Identities Comming from Integrals

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This is an exercise on integrals and identities. The solutions where suggested by two members of the the MATHEMATICS StackExchange Forum [1] from a question that I posted.

We provide a function

$$f(\theta) = \frac{C d\theta}{\sin\theta\sqrt{\sin^2\theta - C^2}}. \tag{1}$$

We integrate this function in three different ways to obtain three different identities. We plot the identities and then finally show how they are linked to each other without using differentiation. The integral of funciton f in equation 1 is a geodesic on a sphere.

(i)

$$\begin{aligned}
\phi &= \int \frac{C d\theta}{\sin\theta\sqrt{\sin^2\theta - C^2}} \\
&= \int \frac{C \sin\theta d\theta}{\sin^2\theta\sqrt{\sin^2\theta - C^2}} \\
&= \int \frac{C \sin\theta d\theta}{(1-\cos^2\theta)\sqrt{1-\cos^2\theta - C^2}},
\end{aligned}$$

call $t = \cos\theta$, then $dt = -\sin\theta d\theta$, now if $k^2 = 1 - C^2$,

$$\phi = -\int \frac{C dt}{(1-t^2)\sqrt{k^2 - t^2}}$$

[1] http://math.stackexchange.com/questions/1512810/evaluating-trigonometric-integrals-of-the-form-int-fracc-d-theta-sin

Now call $t = 1/u$, $dt = -du/u^2$, so

$$\phi = C \int \frac{du/u^2}{(1 - 1/u^2)\sqrt{k^2 - 1/u^2}} = C \int \frac{udu}{(u^2 - 1)\sqrt{u^2k^2 - 1}}. \qquad (2)$$

Let us now make the substitution $v^2 = u^2k^2 - 1$, then $vdv = k^2udu$ and $u^2 - 1 = (v^2 + 1)/k^2 - 1 = (v^2 - k^2 + 1)/k^2$, so

$$\phi = C \int \frac{vdv/k^2}{[(v^2 - k^2 + 1)/k^2]v} = C \int \frac{dv}{[(v^2 + C^2]} = \int \frac{dv}{[(v/C)^2 + 1]}$$

Now if $z = v/C$, $dz = dv/C$, and then

$$\phi = C \int \frac{dz}{z^2 + 1} = C \arctan z + C_1.$$

We now do backward substitution:

$$\begin{aligned}
\phi &= C \tan^{-1} z + C_1 \\
&= C \tan^{-1}(v/C) + C' \\
&= C \tan^{-1}(\sqrt{(k^2u^2 - 1)}/C) + C_1 \\
&= C \tan^{-1}(\sqrt{k^2/t^2 - 1}/C) + C_1 \\
&= C \tan^{-1}(\sqrt{k^2 - t^2}/Ct) + C_1 \\
&= C \tan^{-1}(\sqrt{k^2 - \cos^2}/C \cos\theta) + C_1 \\
&= C \tan^{-1}(\sqrt{1 - C^2 - \cos^2\theta}/C \cos\theta) + C_1 \\
&= C \tan^{-1}(\sqrt{\sin^2\theta - C^2}/C \cos\theta) + C_1 \qquad (3)
\end{aligned}$$

(ii) On the other hand we could solve the integral following a different path:

$$\begin{aligned}
\phi &= \int \frac{Cd\theta}{\sin\theta\sqrt{\sin^2\theta - C^2}} \\
&= \int \frac{C\csc^2\theta d\theta}{\csc^2\theta \sin\theta\sqrt{\sin^2\theta - C^2}} \\
&= \int \frac{C\csc^2\theta d\theta}{\sqrt{1 - C^2\csc^2\theta}} \\
&= \int \frac{C\csc^2\theta d\theta}{\sqrt{1 - C^2(1 + \cot^2\theta)}}
\end{aligned}$$

Let us now use the following change of variables: $z = C\cot\theta$, so $dz = -C\csc^2\theta d\theta$, and

$$\begin{aligned}\phi &= \int \frac{-dz}{\sqrt{1-C^2-z^2}} \\ &= -\int \frac{dz}{\sqrt{a^2-z^2}} \quad \text{with} \quad a^2 = 1-C^2\end{aligned}$$

Now call $z = aw$, so $dz = adw$, $a > 0$ and

$$\begin{aligned}\phi &= -\int \frac{adw}{\sqrt{a^2-a^2w^2}} \\ &= -\int \frac{dw}{\sqrt{1-w^2}} \qquad (4)\end{aligned}$$

Call $w = \sin\alpha$, then $1-w^2 = \cos^2\alpha$, $dw = \cos\alpha d\alpha$ $-\pi/2 \leq \alpha \leq \pi/2$, and so, since in the domain of α, $\sec\alpha \geq 0$,

$$\phi = -\int d\alpha = \alpha + C_2. \qquad (5)$$

and

We now do backward substitution.

$$\begin{aligned}\phi &= -\alpha + C_2 \\ \phi &= -\sin^{-1} w + C_2 \\ \phi &= -\sin^{-1}(z/a) + C_2 \\ \phi &= -\sin^{-1}\left(\frac{C\cot\theta}{\sqrt{1-C^2}}\right) + C_2\end{aligned}$$

(iii) Use Wolfram Alpha to get other identities.

Challenge: Show that all those identities are inded identities.