Difference between phase and group velocity

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1 Introduction

Just a few concepts. No more.

2 Phase and Group Velocities by example

Think about two monochromatic waves added together. These are

$$f(x,t) = A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t)$$

This needs a 3D plot (do it later). Assume two fixed frequencies and wave numbers ω_1, ω_2 and k_1, k_1 respectively.

We want to write these two waves as a wave package. The trick is to convert the sum into a product (if you want you can input the expression into Wolfram Alpha The result is

$$f(x,t) = 2A\cos(k_- x - \omega_- t)\ \cos(k_+ x - \omega_+ t) \tag{2 .1}$$

where

$$\begin{aligned} k_+ &= \frac{k_1 - k_2}{2} \qquad k_+ = \frac{k_1 + k_2}{2} \\ \omega_+ &= \frac{\omega_1 - \omega_2}{2} \qquad \omega_+ = \frac{\omega_1 + \omega_2}{2} \end{aligned}$$

where k_- is the semi-difference and k_+ is the semi-sum (or average), and similar for ω_- and ω_+.

If $k_1 \approx k_2$ the difference is very small the pattern is that of the phenomena "bitting". That is a wave package with two distinctive frequencies/wavenumbers.

Let us compute the main parameters in the representation 2 .1. Assume $k_1 = 11, k_2 = 10, \omega_1 = 5$, and $\omega_2 = 4$.

Then

$$k_- = 0.5 \;\; \omega_- = 0.5 \quad k_+ = 10.5 \;\; \omega_+ = 4.5.$$

Let us compute the periods $T = 2\pi/\omega$ and the wavelengths $\lambda = 2\pi/k$. That is for periods

$$T_- \approx 12.57\,\mathrm{s} \quad T_+ \approx 1.4\,\mathrm{s}$$

and for wavelengths

$$\lambda_- \approx 12.57\,\mathrm{m} \quad \lambda_+ \approx 0.6\,\mathrm{m}$$

Figure 1 shows that this is the case. The long periods/wavelengths are a bit more than 12 (in the plot see that a period/wavelength goes about between minus six and six).

It is harder from the figure to estimate the short period/wavelength. But it is clear that the wavelength is shorter than the period.

Most important here is to establish the velocity of an imaginary point moving along a crest of a high or low frequency/wavenumber.

The speed of the high frequency component is

$$v_\phi = \frac{\omega_+}{k_+} = 2.33\,\mathrm{m/s} \tag{2 .2}$$

while that of the low frequency component is

$$v_g = \frac{\omega_-}{k_-} = 1.0\,\mathrm{m/s} \tag{2 .3}$$

These can be computed from the contours (straight diagonal lines) in the base of the plot. For example, consider the high frequency stripes in the $x - t$ plane. The one that starts at the x-axis at $x = 2$, reaches the t-axis at $t \approx 4$. The time to the lower corner is $4 - (-7) = 11$ s, the space traveled is $7 - 2 = 5$ m, the velocity (slope) of the line is $v \approx 11/5 = 2.2$. Similarly look at the low frequency strips (with clear different slope). Take that one that

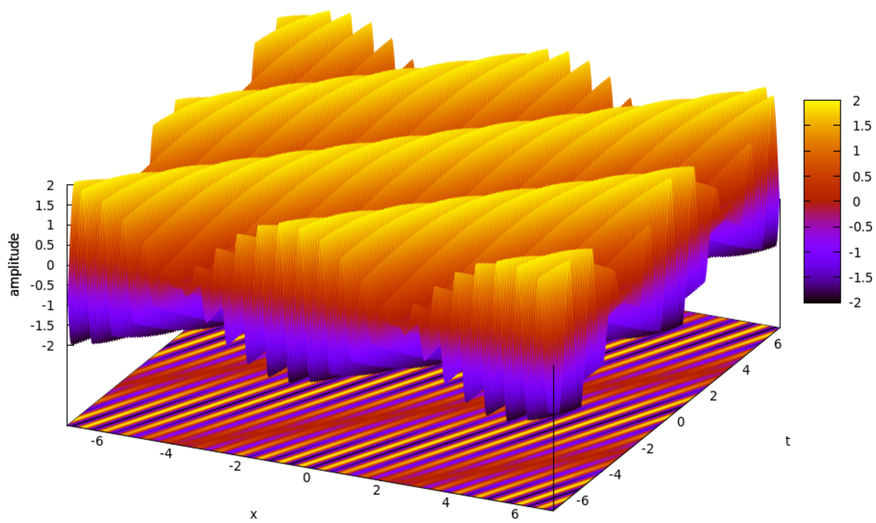

Figure 1: Plot of the function $f(x, y)$ defined in this context.

starts at the x-axis at $x = 2$, reaches the t-axis at $t = -2$. The total time is $(-2 - (-7)) = 5$ s, and the total distance traveled is $7 - 2 = 5$ m. That is for a velocity of $v = 5/5 = 1.0$. So the slow of the thick (low frequency) contours is v_g while that of the thin (high frequency) contours is v_ϕ.

The term v_ϕ means *phase velocity* while the term v_g is group velocity. From the expressions 2 .2 and 2 .3 we could say that taking the limit as the Δk and $\Delta\omega$ go to zero,

$$v_\phi = \frac{\omega}{k} \qquad v_g = \frac{\partial\omega}{\partial k}$$

However this needs a bit of mathematical formalism. We define the phase as

$$\phi = kx - \omega t \tag{2 .4}$$

and from this we want to implicitly find the (phase) velocity. We keep the phase constant and want to find dx/dt keeping ϕ constant. Taking derivatives with respect to t we see that

$$0 = k\frac{\partial x}{\partial t} - \omega$$

or

$$\frac{\partial x}{\partial t} = \frac{\omega}{k}.$$

That is

$$\left(\frac{\partial x}{\partial t}\right)\Bigg|_{\phi=\text{constant}} = \frac{\omega}{k}.$$

3 Phase and Group Velocities a Fourier Approach tied to a wave equation

I used one dimension for simplicity but this can be extended to several dimensions in the same manner. The previous section showed an example with two waves traveling in a one dimensional space, with constant amplitude. We talked about making the frequencies or wavenumbers getting as closed as possible to zero, taking limits and getting an idea about what group and phase velocities are. However in order to do this we need a package of waves along a continuum. The way to describe this is by using Fourier transforms. Let us assume a group of plane waves with real coefficients in the form

$$A(k, \omega)e^{j(kx-\omega t)},$$

with $j = \sqrt{-1}$. A superposition (on frequencies and wavenumbers) of these wavelets can be defined by the package

$$f(x, t) = \int dk\, d\omega A(k, \omega)e^{j(kx-\omega t)} \qquad (3\ .5)$$

So we went from a package of just two monochromatic wavelets to a continuum of them. The interesting thing as that if $x = x(t)$ then the variables k and ω are not independent. This is no clear yet, but a way to see why this happens is that we are interested in wave phenomena and whatever wave function $u(x,t)$, satisfies some differential equation (let us say the acoustic wave equation) and this equation is a differential equation. The differential equation can be converted from (x,t) to (k,w) by taking Fourier transforms. Than then we find what is known as the dispersion relation where $\omega = \omega(k)$. For example in the homogeneous acoustic wave equation (for one dimension) we see that

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2(x)} \frac{\partial^2 u(x,t)}{\partial t^2}$$

Taking the Fourier transform in x and t,

$$k^2 = \frac{\omega^2}{c^2},$$

where

$$\omega = \pm ck,$$

so ω is a linear function of k with the wavespeed c as the constant of proportionality. We deal with positive frequencies $\omega > 0$. In general ω does not need to be a linear function of k. When this happens there occurs the phenomenon of *dispersion.* So let us then assume in general that $\omega = \omega(k)$, and rewrite equation 3 .5 as

$$f(x,t) = \int dk A(k,\omega) e^{j(k\,x - \omega(k)t)}$$

where we eliminate the integration over ω since this is function of k. The concept of constant how different frequencies travel is valid here. For a fixed frequency, ω and constant phase, we have

$$\phi = kx - \omega t = \text{constant}$$

and taking derivative of x with respect to t implicitly we find

$$v_\phi = \frac{\partial x}{\partial t} = \frac{\omega}{k}.$$

However the whole package could be moving at a different velocity. Which velocity? We are interested in how the peak of the package moves. The peak of the package is obtained using the method of *stationary phase* . That occurs when the phase is stationary. In other words. We want to evaluate the integral for $f(x,t)$ using the method of stationary phase and we assert that this happens when

$$\frac{\partial \phi}{\partial k} = 0.$$

The phase $\phi(k)$ is given by the function

$$\phi(k) = kx - \omega(k)t.$$

The method of stationary phase then says that

$$x - \frac{\partial \omega}{\partial k} t = 0.$$

from which we find

$$\frac{\partial \omega}{\partial k} = \frac{x}{t} = v_g$$

the (group) velocity of the peak of the package.

4 The Physics of Light Approach

While the previous approach was based on pure mathematical formulations, the group velocity came from the physics of light dispersion. According to Peatros and Ware [1], the *group velocity* concept was introduced by Rayleigh (John Strutt). I present here the point of view of Peatros and Ware.

Think about a field F (electric, magnetic, or other in general) as a function of the three dimensional position vector $\boldsymbol{r}$ and frequency ω. Given this field at some postion $\boldsymbol{r}_0$, we want to understand the field at a close position $\boldsymbol{r}_0 + \Delta\boldsymbol{r}$. We assume a plane wave and from Fourier theorem

$$F(\boldsymbol{r}_0 + \Delta\boldsymbol{r}, \omega) = F(\boldsymbol{r}_0, \omega)\mathrm{e}^{j\boldsymbol{k}(j\omega)\cdot\boldsymbol{r}}. \tag{4 .6}$$

where $\boldsymbol{k}(\omega)$ is the wave number. The wave number in the propagation of light is defined as the refaction index normalized by the speed of light and scaled by the frequency.

[1] http://optics.byu.edu/BYUOpticsBook_2008.pdf

Think initially that $\boldsymbol{k}$ is the wave number which has the direction of the wave propagation and the magnitud of

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{cT} = \frac{\omega}{v}$$

where T is the wave period and v is the phase velocity of the propagating wave at the frequency ω. Now, the refraction index is $n = c/v$ where c is the speed of light in the vacuum. So $v = c/n$ and

$$k = \frac{n\omega}{c}.$$

Note that k could be a complex number. If this is the case, this will introduced an attenuating or amplifying factor.

The refraction index could depend on frequency (if it does not, then the medium is said to be non-dispersive). Think about light going through a prism. Different frequencies split into different' directions providing the colors of the rainbow. That is a manifestation of the dispersion phenomena. We then can rewrite, using the Fourier transform, equation 4 .6 as

$$\begin{aligned} F(\boldsymbol{r}_0 + \boldsymbol{r}, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\boldsymbol{r}_0 + \Delta\boldsymbol{r}, \omega)\, \mathrm{e}^{-\omega t} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\boldsymbol{r}_0, \omega)\, \mathrm{e}^{j(\boldsymbol{k}(\omega)\cdot\Delta\boldsymbol{r} - \omega t)} d\omega \end{aligned} \qquad (4\ .7)$$

The exponent is known as the *phase delay* for the pulse propagation. The wavenumber function $\boldsymbol{k}(\omega)$, known as the *dispersion relation* is usually expanded in a Taylor series around a "center" frequency called the *carrier* frequency $\bar{\omega}$ (as in the firsts section), as follows:

$$\boldsymbol{k} = \boldsymbol{k}(\bar{\omega}) + \frac{\partial \boldsymbol{k}}{\partial \omega}(\omega - \bar{\omega}) + \frac{1}{2}\frac{\partial^2 \boldsymbol{k}}{\partial \omega^2}(\omega - \bar{\omega})^2 + \cdots$$

We will consider this expansion up to the linear term in equation 4 .7. That is,

$$F(\boldsymbol{r}_0 + \boldsymbol{r}, t) = \mathrm{e}^{j\left[\boldsymbol{k}(\bar{\omega} - \bar{\omega}\frac{\partial \boldsymbol{k}}{\partial \omega}(\bar{\omega})\right]\cdot\Delta\boldsymbol{r}} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\boldsymbol{r}_0, \omega)\, \mathrm{e}^{j\left[\omega t - \frac{\partial \boldsymbol{k}}{\partial \omega}(\bar{\omega})\right]\cdot\Delta\boldsymbol{r}} d\omega.$$

We simplify the notation using

$$\Delta t = \frac{\partial \boldsymbol{k}}{\partial \omega}(\bar{\omega}) \cdot \Delta \boldsymbol{r}, \tag{4 .8}$$

so we write

$$F(\boldsymbol{r}_0 + \boldsymbol{r}, t) \quad = \quad \mathrm{e}^{j[\boldsymbol{k}(\bar{\omega})\cdot\Delta\boldsymbol{r} - \bar{\omega}\Delta t]} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\boldsymbol{r}_0, \omega)\ \mathrm{e}^{j\omega(t-\Delta t)} d\omega.$$

We recognize the Fourier transform at the end of this expression and write

$$F(\boldsymbol{r}_0, t) = \mathrm{e}^{j[\boldsymbol{k}(\bar{\omega})\cdot\Delta\boldsymbol{r}) - \bar{\omega}\Delta t]} F(\boldsymbol{r}_0, t - \Delta t) \tag{4 .9}$$

This accounts for a translation by Δt of the original signal and a modulation described by the complex coefficient. From equation 4 .8 and taking limits as $\Delta \boldsymbol{r}$ goes to 0, we see that

$$\frac{\partial t}{\partial x_i} = \frac{\partial k_i}{\partial \omega}(\bar{\omega}).$$

If k is the magnitude of $\boldsymbol{k}$ and we consider the wave moving along the direction of $\boldsymbol{k}$ then we can just write

$$\frac{\partial k}{\partial \omega}(\bar{\omega}) = v_g^{-1}$$

Where v_g is the group velocity of the package. The phase velocity is carried by keeping the phase on the first factor of 4 .9 constant. That is

$$\frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{\omega}{k}$$

along the direction of $\boldsymbol{k}$. Observe that there is no anisotropy and the dispersion is only carried due to the refraction coefficient $n(\omega)$ which is a scalar. So, group and phase velocity are in the same direction but have different magnitudes.