*$. The coordinates in this "mixed" space are of the form (shown below)

$$c^i_j = x^i\, y_j \tag{7.59}$$

and they will transform according to the rule

$$\bar{c}^i_j = \bar{x}^i\, \bar{y}_j = (a^{-1})^i_k\, x^k b_j^l\, y_l = (a^{-1})^i_k\, b_j^l\, x^k\, y_l = (a^{-1})^i_k\, b_j^l\, c^l_k \tag{7.60}$$

Let's see why the coordinates in the space $V \times W^*$ are as in equation (7.59).

The set $(\boldsymbol{e}_i, \boldsymbol{f}^j)$ is a basis for $V \times W^*$ (not shown here). Therefore; any element of $V \times W^*$ can be written as

$$(\boldsymbol{x}, \boldsymbol{y}) = c^j_i(\boldsymbol{e}_i, \boldsymbol{f}^j) \tag{7.61}$$

for some c^j_i, but

$$\begin{aligned} (\boldsymbol{x}, \boldsymbol{y}) &= (x^i\, \boldsymbol{e}_i, y_j \boldsymbol{f}^j) \\ &= x^i\, y_j(\boldsymbol{e}_i, \boldsymbol{f}^j) \end{aligned} \tag{7.62}$$

so in fact $c^i_j = x^i\, y_j$ as proposed by equation (7.59).

In general. Let

$$V_1, V_2, \, ...\, V_n \tag{7.63}$$

be a set of n finite dimensional vector spaces with dimensions $i_1, i_2, \, ...\, i_n$ respectively and with basis

$$\{\boldsymbol{e}_{1_i}\}\ \{\boldsymbol{e}_{2_i}\}\,,\ ...\ \{\boldsymbol{e}_{n_i}\} \tag{7.64}$$

Let us define the contravariant basis of the corresponding dual spaces $V_1^*, V_2^*, \, ...\, V_n^*$ as

$$\{\boldsymbol{e}_1{}^i\}\ \{\boldsymbol{e}_2{}^i\}\,,\ ...\ \{\boldsymbol{e}_n{}^i\} \tag{7.65}$$

and construct the product of p V-spaces with q V^*-spaces

$$V_{\alpha_1} \times V_{\alpha_2} \, ... \, \times V_{\alpha_p} \, \times V^*_{\beta_1} \otimes V^*_{\beta_2} \, ... \otimes V^*_{\beta_p}. \tag{7.66}$$

$p, q < n$. We use the Greek letters α_i to denote objects related to the space V_{α_i} and β_i to denote objects related to the space $V^*_{\beta_i}$. A base element of the space (7.66) would be of the form

$$(\boldsymbol{e}_{\alpha_1 k_1}, \boldsymbol{e}_{\alpha_2 k_2}, \, ... \, \boldsymbol{e}_{\alpha_p k_p}, \boldsymbol{e}^{l_1}_{\beta_1}\, \boldsymbol{e}^{l_2}_{\beta_2} \, ... \, \boldsymbol{e}^{l_q}_{\beta_q}), \tag{7.67}$$

where $1 \leq k_j \leq i_j$ and $1 \leq l_j \leq i_j$. Any vector of the space (7.66) can be seen as

$$c^{\,l_1\ l_2\ ...\ l_p}_{m_1 m_2 \, ... \, m_q} = (x^{l_1}_{\alpha_1}\, x^{l_2}_{\alpha_2} \, ... \, x^{l_p}_{\alpha_p})\ (y_{\beta_1 m_1}\, y_{\beta_2 m_2} \, ... \, y_{\beta_q m_q}) \tag{7.68}$$

and it transforms under a change of basis as follows:

$$\begin{aligned} \bar{c}^{\,l_1\ l_2 \, ... \ l_p}_{m_1 m_2 ... m_q} \ &= \ (a^{-1}_{\alpha_1})^{l_1}_{s_1}\, x^{s_1}_{\alpha 1}\, (a^{-1}_{\alpha_2})^{l_2}_{s_2}\, x^{s_2}_{\alpha_2} \, ... \, (a^{-1}_{\alpha_p})^{l_p}_{s_p}\, x^{s_p}_{\alpha_p} \\ \cdot \quad & (a_{\beta_1})^{t_1}_{m_1}\, y_{\beta_1 t_1}\, (a_{\beta_2})^{t_2}_{m_2}\, y_{\beta_2 t_2} \, ... \, (a_{\beta q})^{t_q}_{m_q}\, y_{\beta_q t_q} \end{aligned} \tag{7.69}$$

where $(a_{\beta i})^{t_i}_{m_i}$ is the matrix for the change of basis in the space V_i from $\bar{\boldsymbol{e}}^{t_i}_{\alpha_i}$ to $\boldsymbol{e}^{t_i}_{\alpha_i}$ and $x^{s_\alpha}_\alpha$ is the s_α–contravariant coordinate in the space V_α. In the same way the symbols $(a^{-1}_{\alpha_i})^{l_i}_{s_i}$ indicate the matrix of transformation between the contravariant base vectors $\bar{\boldsymbol{e}}^{s_i}_{\beta_i}$

and $\boldsymbol{e}^{s_i}_{\beta_i}$ and $y_{\beta_i t_i}$ is the t_i–covariant coordinate of the space $V^*_{\beta_i}$.

We found then that the objects $c^{s_1 s_2 \dots s_p}_{t_1 t_2 \dots t_q}$ transform (as the basis of the space (7.66) transform "multi–linearly") following the rule

$$\bar{c}^{l_1\ l_2 \dots\ l_p}_{m_1 m_2 \dots m_q} \quad = \quad (A^{-1})^{l_1 l_2 \dots l_p}_{s_1 s_2 \dots s_p}\ (A)^{t_1\ t_2 \dots\ t_q}_{m_1 m_2 \dots m_q}\ c^{s_1\ s_1 \dots s_p}_{t_1\ t_2 \dots t_q}, \tag{7.70}$$

where

$$(A^{-1})^{l_1\ l_2 \dots l_p}_{s_1 s_2 \dots s_p} = (a^{-1}_{\alpha_1})^{l_1}_{s_1}\ (a^{-1}_{\alpha_2})^{l_2}_{s_2} \dots (a^{-1}_{\alpha_p})^{l_p}_{s_p}, \tag{7.71}$$

$$(A)^{t_1\ t_2 \dots\ t_q}_{m_1 m_2 \dots m_q} = (a_{\beta_1})^{t_1}_{m_1}\ (a_{\beta_2})^{t_2}_{m_2} \dots (a_{\beta q})^{t_q}_{m_q} \tag{7.72}$$

and

$$c^{s_1\ s_1 \dots s_p}_{t_1\ t_2 \dots t_q} = (x^{s_1}_{\alpha_1}\ x^{s_2}_{\alpha_2} \dots x^{s_p}_{\alpha_p})\ (y_{\beta_1 t_1}\ y_{\beta_2 t_2} \dots y_{\beta_q t_q}). \tag{7.73}$$

The set of numbers in equation (7.73) is referred to as *a tensor of contravariant order p and covariant order q.*

8 Examples

8.1 Tensors associated to a surface

An $(n-1)$–dimensional hyper–surface in the n dimensional Euclidean space is parameterized by $n-1$ parameters $u_1, u_2, \dots u_{n-1}$. Pick any vector $\boldsymbol{x}$ on the surface and let us assume that the surface is differentiable in $\boldsymbol{x}$. The vectors $\boldsymbol{e}_i = \partial \boldsymbol{x}/\partial u_i$ are linearly independent and form a basis for the tangent hyper–plane. Every vector in the tangent hyper–plane can be written as $\boldsymbol{x} = c^i \partial \boldsymbol{x}/\partial u_i$. Here c^i are the contravariant coordinates of $\boldsymbol{x}$ in the basis $\partial \boldsymbol{x}/\partial u_i$. The covariant coordinates are found by the rule

$$c_i = \boldsymbol{e}_i \cdot \boldsymbol{e}_j\, c^j = g_{ij} c^j = \frac{\partial \boldsymbol{x}}{\partial u_i} \cdot \frac{\partial \boldsymbol{x}}{\partial u_j}\, c^j, \tag{8.74}$$

where g_{ij} is a second order covariant tensor, known as the metric tensor.

Now let us take a curve in the hyper–surface with a parameter s. The tangent vector to the surface along this curve is given by

$$\frac{d\boldsymbol{x}}{ds} = \frac{\partial \boldsymbol{x}}{\partial u_i}\frac{\partial u_i}{\partial s}. \tag{8.75}$$

Therefore we can interpret the contravariant vectors $c^i(s)$ as the components of the tangent to the surface along the curve $\boldsymbol{x}(s)$ with respect to the basis $\boldsymbol{e}_i$.

8.2 Field tensors

Let us recall that a linear transformation between the covariant basis $\bar{\boldsymbol{e}}_i = a^i_j \boldsymbol{e}_j$ induces a linear transformation between the contravariant vectors $\bar{x}^i = (a^{-1})^i_j\, x^j$. Now let us assume a non-linear transformation between the contravariant coordinates

$$\bar{x}^i = f^i(x^j). \tag{8.76}$$

We further assume that x^j are in an Euclidean space where the basis are orthogonal, so no distinction is made between covariant and contravariant coordinates. That is $x^j = x_j$.

The coordinate transformation in equation (8.76) should be invertible and locally differentiable. Then for the region of interest,

$$\det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) \neq 0 \tag{8.77}$$

We fix a point $\boldsymbol{x}_0$ and approximate the coordinates close to that point by using the first term of the Taylor series expansion. That is

$$\bar{x}^i = \bar{x}^i_0 + \left.\frac{\partial f_i}{\partial x_j}\right|_{\boldsymbol{x}=\boldsymbol{x}_0}(x^j - x^j_0). \tag{8.78}$$

Let us define the new contravariant coordinates

$$\begin{aligned} d\bar{x}^i &= \bar{x}^i - \bar{x}^i_0 \\ dx^j &= x^i - x^i_0 \end{aligned} \tag{8.79}$$

so

$$d\bar{x}^i = \frac{\partial f_i}{\partial x_j} dx^j = \frac{\partial \bar{x}_i}{\partial x_j} dx^j. \tag{8.80}$$

Then we interpret dx^i as contravariant coordinates that follow the rule (8.80) and identify $\partial\bar{x}^i/\partial x^j$ with (a^{-1}) in equation (5.30). In this sense, the vectors dx^i are contravariant tensors of order one attached to the point $\boldsymbol{x}_0$. We then find that the no linear transformation between Cartesian coordinates (8.76) induces a contravariant tensor for each fixed point $\boldsymbol{x}_0$ of the Euclidean vector space.

From equation 5.30 we have the relation

$$\boldsymbol{e}^i = (a^{-1})^j_i \bar{\boldsymbol{e}}^j \tag{8.81}$$

so if the vectors $\bar{\boldsymbol{e}}^j$ represent the cannonical basis, then

$$(\boldsymbol{e}^i)_k = \frac{\partial \bar{x}^i}{\partial x_j} \delta^j_k = \frac{\partial \bar{x}^i}{\partial x_k}. \tag{8.82}$$

That is the contravariant base vectors are the columns of the Jacobian matrix, or the gradient of the scalar fields ∇f_i.

The covariant base vectors are orthonormal to this. While some are in a normal direction the others are in a corresponding tangential direction as interpreted previously. The duality between the two groups is clear if we write

$$\delta^j_i = \frac{\partial x_i}{\partial \bar{x}_k} \frac{\partial \bar{x}_k}{\partial x_j}. \tag{8.83}$$

The vector space V is spanned by the vectors

$$\frac{\partial x_i}{\partial \bar{x}_1}, \frac{\partial x_i}{\partial \bar{x}_2}, \dots, \frac{\partial x_i}{\partial \bar{x}_n} \tag{8.84}$$

and the vector space V^* is spanned by the vectors

$$\frac{\partial \bar{x}_i}{\partial x_1}, \frac{\partial \bar{x}_i}{\partial x_2}, \dots, \frac{\partial \bar{x}_i}{\partial x_n}. \tag{8.85}$$

Given that V^* is an inner product space, these base vectors should be interpreted as vectors over which V is projected. For example the functional corresponding to
$\boldsymbol{e}^1 = \partial \bar{x}_1 / \partial x_i$ is

$$\begin{aligned} \phi : \quad & \mathrm{V} \longrightarrow \mathbb{R} \\ & dx \longmapsto \frac{\partial \bar{x}_1}{\partial x_i}\, dx_i = d\bar{x}_1. \end{aligned} \tag{8.86}$$

Therefore we can identify the space V^* as the space of differentials and its basis can be represented as

$$dx_1, dx_2, \dots, dx_n. \tag{8.87}$$

If we expand the Taylor series for f_i up to second order terms we have

$$\bar{x}^j = \bar{x}^j_0 + \frac{\partial \bar{x}^j}{\partial x_k}\, dx^k + \frac{\partial \bar{x}^j}{\partial x^k\, \partial x^l}\, dx^k\, dx^l, \tag{8.88}$$

or in other words

$$d\bar{x}^j = \frac{\partial \bar{x}^j}{\partial x_k}\, dx^k + \frac{\partial \bar{x}^j}{\partial x^k\, \partial x^l}\, dx^k\, dx^l, \tag{8.89}$$

Let us call

$$d(d\bar{x}^j) = d\bar{x}^j - \frac{\partial \bar{x}^j}{\partial x_k}\, dx^k \tag{8.90}$$

then

$$d(d\bar{x}^j) = \frac{\partial \bar{x}^j}{\partial x^k\, \partial x^l}\, dx^k\, dx^l. \qquad (8.91)$$

This approach induces a second order covariant tensor for $\partial \bar{x}^j / \partial x^k \partial x^l$. However the machinery to consider higher order tensor was developed above, so we now how to build tensors in the products $V \times W$, $V^* \times W$ $V \times W^*$ and $V^* \otimes W^*$.

In general. A tensor field (attached to the point $\boldsymbol{x}_0$) of contravariant order p and covariant order q is defined as the set of elements (the A's) of the vector space $V^p \times (V^*)^q$ that satisfies the transformation rule

$$\bar{A}^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}(\bar{\boldsymbol{x}}) = \frac{\partial \bar{x}^1}{\partial x^m} \frac{\partial \bar{x}^{i_1}}{\partial x^{m_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{m_2}} \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{m_p}} \frac{\partial x^{n_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{n_2}}{\partial \bar{x}^{j_2}} \cdots \frac{\partial x^{n_q}}{\partial \bar{x}^{j_q}} A^{m_1 m_2 \dots m_p}_{n_1 n_2 \dots n_q}(\boldsymbol{x}). \qquad (8.92)$$

Here $V^p = V \times V \times V \dots \times V$ p–times and $(V^*)^q = V^* \otimes V^* \otimes V^* \dots \otimes V^*$ q–times .

8.3 other type of tensors

A *relative tensor* or *weighted tensor* of weight M is defined as those elements satisfying the transformation rule

$$\bar{A}^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}(\bar{\boldsymbol{x}}) = \gamma^M \frac{\partial \bar{x}^1}{\partial x^m} \frac{\partial \bar{x}^{i_1}}{\partial x^{m_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{m_2}} \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{m_p}} \frac{\partial x^{n_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{n_2}}{\partial \bar{x}^{j_2}} \cdots \frac{\partial x^{n_q}}{\partial \bar{x}^{j_q}} A^{m_1 m_2 \dots m_p}_{n_1 n_2 \dots n_q}(\boldsymbol{x}), \qquad (8.93)$$

where

$$\gamma = \det \frac{\partial \bar{x}^i}{\partial x^j}.$$

If $M = 0$ then the tensor $A^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}$ is called *absolute*. Appendix A shows that the alternating tensor is indeed a relative tensor with weight -1. When the space V is an Euclidean space with Cartesian coordinates (orthogonal coordinates) then $g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$ and there is not distinction between contravariant and covariant coordinates. The tensors here are called *Cartesian tensors*. Tensors with respect to linear transformations are not necessary tensors with respect to the more general transformation f_i. In Cartesian tensors the transformation are rotations. They are not necessarily tensors under more general transformations. As an example, the alternating tensor is a relative tensor of weight -1. If the transformation is orthogonal (a rotation or a reflection) then $\gamma = 1$ and the transformation for absolute tensors apply. However if the transformation is not orthogonal and $\gamma \neq 1$ then the transformation for absolute tensors will not apply.

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A The alternating tensor

I define the alternating tensor. This tensor in the 3–D Euclidean space is known as the Levi-Civitta tensor. Some immediate applications of it are illustrated.

A.1 permutation

A permutation —in the n–dimensional space— is a bijective function $a(n)$ between the set of numbers $\{1, 2, ..., n\}$ and itself. This will be noted as $(a_1a_2....a_n)$. A transposition is the interchange of two elements. For example by interchanging the first and last element of the previous permutation we would obtain $(a_na_2....a_1)$, which is another permutation.

It can be shown that any permutation can be transformed into the identity function by a finite number of transpositions. The minimum number of transpositions required to convert a permutation $(a_1 a_2 .... a_n)$ into the identity function is called the order of the permutation and we noted as $o(a_1 a_2 .... a_n)$. That order is a natural number. If that number is even we say that the permutation is even, if that number is odd we say that the permutation is odd.

A.2 the characteristic function of a set

Given a set , we define the characteristic function $\chi_{_A}$ of A as

$$\chi_{_A}(x) = \begin{cases} 1 & \text{if } x \in \text{A} \\ 0 & \text{otherwise} \end{cases}$$

A.3 alternating tensor

Let us assume that $\{i_1, i_2, ..., i_n\}$ is a set of numbers between 1 and n with possible repetitions, and let us call A, the set of permutations corresponding to the first n natural numbers. The tensor ‡

$$\epsilon^{i_1 i_2 .....i_n} = \epsilon_{i_1 i_2 .....i_n} = \chi_{_A}(i_1, i_2, ..., i_n)(-1)^{o(i_1 i_2, ...i_n)} \tag{A.94}$$

is defined as the *alternating tensor*. While $o(i_1 i_2 .... i_n)$ is not a well defined function for mappings that are not permutations, the characteristic function will control the value of $\epsilon_{i_1 i_2 .....i_n}$ so that this is a well defined number (that is, we could give an order of 0 to a sequence of objects with repetitions) . In this case any finite number assigned to $o(i_1 i_2 .... i_n)$ would work fine. For example

$$\epsilon_{ij} = \epsilon^{ij} = \begin{cases} 1 & \text{if} \quad i = 1,\ j = 2 \\ -1 & \text{if} \quad i = 2,\ j = 1 \\ 0 & \text{if} \quad i = j \end{cases}$$

In the three–dimensional space, the tensor ϵ_{ijk} is known as the Levi–Civitta tensor.

‡We show at the end that this symbol is actually a tensor.

A.4 examples

Here I present some direct applications of the alternating tensor

- **the determinant:** I start by using Munkre's [2] definition of determinant and show how this definition is equivalent to McConnell's [4] definition.

  Munkre's definition of determinant is as follows: A functon that assigns, to each n by n matrix A, a real number denoted $\det A$, is called a **determinant function** if it satisfies the following axioms:

  1. If B is the matrix obtained by exchanging any two rows of A, then $\det B = -\det A$.
  2. Given i, the function $\det A$ is linear as a function of the i^{th} row alone.
  3. $\det I_n = 1$.

  Using the notation $A = (a_{ij})$. The i^{th} raw of A can be written as

$$a_{ij_i}\, \boldsymbol{e}^{j_i} \tag{A.95}$$

  with $\boldsymbol{e}^{j_i}$ the cannonical vector with the k component given by δ^{kj_i}. Using the second axiom of the Munkre's definition, that is applying linearity we can expand the determinant with respect to the first, second up to the last row in that order and obtain the expression

$$\det A = a_{1j_1} a_{2j_2} ... a_{nj_n} \; \det \begin{pmatrix} \boldsymbol{e}^{j_1} \\ \boldsymbol{e}^{j_2} \\ . \\ . \\ . \\ \boldsymbol{e}^{j_n} \end{pmatrix} \tag{A.96}$$

  If we start with the identity I_n matrix and permute rows as to obtain the matrix above which rows are as the cannontical vectors $\boldsymbol{e}^{j_1}, \boldsymbol{e}^{j_2}, ..., \boldsymbol{e}^{j_n}$, then by using axiom (1) and (3) we obtain $-1^{o(j_1 j_2 \cdots j_n)}$. Of course it could happen that some factors could have repeated base vectors $\boldsymbol{e}^{j}$, in which case its contribution is zero (this is an easy consequence of axiom (1)). In other words, we arrive to McConnell's [4] definition of determinant:

$$\det(a_{ij}) = \epsilon^{j_1 j_2 \dots j_n} a_{1j_1} a_{2j_2} \dots a_{nj_n}. \tag{A.97}$$

- **the cross product:** Given two vectors $\boldsymbol{x} = (x_i)$, and $\boldsymbol{y} = (y_i)$ in the three–dimensional space, we define the i–th component of the cross "$\times$" product as

$$(\boldsymbol{x} \times \boldsymbol{y})_i = \epsilon^{ijk} x_j \, y_k. \tag{A.98}$$

The generalization of the cross product to $n \geq 2$ dimensions is given by the formula

$$\times \, [\, \boldsymbol{x}_1 \, \boldsymbol{x}_2 \dots\dots \boldsymbol{x}_{k-1} \, \boldsymbol{x}_{k+1} \dots\dots \boldsymbol{x}_n]_i = \epsilon^{i_1 i_2 .. i_{k-1} \, i \, i_{k+1} \dots i_n} x_{i_1 1} x_{i_2 2} \dots\dots x_{i_{k-1} k-1} x_{i_{k+1} k+1} \dots . x_{i_n n}. \tag{A.99}$$

An interesting application of this definition is the interpretation of the inverse of a matrix A with columns $\boldsymbol{e}^j$. Let us call the raws of the inverse matrix $\boldsymbol{e}_i$ so that we have the identity

$$\boldsymbol{e}_i \cdot \boldsymbol{e}^j = \delta_{ij} \tag{A.100}$$

For convenience in notation let us refer to the i-th component of the j-th column of A as a_i^j and the i-th component of $\boldsymbol{e}_k$ as e_k^i. Then from the definition of determinant (A.97) we have

$$\begin{aligned} e_k^i \det A &= e_k^i \epsilon^{i_1 i_2 .. i_{k-1} \, i_k \, i_{k+1} \dots i_n} a_{i_1}^1 a_{i_2}^2 \dots\dots a_{i_{k-1}}^{k-1} a_{i_k}^k a_{i_{k+1}}^{k+1} \dots . a_{i_n}^n \\ &= \epsilon^{i_1 i_2 .. i_{k-1} \, i_k \, i_{k+1} \dots i_n} a_{i_1}^1 a_{i_2}^2 \dots\dots a_{i_{k-1}}^{k-1} \, a_{i_k}^k \, e_k^i \, a_{i_{k+1}}^{k+1} \dots . a_{i_n}^n \\ &= \epsilon^{i_1 i_2 .. i_{k-1} \, i_k \, i_{k+1} \dots i_n} a_{i_1}^1 a_{i_2}^2 \dots\dots a_{i_{k-1}}^{k-1} \, \delta_{i_k}^i \, a_{i_{k+1}}^{k+1} \dots . a_{i_n}^n \\ &= \epsilon^{i_1 i_2 .. i_{k-1} \, i \, i_{k+1} \dots i_n} a_{i_1}^1 a_{i_2}^2 \dots\dots a_{i_{k-1}}^{k-1} a_{i_{k+1}}^{k+1} \dots . a_{i_n}^n \\ &= \times [A^1 A^2 \dots\dots A^{k-1} A^{k+1} \dots\dots A^n]_i \end{aligned} \tag{A.101}$$

where A^j is the j-th column of A. Now given that the j-th column of A is $\boldsymbol{e}^j$ we found that

$$\boldsymbol{e}_k = \frac{\times[\boldsymbol{e}^1\boldsymbol{e}^2.....\boldsymbol{e}^{k-1}\boldsymbol{e}^{k+1}.....\boldsymbol{e}^n]}{\det A}. \tag{A.102}$$

This means that the k-th raw of A^{-1} (that is e_k) is the generalized cross product of all other indexed columns of A "normalized" by the determinant of A.

In solving the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ we know that

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} \tag{A.103}$$

so that by using equation (A.102) we find that

$$x_i = \boldsymbol{e}_i \cdot \boldsymbol{b} = \frac{\times[\boldsymbol{e}^1\boldsymbol{e}^2.....\boldsymbol{e}^{i-1}\boldsymbol{e}^{i+1}.....\boldsymbol{e}^n] \cdot b}{\det A}. \tag{A.104}$$

from definition (A.99) it is seen that

$$\times[\boldsymbol{e}^1\boldsymbol{e}^2.....\boldsymbol{e}^{i-1}\boldsymbol{e}^{i+1}.....\boldsymbol{e}^n] \cdot b \tag{A.105}$$

is the determinant of the matrix A after replacing its column A^i by $\boldsymbol{b}$. Equation (A.104) is Cramer's rule.

A.5 The differential form of *kth degree*

A differential form of the *kth degree* on an n–dimensional space —or manifold. This is part of the theory of global differential geometry— with coordinates $x_1, x_2, ...x_n$ is an expression of the form

$$a_{i_1 i_2 .... i_k}(\boldsymbol{x})\; dx^{i_1} dx^{i_2} .... dx^{i_k}. \tag{A.106}$$

Sometimes this is written with the help of the wedge product sign " $\wedge$ " as

$$a_{i_1 i_2 .... i_k}(\boldsymbol{x})\; dx^{i_1} \wedge dx^{i_2} \wedge .... \wedge dx^{i_k} \tag{A.107}$$

to indicate that the product of differentials is *anticommutative*, that is

$$dx_i \wedge dx_j = -dx_j \wedge dx_i. \tag{A.108}$$

We will omit the wedge sign " $\wedge$ "but still we will honor the anticommuation rule.

Each term in the differential form (A.106) is anti-symmetric in all its indices, so that this can be writen as

$$a_{i_1 i_2 .... i_k}(\boldsymbol{x})\; dx^{i_1} dx^{i_2} .... dx^{i_k} = \epsilon^{i_1 i_2 .... i_k}\; a_{i_1 i_2 .... i_k}(\boldsymbol{x})\; dx^{j_1} dx^{j_2} .... dx^{j_k} \tag{A.109}$$

where $j_1 < j_2 < .... < j_k$ is an arrangement of the indices $i_1 i_2 .... i_k$ in strict increasing order.

The first interesting property of the *kth degree* differential form the transformation under change of coordinates. Let us assume a change of coordinates $x_i = x_i(y_1, y_2, ..., y_n)$. Then

$$a_{i_1 i_2 .... i_k}(\boldsymbol{x})\; dx^{i_1} dx^{i_2} .... dx^{i_k} = \epsilon^{i_1 i_2 .... i_k}\; a_{i_1 i_2 .... i_k}(\boldsymbol{y})\; \frac{\partial x_{i_1}}{\partial y_{j_1}} \frac{\partial x_{i_2}}{\partial y_{j_2}} ..... \frac{\partial x_{i_k}}{\partial y_{j_k}}\; dy^{j_1} dy^{j_2} .... dy^{j_k} \tag{A.110}$$

where $j_1 < j_2 < .... < j_k$. Then by the definition of determinant (A.97) we have

$$\frac{\partial(x_{i_1}, x_{i_2}, ..., x_{i_k})}{\partial(y_{j_1}, x_{j_2}, ..., x_{j_k})} = \epsilon^{i_1 i_2 .... i_k}\; \frac{\partial x_{i_1}}{\partial y_{j_1}} \frac{\partial x_{i_2}}{\partial y_{j_2}} ..... \frac{\partial x_{i_k}}{\partial y_{j_k}}\; dy^{j_1} dy^{j_2} .... dy^{j_k}, \tag{A.111}$$

so

$$a_{i_1 i_2 .... i_k}(\boldsymbol{x})\; dx^{i_1} dx^{i_2} .... dx^{i_k} = a_{i_1 i_2 .... i_k}(\boldsymbol{y})\; \frac{\partial(x_{i_1}, x_{i_2}, ..., x_{i_k})}{\partial(y_{j_1}, x_{j_2}, ..., x_{j_k})}\; dy^{j_1} dy^{j_2} .... dy^{j_k}. \tag{A.112}$$

The sum should be done in a way that $j_1 < j_2 < .... < j_k$.

Another interesting definition in the global differential geometry is the exterior derivative. Let us formulate it.

The *exterior derivative* of a differential form $\alpha = a_{i_1 i_2 .... i_k}(\boldsymbol{x})\, dx^{i_1} dx^{i_2} .... dx^{i_k}$ is defined as the $(k+1)$st differential form

$$d\alpha = \left(\frac{\partial a_{i_1 i_2 .... i_k}(\boldsymbol{x})}{\partial x_i} dx^i \right)\, dx^{i_1} dx^{i_2} .... dx^{i_k}. \tag{A.113}$$

For instance, the exterior derivative of a form of degree zero — a scalar— function $a(\boldsymbol{x})$, is given by the first degree form

$$\frac{\partial a_i(\boldsymbol{x})}{\partial x_i}\, dx_i. \tag{A.114}$$

We identify the coefficients on this expressions with the gradient of $a_i(\boldsymbol{x})$. We now find the exterior derivative of the first degree form $a_i(\boldsymbol{x})\, dx^i$. Let us first define the generalized Kronecker delta symbol

$$\delta^{i_1 i_2 ...... i_k}_{j_1 j_2 ..... j_k} = \begin{cases} 1 & \text{if}(i_1 i_2 ...... i_k) \text{ is an even permutation of } (j_1 j_2 ...... j_k) \\ -1 & \text{if}(i_1 i_2 ...... i_k) \text{ is an odd permutation of } (j_1 j_2 ...... j_k) \\ 0 & \text{otherwise} \end{cases} \tag{A.115}$$

Then

$$\begin{aligned} a_{i,j} dx^j dx^i &= \frac{1}{2}(a_{i,j} - a_{j,i}) dx^j dx^i \\ &= \frac{1}{2}\delta^{pq}_{ij}\, a_{p,q}\, dx^j dx^i \\ &= \frac{1}{2}\delta^{kpq}_{kij} a_{p,q} dx^j dx^i \\ &= \frac{1}{2}\epsilon^{kij}\epsilon_{kpq} a_{p,q} dx^j dx^i \\ &= \frac{1}{2}\epsilon^{ijk}\epsilon_{kqp} a_{p,q} dx^i dx^j. \end{aligned} \tag{A.116}$$

An alternative way to show this is by realizing that

$$\begin{aligned} a_{i,j} - a_{j,i} &= a_{m,l}\delta_i^m \, \delta_j^l - a_{m,l}\delta_j^m \, \delta_i^l \\ &= a_{m,l} \, \epsilon_{kij} \, \epsilon^{kml}, \end{aligned} \tag{A.117}$$

where the "ed"–rule was used.

In equation (A.116) we see that for the three–dimensional space the external derivative can be identified with the curl in the last equation. That is

$$a_{i,j} dx^j dx^i = \epsilon^{ijk} \, (\nabla \times a)_k \, dx^i dx^j. \tag{A.118}$$

A.6 The alternating tensor is a relative tensor with weight -1

Here we show that the alternating tensor (here defined as an n order covariant tensor) is a relative tensor with weight 1.

Let us see.

$$\epsilon_{i_1 i_2 \ldots . i_p .. i_q \ldots . i_n} a^{i_1}_{j_1} a^{i_2}_{j_2} \ldots . a^{i_p}_{j_p} .. a^{i_q}_{j_q} \ldots . a^{i_n}_{j_n} = \epsilon_{i_1 i_2 \ldots . i_q .. i_p \ldots . i_n} a^{i_1}_{j_1} a^{i_2}_{j_2} .. a^{i_q}_{j_p} .. a^{i_p}_{j_q} \ldots a^{i_n}_{j_n},$$

where we renamed the dummy index i_p as i_q and i_q as i_p. Now we commute $a^{i_q}_{j_p}$ with $a^{i_p}_{j_q}$ and find

$$\begin{aligned} \epsilon_{i_1 i_2 \ldots . i_p .. i_q \ldots . i_n} a^{i_1}_{j_1} a^{i_2}_{j_2} \ldots . a^{i_p}_{j_p} .. a^{i_q}_{j_q} \ldots . a^{i_n}_{j_n} &= \epsilon_{i_1 i_2 \ldots . i_q .. i_p \ldots . i_n} a^{i_1}_{j_1} a^{i_2}_{j_2} \ldots . a^{i_p}_{j_q} .. a^{i_q}_{j_p} \ldots . a^{i_n}_{j_n} \\ &= -\epsilon_{i_1 i_2 \ldots . i_p .. i_q \ldots . i_n} a^{i_1}_{j_1} a^{i_2}_{j_2} \ldots . a^{i_p}_{j_q} .. a^{i_q}_{j_p} \ldots . a^{i_n}_{j_n}, \end{aligned}$$

the minus sign in the second step comes from interchanging the indexes i_p and i_q in the alternating tensor. We proved then that

$$\epsilon_{i_1 i_2 \ldots . i_p .. i_q \ldots . i_n} a^{i_1}_{j_1} a^{i_2}_{j_2} \ldots . a^{i_p}_{j_p} .. a^{i_q}_{j_q} \ldots . a^{i_n}_{j_n} \tag{A.119}$$

is completely skew–symmetric in its (free) indexes j_k therefore

$$\begin{aligned}\epsilon_{i_1 i_2 \ldots . i_n} a^{i_1}_{j_1} a^{i_2}_{j_2} \ldots . a^{i_n}_{j_n} &= \epsilon_{i_1 i_2 \ldots . i_n} a^{i_1}_{1} a^{i_2}_{2} .. a^{i_n}_{n} \epsilon_{j_1 j_2 \ldots . j_n} \\ &= \det(a_{ij}) \epsilon_{j_1 j_2 \ldots . j_n}. \qquad (A.120)\end{aligned}$$

so

$$\epsilon_{j_1 j_2 \ldots . j_n} = \det(a_{ij})^{-1} \epsilon_{i_1 i_2 \ldots . i_n} a^{i_1}_{j_1} a^{i_2}_{j_2} \ldots . a^{i_n}_{j_n} \qquad (A.121)$$

Therefore $\epsilon_{i_1 i_2 \ldots . i_n}$ is a relative n–order covariant tensor with weight -1. Here the matrix A should chosen as the matrix of coordinate transformation (or transformation from the old base to the new base). The n–order contravariant tensor $\epsilon_{i_1 i_2 \ldots . i_n}$ is a relative tensor of weight 1

In the Euclidean space, the coordinate transformations are orthogonal therefore $\det(a_{ij}) = 1$. The tensors in the Euclidean space are known as Cartesian tensors, then $\epsilon_{i_1 i_2 \ldots . i_n}$ is a Cartesian tensor. In the Euclidean space there is not distinction between the words covariant or contravariant.