Problems On Signal Analysis. From Oppenheim and Schafer's book

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Preliminary Discussion

What does it mean

$$y[n] = x[-n+2]?$$

In any interpretation a numerical evaluation validates the answer. For example, if $n = 20$, then $-n + 2 = -18$, so the mapping for y is such that the indices map as

$$20 \mapsto -18.$$

The confusion rises because the order of operations. Think as programming in a computer, first change the sign of n then add 2. This means, flip the x axis and add two. Of course you can factor $-n + 2 = -(n - 2)$, and then you can subtract two and flip the axis. That would be $20 - 2 = 18$, and flip it to -18. So either

(a) flip the axis, add two

(b) subtract two and filp the axis.

Both produce the same result.

In all cases that come to my mind, y can be seen as a composition of two functions. For example

$$\begin{aligned} g \quad &: \quad \mathbb{Z} \to \mathbb{R} \text{ or } \mathbb{C} \\ & \quad\;\; n \mapsto x[n] \end{aligned}$$

where $\mathbb{R}$ is the field of real numbers and $\mathbb{C}$ is the field of complex numbers, and

$$\begin{aligned} f \quad &: \quad \mathbb{Z} \to \mathbb{R} \text{ or } \mathbb{C} \\ & \quad\;\; n \mapsto -n+2 \end{aligned}$$

So

$$y[n] = f \circ g\ (n) = f[g[n]]$$

Then all we know about function composition on the continuum can be used in the discrete. For example, if we take a derivative, the chain rule can be applied and so. Usually f is used as an operator $f = T$ (T stands for transformation) defined as

$$\begin{aligned} T : X \subset \ell^p &\to Y \subset \ell^q \\ (x_n)_{n\in\mathbb{Z}} &\mapsto T[x[.]] = (y_n)_{n\in\mathbb{Z}} \end{aligned}$$

where the space ℓ^p is the space of infinite sequence of reals ($\mathbb{R}$) or complex ($\mathbb{C}$) numbers, convergent under the p–norm. That is, such that if $w = (w_n)_{n\in\mathbb{Z}}$

$$\|w\|_p = \left(\sum_{n\in\mathbb{Z}} |w_n|^p \right)^{1/p} < \infty$$

(here q could be equal or not to p) and both p and q should be real positive numbers.

1.1 A few words about sums

Wikipedia [1] has an extense discussion of the meaning of the symbol $\sum$ to represent a sum.

I want to emphasize that from all those symbols, a unified way to bound them together is with the symbol

$$\sum_{x\in S} A_x$$

for some set S.

For example

$$\sum_{i=1}^{6} A_i = \sum_{i=6}^{1} A_i = \sum_{i\in S} A_i \quad , \quad S = \{1, 2, 3, 4, 5, 6\}.$$

Some textbooks say that

$$\sum_{n}^{m} a_k = 0 \quad \text{if} \quad n > m.$$

I will ignore this definition. It is missleading. Making a sum from top to bottom or from bottom to top, should not change the result.

If the set is finite, it is a true set in the sense that the order of listing does not change the set. If the set S is infinite then we can not commute the set. If S is infinte countable (with index over all natural or integer numbers, for example) the sum is not necessary commutative. In this case the symbol

$$\sum_{i\in\mathbb{Z}} a_i$$

is not well defined, unless an order is stablished. Here the notation

$$\sum_{i=-\infty}^{\infty} a_i$$

[1] http://en.wikipedia.org/wiki/Summation

is better suited. Andreas ask: Is,

$$\sum_{i=-\infty}^{\infty} a_i = \sum_{i=\infty}^{-\infty} a_i$$

We are changing the order, right? and we say that infinite series do not necessary commuted. Why this should be true? I am not sure. First, to do anything they should converge. Here is something that Euler found very long time ago.

$$\sum_{i=0}^{\infty}(-1)^i = 1 - 1 + 1 - 1 + 1 \cdots .$$

This is not even asociative. If you let asociation it converges to two different numbers.

$$(1-1) + (1-1) + (1-1) \cdots = 0 + 0 + 0 \cdots = 0.$$

but

$$1 + (-1+1) + (-1+1) + (-1+1) = 1 + 0 + 0 + 0 \cdots = 1.$$

If further you let them commute, you can built sequences that converge to anything you want.

If S is a continuum, the sum is an integral. That is

$$\int_S f(x)dx = \sum_{x \in S} f d\mu$$

here μ is a measure. The measure depends on the type of integration and its study is beyond the scope of these notes. However if you insist, the measure $d\mu$ is like a weight. So this is a weighted sum with the weight determined by $d\mu$. If you find time google for "measure theory". The simplest measure on the continuum is that of the Riemman measure and it is all you learn in Calculus and what is needed and used in more than 99 percent of the cases. Another way to understand this theory is by looking at "probability theory". The measure is like a density distribution of the space of models.

If you are in the continuum, you know that

$$\int_a^b f(x)dx = -\int_b^a f(x)dx,$$

so your question is well founded. Down in these notes there is the problem about commutativity of convolution. I address there both the continuum and the discrete spaces.

Problem 2.22

In this chapter we introduced a number of general properties of systems. In particular, a system may or may not be

(1) Memoryless

(2) Time-invariant

(3) Linear

(4) Causal

(5) Stable

Determine which of these properties hold and which do not hold for each of the following systems. Justify your answers. In each example $y(t)$ or $y[n]$ denotes the system output, and $x(t)$ or $x[n]$ is the system input.

(a) $y(t) = e^{x(t)}$

(b) $y[n] = x[n]x[n-1]$

(c) $y(t) = \frac{dx(t)}{dt}$

(d) $y[n] = x[-n]$

(e) $y[n] = x[n-2] - 2\,x[n-17]$

(f) $y(t) = x(t-1) - x(1-t)$

(g) $y(t) = \sin(6t)\,x(t)$

(h) $y[n] = \sum_{k=n-2}^{n+4} x[k]$

(i) $y[n] = nx[n]$

(j) $y(t) = \int_{-\infty}^{3t} x(\tau)d\tau$

(k) $y[n] = \varepsilon\nu\{x[n]\}$

(l) $y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-100), & t \geq 0 \end{cases}$

(m) $y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-100), & x(t) \geq 0 \end{cases}$

(n) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n+1], & n \leq -1 \end{cases}$

(o) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$

(p) $y(t) = x(t/2)$

(q) $y[n] = x[2n]$

Solution

1 **Memoryless:** Output $y[n]$ (or $y(t)$) depends ONLY on input $x[n]$ (on $x(t)$).

(a) $y(t) = \mathrm{e}^{x(t)}$, *ans=yes*

(b) $y[n] = x[n]x[n-1]$, *ans=no*

(c) $y(t) = \frac{dx(t)}{dt}$, *ans=yes*

(d) $y[n] = x[-n]$, *ans=no*

(e) $y[n] = x[n-2] - 2\,x[n-17]$, *ans=no*

(f) $y(t) = x(t-1) - x(1-t)$, *ans=no*

(g) $y(t) = \sin(6t)\,x(t)$, *ans=yes*

(h) $y[n] = \sum_{k=n-2}^{n+4} x[k]$, *ans=no*

(i) $y[n] = nx[n]$, *ans=yes*

(j) $y(t) = \int_{-\infty}^{3t} x(\tau)d\tau$, *ans=n*

(k) $y[n] = \varepsilon\nu\{x[n]\} = \frac{x[n]+x[-n]}{2}$, *ans=no*

(l) $y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-100), & t \geq 0 \end{cases}$, *ans=no*

(m) $y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-100), & x(t) \geq 0 \end{cases}$, *ans=no*

(n) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n+1], & n \leq -1 \end{cases}$, *ans=no*

(o) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$, *ans=yes*

(p) $y(t) = x(t/2)$, *ans=no*

(q) $y[n] = x[2n]$, *ans=no*

2 **Time Invariant:** (shift invariant)

(a) $y(t) = \mathrm{e}^{x(t)}$, $ans = yes$, $y(t-t_0) = \mathrm{e}^{x(t-t_0)}$

(b) $y[n] = x[n]x[n-1]$, *ans=yes*

(c) $y(t) = \frac{dx(t)}{dt}$, *ans=yes*

(d) $y[n] = x[-n]$, *ans=no*

Apply a shift to the input $x[-n-k]$, the output of this is $x[-(n+k)] = y[n+k]$. That is a right shift to the input produces the same shift, but in the opposite direction to the output. So it is not time invariant.

In other words $T[x[n-k]] = x[-(n-k)] = x[-n+k]$. When the input shifts a system k units to the right, the operator flips the system and shift it k units to the left.

Yet another way to understand this problem is as follows. There are two operators F which flips a sequence $F[x[n]] = x[-n]$, and S_k which shift a sequence. That is $S_k[x[n]] = x[n-k]$. Time invariance means that the flip operator F commutes with the shift operator S. That is:

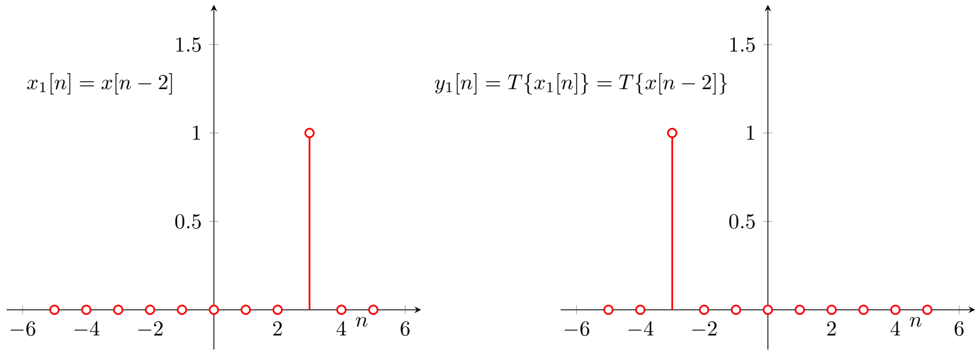

Figure 2: Testing the output for shifted input. (Left) Input $x_1[n] = x[n-2]$, (Right) output $y_1[n] = T\{x_1[n]\} = T\{x[n-2]\}$

$(S_k \circ F)([x[n]) = (F \circ S_k)(x[n])$ or, what is the same as $(S_k(F[x[n])) = F(S_k[x[n]])$ Let us check

$$S_k(F[x[n]] = S_k(x[-n]) = x[-n-k],$$

on the other hand

$$F(S_k[x[n]) = F(x[n-k]) = x[-n+k],$$

So $S \circ F \neq F \circ S$ and the system is not shift invariant.
Figure 1 shows the reflection operator $T[x[n]] = x[-n]$, for an impulse $x[n] = \delta[x-1]$.

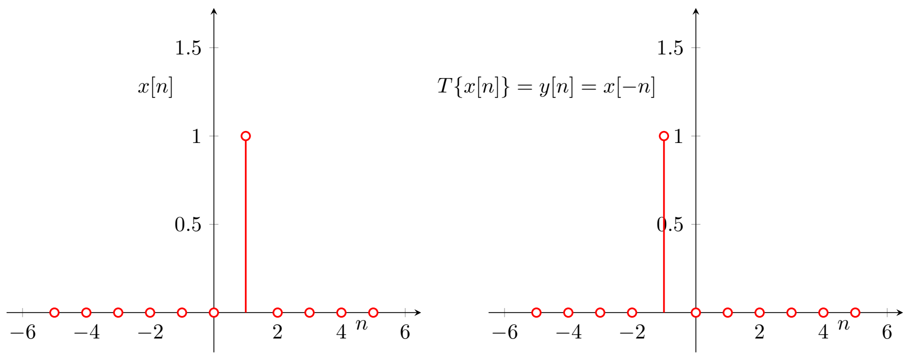

Figure 1: Relfection system $y[n] = T\{x[n]\} = x[-n]$. (Right) Sample input $x[n] = \delta[n-1]$, (left) output $y[n]$

Figure 2 shows the effect of shifting and then reflecting the impulse by two units; that is, the input is $\delta[n-1]$, the shift is $\delta[n-3]$, and the reflection is $\delta[-n+3]$. Finally Figure 3, shows a different situation. First we reflect from $\delta[n-1]$ to $\delta[1-n]$ then we shift two units to the right. This is $\delta[-n-1]$.

(e) $y[n] = x[n-2] - 2\,x[n-17]$ *ans=yes*

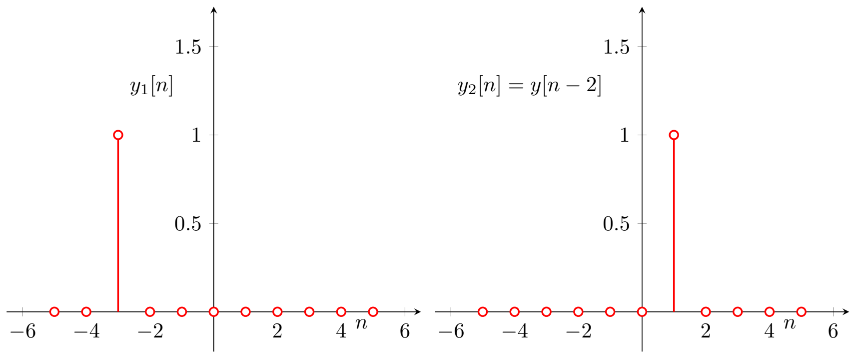

Figure 3: Compare output $y_1[n]$ from shifted input $x_1[n]$ (left) with shifted output $y_2[n] = y[n-2]$ (right) shows that the system is not time invariant.

(f) $y(t) = x(t-1) - x(1-t)$ *ans=no*
Apply a shift of t_0 to the input $x(t-1-t_0) - x(1-t-t_0) \neq y(t-t_0) = x(t-t_0-1) - x[1-(t-t_0)]$

(g) $y(t) = \sin(6t)\, x(t)$, *ans=no*
$\sin(6t)\, x(t-t_0) \neq y(t-t_0)$.

(h) $y[n] = \sum_{k=n-2}^{n+4} x[k]$ *ans=yes*
$\sum_{k=n-2}^{n+4} x[k-l] = \sum_{k=n-2-l}^{n+4-l} x[k] = y[n-l]$.

(i) $y[n] = nx[n]$, *ans=no*
$nx[n-k] \neq y[n-k]$.

(j) $y(t) = \int_{-\infty}^{3t} x(\tau) d\tau$, *ans=no*
Let us evaluate $\int_{-\infty}^{3t} x(\tau - \tau_0) d\tau$. We perform a change of variable $u = \tau - \tau_0$, At $\tau = -\infty$, $u = -\infty$, but at $\tau = 3t$, $u = 3t - \tau_0$, so $\int_{-\infty}^{3t} x(\tau - \tau_0) d\tau = \int_{-\infty}^{3t-\tau_0} x(u) du \neq y(t - \tau_0) = \int_{-\infty}^{3(t-\tau_0)} x(\tau) d\tau$

(k) $y[n] = \varepsilon\nu\{x[n]\} = \frac{x[n]+x[-n]}{2}$ *ans=no*
Since by shifting x by k units we have $\frac{x[n-k]+x[-n-k]}{2} \neq \frac{x[n-k]+x[-n+k]}{2} = y[n-k]$.

(l) $y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-100), & t \geq 0 \end{cases}$
Let us shift the input by t_0. Then evaluate the expression on the right. That is

$$\begin{cases} 0, & t < 0 \\ x(t-t_0) + x(t-100-t_0), & t \geq 0 \end{cases}$$

However $y(t-t_0)$ is $\begin{cases} 0, & t < t_0 \\ x(t-t_0) + x(t-100-t_0), & t \geq t_0 \end{cases}$

So the answer is *no* .

(m) $y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-100), & x(t) \geq 0 \end{cases}$
The answer is *no* .
Changing t by $t - t_0$ in the input, will produce

$\begin{cases} 0, & x(t)<0 \\ x(t-t_0)+x(t-100-t_0), & x(t)\geq 0 \end{cases}$ which is different from
$y(t-t_0)=\begin{cases} 0, & x(t-t_0)<0 \\ x(t-t_0)+x(t-t_0-100), & x(t-t_0)\geq 0 \end{cases}$.

(n) $y[n]=\begin{cases} x[n], & n\geq 1 \\ 0, & n=0 \\ x[n+1], & n\leq -1 \end{cases}$

Change $x[n]$ by $x[n-k]$, then we get $\begin{cases} x[n-k], & n\geq 1 \\ 0, & n=0 \\ x[n-k+1], & n\leq -1 \end{cases}$

This is not $y[n-k]$ for the same reasons of problem (l). So the answer is *no.*

(o) $y[n]=\begin{cases} x[n], & n\geq 1 \\ 0, & n=0 \\ x[n], & n\leq -1 \end{cases}$ The answer is *no* . See previous problem.

(p) $y(t)=x(t/2)$ Shift the input x by t_0, that is $x(t/2-t_0)$, however $y(t-t_0)=x(t/2-t_0/2)\neq x(t/2-t_0)$. So the answer is *no* .

(q) $y[n]=x[2n]$
Similarly to the previous exercise, shifting in the intput by t_0 we have $x[2n-t_0]\neq y[n-t_0]=x[2(t-t_0)]$. So the answer is *no*

3 Linear

(a) $y(t)=e^{x(t)}$
$e^{x_1(t)+x_2(t)}=e^{x_1(t)}e^{x_2(t)}\neq e^{x_1(t)}+e^{x_2(t)}$. So the answer is *no* .

(b) $y[n]=x[n]x[n-1]$. The answer is *no* .

(c) $y(t)=\frac{dx(t)}{dt}$
since $\frac{d[ax_1(t)+bx_2(t)]}{dt}=a\frac{dx_1(t)}{dt}+b\frac{dx_2(t)}{dt}$, the system is linear. The answer is *yes*

(d) $y[n]=x[-n]$, *ans=yes*

(e) $y[n]=x[n-2]-2\,x[n-17]$ *ans=yes*

(f) $y(t)=x(t-1)-x(1-t)$ *ans=yes*

(g) $y(t)=\sin(6t)\,x(t)$ *ans=yes*

(h) $y[n]=\sum_{k=n-2}^{n+4}x[k]$ *ans=yes*

(i) $y[n]=nx[n]$ *ans=yes*

(j) $y(t)=\int_{-\infty}^{3t}x(\tau)d\tau$ *ans=yes*

(k) $y[n]=\varepsilon\nu\{x[n]\}$ *ans=yes*

(l) $y(t)=\begin{cases} 0, & t<0 \\ x(t)+x(t-100), & t\geq 0 \end{cases}$

ans=yes

(m) $y(t)=\begin{cases} 0, & x(t)<0 \\ x(t)+x(t-100), & x(t)\geq 0 \end{cases}$
If changing x by x_1+x_2 we find

$\begin{cases} 0, & x_1(t)+x_2(t)<0 \\ x_1(t)+x_2(t)+x_1(t-100)+x_2(t-100), & x_1(t)+x_2(t)\ge 0 \end{cases}$ yet

$y_1(t)+y_2(t)=\begin{cases} 0, & x_1(t)<0 \\ x_1(t)+x_1(t-100), & x_1(t)\ge 0 \end{cases}$ +

$\begin{cases} 0, & x_2(t)<0 \\ x_2(t)+x_2(t-100), & x_2(t)\ge 0 \end{cases}$

$$= \begin{cases} 0, & x_1(t)<0 \text{ and } x_2(t)<0 \\ x_1(t)+x_1(t-100), & x_1(t)\ge 0 \text{ and } x_2(t)<0 \\ x_2(t)+x_2(t-100), & x_2(t)\ge 0 \text{ and } x_1(t)<0 \\ x_1(t)+x_2(t)+x_1(t-100)+x_2(t-100), & x_2(t)\ge 0 \text{ and } x_1(t)\ge 0 \end{cases}$$

To show that the two expressions above are different, we pick $x_1(t)=t$ and $x_2(t)=-t$. Since $x_1(t)+x_2(t)=0$, the first expression is $y(t)=t-100-t+100=0$. The second expression is $y_1(t)+y_2(t)=\begin{cases} 2t-100 & t>0 \\ -2t+100 & t<0 \\ 0 & t=0 \end{cases}$.

So the answer is *ans=no*

(n) $y[n]=\begin{cases} x[n], & n\ge 1 \\ 0, & n=0 \\ x[n+1], & n\le -1 \end{cases}$
ans=yes

(o) $y[n]=\begin{cases} x[n], & n\ge 1 \\ 0, & n=0 \\ x[n], & n\le -1 \end{cases}$
ans=yes

(p) $y(t)=x(t/2)$
ans=yes

(q) $y[n]=x[2n]$
ans=yes
Here is a proof as presented by mathematicians.
Let T the operator

$$T(x[n])=x[2n]$$

Let $x[n]=a\,x_1[n]+b\,x_2[n]$ with a and b are scalars (real or complex). Then

$$T(ax_1[n]+bx_2[n])=(ax_1+bx_2)[2n]=ax_1[2n]+bx_2[2n]=aT(x_1[n])+bT(x_2[n]).$$

That is, the operator T is linear. All proofs could be presented in this way, but they become lengthy.

4 **Causal** The output depends on values of the input and present and past times. All memoryless systems are causal, as a consequence.

(a) $y(t)=\mathrm{e}^{x(t)}$ *ans=yes*

(b) $y[n]=x[n]x[n-1]$ *ans=yes*

(c) $y(t) = \frac{dx(t)}{dt}$ *ans=yes*

(d) $y[n] = x[-n]$ *ans=no*

(e) $y[n] = x[n-2] - 2\,x[n-17]$ *ans=yes*

(f) $y(t) = x(t-1) - x(1-t)$ *ans=no*

(g) $y(t) = \sin(6t)\,x(t)$ *ans=yes*

(h) $y[n] = \sum_{k=n-2}^{n+4} x[k]$ *ans=no*

(i) $y[n] = n\,x[n]$ *ans=yes*

(j) $y(t) = \int_{-\infty}^{3t} x(\tau)dt$ *ans=no*

(k) $y[n] = \varepsilon\nu\{x[n]\}$ *ans=no*

(l) $y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-100), & t \geq 0 \end{cases}$ *ans=yes*

(m) $y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-100), & x(t) \geq 0 \end{cases}$ *ans=yes*

(n) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n+1], & n \leq -1 \end{cases}$ *ans=no*

(o) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$ *ans=yes*

(p) $y(t) = x(t/2)$ *ans=no*
Note that if $t < 0$, $t/2 > t$.

(q) $y[n] = x[2n]$ *ans=no*

5 **Stable:** I believe the text does a poor job of explaining this. Other books by the same authors or Wikipedia do a better job. Here a definition fom "Discrete-Time Signal Processing" by Oppenheim, Schaffer and Buck. Discrete-Time Signal Processing [2] " A system is stable in the bounded-input, bounded-output (BIBO) sense if and only if every bounded input sequence produces a bounded output sequence. The input $x[n]$ is bounded if there exists a fixed positive finite value B_x, such that

$$|x[n]| \leq B_x < \infty, \quad \forall n$$

" Stability requires, that for every boundedc input, there exists a fixed positive finite value B_y such that

$$|y[n]| \leq B_y < \infty, \quad \forall n\text{"}$$

[2] http://www.ingelec.uns.edu.ar/pds2803/materiales/librospdf/oppenheim/toc.htm

In mathematical language a system is BIBO if for each input with $\|x\|_\infty < \infty$ the output has a finite $\ell - \infty$ norm. $\|y\|_\infty < \infty$.

(a) $y(t) = \mathrm{e}^{x(t)}$.
If $\|x(t)\|_\infty < B_x$, then $\|\mathrm{e}^{x(t)}\|_\infty < e^{B_x} = B_y$. So, *ans=yes*

(b) $y[n] = x[n]x[n-1]$, *ans=yes* $B_y = B_x^2$.

(c) $y(t) = \frac{dx(t)}{dt}$. *ans=no*
Think about a semicircle $x(t) = \sqrt{1-t^2}$, what happens , as $t \to 1$ the slope is vertical. While the semicircle is all bounded, its derivative is not. In simbols, if $x(t)$ are points in the upper part of the semi–circles, the points $y(t) = -t/\sqrt{1-t^2}$, do not have abound as t is closed to 1.

(d) $y[n] = x[-n]$ *ans=yes* $B_y = B_x$.

(e) $y[n] = x[n-2] - 2\,x[n-17]$ *ans=yes* $B_y = 3B_x$.

(f) $y(t) = x(t-1) - x(1-t)$ *ans=yes* $B_y = 2B_x$.

(g) $y(t) = \sin(6t)\,x(t)$ *ans=yes* $B_y = B_x$.

(h) $y[n] = \sum_{k=n-2}^{n+4} x[k]$ *ans=yes* $B_y = 6B_x$.

(i) $y[n] = nx[n]$ *ans=no.*
Pick

$$x[n] = \frac{1}{\sqrt{n}}, \tag{1.1}$$

Obviously

$$\left|\frac{1}{\sqrt{n}}\right| < 1 = B_x. \tag{1.2}$$

However

$$y[n] = \sqrt{n} \tag{1.3}$$

which grows without bounds.

(j) $y(t) = \int_{-\infty}^{3t} x(\tau)d\tau$ *ans=no*
Pick $x(t) = 1$, for $t > 0$ or 0 if $t < 0$. So $y(t) = 3t$, which grows without bounds as $t \to \infty$. Here $\|x\|_\infty < 1$, so $B_x = 1$ works.

(k) $y[n] = \varepsilon\nu\{x[n]\}$ *ans=yes*
The even or odd parts of a dataset are a subset of it, so if the set is bound, so it is its subset. That is, pick $B_y = B_x$.

(l) $y(t) = \begin{cases} 0, & t < 0 \\ x(t) + x(t-100), & t \geq 0 \end{cases}$ *ans=yes* $B_y = 2B_x$.

(m) $y(t) = \begin{cases} 0, & x(t) < 0 \\ x(t) + x(t-100), & x(t) \geq 0 \end{cases}$ *ans=yes* $B_y = 2B_x$.

(n) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n+1], & n \leq -1 \end{cases}$ *ans=yes* $B_y = B_x$.

(o) $y[n] = \begin{cases} x[n], & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$ *ans=yes* $B_y = B_x$.

(p) $y(t) = x(t/2)$ *ans=yes* $B_y = B_x$.

(q) $y[n] = x[2n]$ *ans=yes* $B_y = B_x$.

Problem 2.26

Determine if each of the following systems is invertible. If it is, construct the inverse system. If it is not, find two input signals to the system that have the same output.

(a) $y(t) = x(t-d)$.

solution: The answer is *yes* . The inverse is $z(y(t)) = y(t+d)$, since $z(y(t)) = y(t+d) = x(t-d+d) = x(t)$.

(b) $y(t) = \cos[x(t)]$

solution: *answer=no*
pick $x_2(t) = x(t) + 2\pi$

(c) $y[n] = n\,x[n]$.

solution: *ans=no*
pick $x_2[0] = x[0] + 2$, $x_2[n] = x[n]$ for $n \neq 0$. If $n = 0$ is not in the domain of x, then the answer is *yes* and the inverse is $z[y[n]] = (1/n)y[n]$, since $z[y[n]] = (1/n)y[n] = (1/n)nx[n] = x[n]$.

(d) $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$.

solution: *answer=yes*
The inverse is $z(y(t)) = \frac{dy(t)}{dt}$, since $z(y(t)) = \frac{dy(t)}{dt} = x(t)$.

(e) $y[n] = \begin{cases} x[n-1] & n \geq 1 \\ 0, & n = 0 \\ x[n], & n \leq -1 \end{cases}$

solution: *answer=no*
Pick $x_1[0] = x[0]+2$, then $y[x_1[n]] = y[x[n]]$, since the point $n = 0$, is insensitive to y through x. However, if 0 is not in the domain the answer would be *yes* . In this case the inverse would be $z[y[n]] = \begin{cases} y[n+1] & n \geq 1 \\ y[n], & n \leq -1 \end{cases}$
Then
$z[y[n]] = \begin{cases} y[n+1] & n \geq 1 \\ y[n], & n \leq -1 \end{cases} = \begin{cases} x[n+1-1] & n \geq 1 \\ x[n], & n \leq -1 \end{cases} = x[n]$, and $n = 0$ is not in the domain of x.

(f) $y[n] = x[n]\,x[n-1]$

solution: *answer=no* . Pick $x_1[n] = -x[n]$, and use $(-1)(-1) = 1$.

(g) $y[n] = x[1-n]$

solution: *answer=yes*
The inverse is $z[y[n]] = y[1-n]$. Let us verify:
$z[y[n]] = y[1-n] = x[1-(1-n)] = x[n]$.

(h) $y(t) = \int_{-\infty}^{t} \mathrm{e}^{-(t-\tau)}\, x(\tau) d\tau$.

solution: *answer=yes*
The inverse is $z[y(t)] = \frac{dy(t)}{dt}$. Let us check
$z[y(t)] = \frac{dy(t)}{dt} = x(t)$.

(i) $y[n] = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x[k]$

solution: Let us note that $y[n] - y[n-1] = x[n]$, so the answer is *yes* with $z[y[n]] = y[n] - y[n-1]$.

(j) $y(t) = \frac{dx(t)}{dt}$.

solution: *answer=no.* Pick $x_1(t) = x(t) + c$, $c \neq 0$.

(k) $y[n] = \begin{cases} x[n+1], & n \geq 0 \\ x[n], & n \leq -1 \end{cases}$

solution: *answer=no.*
Since $x[0]$ is not involved in $y[n]$, pick $x_1[0] = x[0] + 2$, and then $y[n]$ would be the same for $x_1[n]$, which as a sequence is different from $x[n]$.

(l) $y(t) = x(2\,t)$

solution: *answer=yes*
The inverse is $z(y(t)) = y(t/2)$, so $z(y(t)) = y(t/2) = x(2\,t/2) = x(t)$.

(m) $y[n] = x[2\,n]$

solution: *answer=no*
Note that no odd sample $x[n]$ is involved in $y[n]$. So, any $x_1[n]$, which agrees with $x[n]$ in its even samples but disagrees in the odd samples would yield the same $y[n]$.

(n) $y[n] = \begin{cases} x[n/2], & n \quad \text{even} \\ 0, & n \quad \text{odd} \end{cases}$

solution: *answer=yes*
The inverse is $z[y[n]] = y[2n]$, since $2n$ is even, $z[y[n]] = y[2n] = x[2n/2] = x[n]$.

Problem: Time Invariant

Is

$$y[n] = \sum_{k=0}^{n-1} x[k] \tag{1.4}$$

Time invariant?

Solution

Andreas solution:

(a) Output for delayed input $x_d[n-n_0]$

$$y_d = y[x_d] = \sum_{k=0}^{n-1} x[k-n_0] = \sum_{k=-n_0}^{n-1-n_0} x[k] \tag{1.5}$$

, after shifting the summation index

(b) This is not the same as ..

Herman solution:

(a)

$$y[n-n_0] = \sum_{k=0}^{n-1-n_0} x[k] = \sum_{k=n_0}^{n-1-n_0} x[k-n_0]$$

(b) This is not the same as ..

Problem: On periodic signals

Specify two different non–negative frequencies f for which the sequence

$$x[n] = \cos(2\pi f n)$$

is periodic with period of $N = 8$ samples.

Solution:

We need to solve the equation

$$x[n] = x[n+N],$$

that is

$$\cos(2\pi f n) = \cos(2\pi f(n+N)),$$

This will require

$$2\pi f n = 2\pi f(n+N) \pm 2k\pi,$$

There is an infinite number of solutions for $k \in \mathbb{N}$. Since we want no negative frequencies we can pick two cases:

(i) k = -1. Here we have

$$2\pi f n = 2\pi f n + 2\pi f N - 2\pi,$$

That is

$$2\pi f N = 2\pi \Rightarrow f = \frac{1}{N} = \frac{1}{8}.$$

(ii) k = -2. Here we find

$$f = \frac{1}{4}.$$

The first frequency $f = 1/8$ is the fundamental, any multiple of it produces also periodic signals. Those are called harmonics.

Problem: Why is a median filter non-linear

First, let us prove this fact and then we discuss it.

Initially we ask the question, what is a median filter? A median filter of length k returns the sample that has the median value for each rolling window of k samples. That is, if the number of k samples is odd, the value that it returns is such that half of the samples are values smaller or equal than it, and the other half of the samples have values larger or equal than it. If the number of k samples is even, usually the two median (samples in the middle of the range) are averaged.

To prove non-linearity the best way is by a counter-example. That is, find an example that violates linearity.

We note that all median filters satisfy the Homogeneity condition of degree 1. That is, if the median of a sequence $x[i]$ is m, the median of the sequence $ax[i]$ is am. Since we are just scaling the whole sequence by a. So, the counter-example has to be on the additivity $T[x[i] + y[i]] \neq T[x[i]] + Y[y[i]]$.

For our example we should not locate the median on the same sample index because of course would be additive. The simplest example should have at least 3 samples and a filter length of 3 samples. All data with one sample fails to be a good counter–example because the median is the only sample and it is linear in this case.

Guess and example such as

$$x = (0, 1, 0) \quad y = (1, 0, 3) \quad x + y = (1, 1, 3)$$

Note that the median in the sequence $x[i]$ is in the second index and the median in the sequence $y[i]$ is in the first index (as recommended above, they should be in different indices for the counter–example to rule out additivity) The median of $x[i]$ is 1, the median of y is 1, but the median of $x + y$ is 1, which is no $1 + 1 = 2$. Maybe we could think about more dramatic examples but this is enough.

Why are median filters important for despiking? Think about some data with a sequence $x[i] = (a_1, a_2, \cdots a_k, \cdots, S, \cdots a_n)$, where all values a_i are limit to a small range in the real interval $[A, B]$, but S is a huge number far from that range, that is either $S \gg B$ or $S \ll A$. S would never be the median of anything because it is an outlier, so it will never be passed by the median filter. The other guys (for a short filter length, say 3 for example) will stay where they are, or if the length of the filter is longer the filter might smooth a bit the data.

How can we, without an example, tell that a median filter is non–linear? Assume data such as the data above.

$$x[i] = (a_1, a_2, \cdots a_k, \cdots, S, \cdots a_n),$$

with a big spike S. Linearity goes hand by hand with continuity. Linear means that small changes in the input, imply small changes in the output. Assume, another data set

$$y[i] = (a_1, a_2, \cdots a_k, \cdots, S^S, \cdots a_n),$$

(assume $S \gg 1$). Then the second data set suffered a big changed, since $S^S \gg S$, that is the energy on $y[i]$ blows up. Still both data sets will have the same output on a median filter. A linear filter with such big change in the input, has to reflect that change (continuity) in the output. Here there is no sensitivity. Tha is, a big change in the input and no change in the output. That cannot be linear.

Things to look for: Why are median filters attached to the L_1 norm and linear filters to the L_2 norm? What are alpha trim mean filters and their relation with median filters. Go to Tarantola's book on line: [3] and page 171. Tarantola's book illustrates with excellent exercises, ways to understand facts between moments (moments are used to find centers of gravity, or inertia and other centers) mean, median, midrange and other measures.

Is Convolution Commutative?

Let us start with the continnum: Define

$$f * g = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau.$$

Now let us to a change of variable $u = t - \tau$, then $du = -d\tau$, and when τ goes between $-\infty$ and ∞, u goes from $-\infty$ to ∞. That is we find

$$f * g = \int_{\infty}^{-\infty} f(t-u)g(u)(-du)$$

Recall, from calculus that reversing the order of integration changes the sign, we have two sign changes, one because $d\tau = -du$ and the other because we want to reverse the order of integration from $(\infty, -\infty)$ to $(-\infty, \infty)$. That is, we can say that

$$f * g = \int_{-\infty}^{\infty} f(t-u)g(u)(du)$$

But if you look at the first definition, this is just $f * g$. So convolution in the continuum is commutative. How about the discrete?

[3]http://www.ipgp.jussieu.fr/ tarantola/Files/Professional/Books

It is not that obvious that we can copy the steps from integrals to sums. Remember the discussion on sums where we say

$$a_1 + a_2 + a_3 = a_3 + a_2 + a_1.$$

That is, if the sum is finite then we can reverse the order and get the same value. Note that there are not sign reversal here as in the continuum.

Let us get to it.

$$f * g = \sum_{m=-\infty}^{\infty} f(m)g(n-m)$$

We can change $n-m$ to a new index u, then $n-m=u$, then $m=n-u$, and if m goes between $-\infty$ to ∞, u has to go between ∞ and $-\infty$. So we have with the new substitution

$$f * g = \sum_{m=-\infty}^{\infty} f(n-u)g(u)$$

but $m = n-u$, that is

$$f * g = \sum_{n-u=-\infty}^{\infty} f(n-u)g(u)$$

n is a finite number so saying that $n-u$ is between $(-\infty, \infty)$ is the same as saying that $-u$ is between $(-\infty, \infty)$. So if $-u$ is between $(-\infty, \infty$, then u is in the range $\infty, -\infty)$. We come to

$$f * g = \sum_{u=\infty}^{-\infty} f(n-u)g(u) = \sum_{u=\infty}^{-\infty} g(u)f(n-u)$$

Here is a key step. Can we reverse the order of the sum, wihtout reversing the sign as we do in integrals? If the sum is finite, the answer is "yes". Remember that $a_1 + a_2 + a_3 = a_3 + a_2 + a_1$ and this can be generalized to all finite sums. If the sum is not finite, then we need absolute convergence, before we can commute terms. Check the theorem here. [4].

Now, since we need to be able to find Fourier transforms, we need at least ℓ_1 sequences. Those are sequences with absolute convergence. So we can reverse the order and so the convolution (for all our practical purposes) is commutative.

[4] http://www.webskate101.com/webnotes/class48.htmld/prp731.html

Homogeneity, additivity and linearity

We showed that in general a function could be homogeneous but not necessarily linear. In one dimensional functions, that is functions

$$f : \mathbb{R} \to \mathbb{R}$$

however homogeneity implies linearity. Here is the simple proof. Pick any x and y in $\mathbb{R}$ (both non-zero, if one is zero there is no much to do). Then find c such that $y = cx$ ($c = y/x$). So

$$\begin{aligned} f(x+y) = f(x+cx) = f[(1+c)x] = (1+c)f(x) &= f(x) + c(f(x) \\ = f(x) + f(cx) &= f(x) + f(y). \end{aligned}$$

We want to know if the opposite is true. That is, if we have additivity, for $f : \mathbb{R} \to \mathbb{R}$, do we have homogeneity? and so linearity? can we find a function in $f : \mathbb{R} \to \mathbb{R}$ which has additivity but no homogeneity?

If such a function f is additive, is it homogeneous? We start with the simplest case of a natural scalar. That is let $\alpha = n$ a natural number so

$$f(\alpha x) = f(nx) = f[(n-1)x + x] = f[(n-1)x] + f(x) = f[(n-2)x + x)] + f(x)$$

and following an inductive alogorithm we see that

$$f(\alpha x) = \alpha f(x).$$

with $\alpha = n$ a natural number. If $\alpha = 0$, then we have $f(\alpha x) = f(0) = f(0+0) = f(0) + f(0) = 2f(0)$. The only way $f(0) = 2f(0)$ is if $f(0) = 0$. so the property counts for $\alpha = 0$. Now, let us assume that $\alpha = -1$. Then using $f(0) = 0$ we find

$$f(0) = f(x-x) = f(x) + f(-x) = 0.$$

so $f(-x) = -f(x)$. So it works well for $\alpha = -1$. The next natural step is to extend the property to all integers $\mathbb{Z}$.

Now, let us assume that $\alpha = -n$ with n a natural number then

$$f(\alpha x) = f(-nx) = f[n(-x)] = nf(-x) = -nf(x),$$

so it works fine for any α in the integers $\mathbb{Z}$. How about $\alpha = 1/q$ with q an integer.

$$q\alpha = 1,$$

so

$$f(x) = f(q\alpha x) = qf(\alpha x),$$

so

$$f(\alpha x) = \frac{1}{q} f(x) = \alpha f(x),$$

so it works fine for $\alpha = 1/q$ with q integer. Now if $\alpha = p/q$ with p and q integers, $q \neq 0$, that is when p is rational then

$$f(\alpha x) = f\left(\frac{p}{q}x\right) = f\left(p\frac{x}{q}\right) = pf\left(\frac{x}{q}\right) = \frac{p}{q}f(x) = \alpha f(x),$$

so it works for $\alpha \in \mathbb{Q}$.

What I claim is that $f(\alpha x) \neq \alpha f(x)$ for α irrational, since the way to obtain an irrational number through rationals is with an infinite series (or sequence) but this breaks the finite additivity required by linear operators. I am looking for the counter-example along these lines. If $f(x)$ is continuous, then assume α is irrational and $\alpha = \lim_{n\to\infty} \alpha_n$ with $\alpha_n \in \mathbb{Q}$, then due to the continuity of f we can take the limit outside of the function argument in the following lines:

$$f(\alpha x) = f(\lim_{n\to\infty} \alpha_n x) = \lim \alpha_n f(x) = \alpha f(x)$$

Now,

$$f(x) = f(x1) = xf(1).$$

So if $f(1) = a$, we have an explicit definition

$$f(x) = ax$$

for all **continuous** functions that satisfy the additifity condtions $f(x+y) = f(x) + f(y)$ on the one dimensional real space.

This [5] link sketches a proof using the fundamental theorem of calculus. The counter example (if it exist on one dimensional real functions) has to be a discontinuous function at any irrational number.

We will show an example that is additive but non-linear. That is, there is at least an element of the domain x and a scalar α such that $f(\alpha x) \neq \alpha f(x)$.

The idea comes from the axiom of choice and Hamel's basis. In fact this link indicates that [6] " The first to realize that it is possible using choice to construct a non-linear additive function was Hamel in 1905 ("Eine Basis aller Zahlen und die unstetigen Losungen der Functionalgleichung: $f(x+y) = f(x) + f(y)$"), Math Ann 60 459-462); indeed, a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ allows us to provide examples."

The idea is to think on the reals in the field of the rational numbers, which we call $\mathbb{R}_{\mathbb{Q}}$. That is, the axiom of choice guarantees the existence of a Hamel basis $B = \{b_i\}_{i\in I}$ and I is a subset of the integer numbers, such that every real x can be uniquely written in the form $x = \sum_{j\in J} \lambda_j b_j$, for some $j \in J \subset I$, with λ_i rational. Here J is a finite set.

(i) The set cardinality of set B is infinity (no shown here). Pick f such that $f(b_k) = 0$ for one $b_k \in B$, $b_k \neq 1$, $b_k \neq 0$, and $f(b_j)$ is anything you want for $j \neq k$.

(ii) Force additivity by defining

$$f(x) = \sum_{i\in K} \lambda_i f(b_i),$$

[5] Execercise 2 of https://www.math.ualberta.ca/ xinweiyu/217.1.13f/217-20130913.pdf

[6] http://mathoverflow.net/questions/57426/are-there-any-non-linear-solutions-of-cauchys-equation-fxy-fxfy-wit

where $x = \sum \lambda_i b_i$ (all sums are finite sums).

Check additivity, if $x = \sum_{i \in J} \lambda_i \alpha_i$, $y = \sum_{i \in K} \gamma_i \alpha_i$, then pick $L = J \cup K$. and

$$x + y = \sum_{i \in L} (\lambda_i + \gamma_i) b_i,$$

where some λ_i or γ_i could be zero (those 0 do not belong to the intersection of $J \cap K$).

$$\begin{aligned} f(x+y) &= \sum_{i \in L} (\lambda_i + \gamma_i) f(b_i) \\ &= \sum_{i \in L} \lambda_i f(b_i) + \sum_{i \in L} \gamma_i f(b_i) \\ &= \sum_{i \in J} \lambda_i f(b_i) + \sum_{i \in K} \gamma_i f(b_i) \\ &= f(x) + f(y), \end{aligned}$$

where in the previous to the last step we did not include those element with 0 coefficients (non in the intersection).

(iii) We show that f is not homogeneous on the element b_k. We know $f(b_k) = 0$. Now, if f is homogeneous in b_k then $f(b_k) = f(b_k 1) = b_k f(1) \neq 0$. This contradicts that $f(b_k) = 0$. So f can not be homogeneous.

The following link [7] provides a proof that the graph (that is, the elements $(x, f(x)) \in \mathbb{R}^2$, is dense in the plane $\mathbb{R}^2$. That is, no matter where in the plane there is always an element of $(x, f(x))$ as near as you want to you. This is an amazing fact.

[7] https://www.math.ualberta.ca/ xinweiyu/217.1.13f/217-20130913.pdf

Zero phase and poles and zeros

Where are the zeros and poles of a zero phase filter?

In general, the spectrum of a signal $h(t)$ is given by its Fourier transform

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\mathrm{e}^{j\omega t}dt \quad , \quad j = \sqrt{-1}.$$

Let us assume that $h(t)$ is real. If the spectrum $H(\omega)$ is real then

$$H(\omega) = \overline{H(\omega)} = H(-\omega),$$

That is H is symmetric in ω. If $H(\omega)$ is written as a Z transform, this means that $H(z) = H(z^{-1})$. Then, expanding $H(z)$ as a series in z, the coefficients are symmetric with respect to 0, or $h(n) = h(-n)$.

So, if there is a zero/pole at some point $p \neq 0$ in the plane, then there is a zero/pole at $1/p$. Figure 4, illustrates this. With zeros c and $1/c$ and poles p and $1/p$.

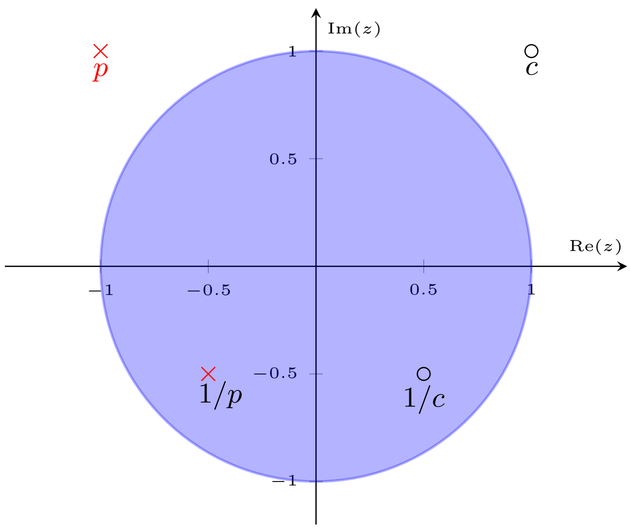

Figure 4: A zero-pole plot for a zero phase filter: A zero c and its reciprocal $1/c$. A pole p and its reciprocal $1/p$.

Poles and zeroes at 0 are matched against corresponding poles and zeroes at infinity.

Zero phase filter has the special property that they do not change the form of the data wavelet. The can scale it, stretched or squeeze it but do not change the form. This is important for many reasons that I will not consider here. Zero phase is a special case of linear phase filters where the phase response is $\alpha\omega$, for α constant. The phase is then linear. If $\alpha = 0$ the slope of the linear filter is zero and the filter is zero phase. Linear filters are zero phase filters and a shift on time/space domain.

Problem with Two Poles

Make a two-pole system with a periodic impulse response such that the period N equals 25 samples. Is this filter stable? What frequency (in Hz) is most enhanced by this filter?

Periodic in time = sampled in frequency. How can I achieve this with two poles?

Solution: A filter of two poles at p_1 and p_2 is

$$\frac{1}{(z-p_1)(z-p_2)} = \frac{A}{p_1 - z} + \frac{B}{p_2 - z}$$

where A and B can be found by partial fraction decomposition.

Then

$$\frac{1}{(z-p_1)(z-p_2)} = \frac{A}{p_1}\sum_{i=0}^{\infty}\left(\frac{z}{p_1}\right)^i + \frac{B}{p_2}\sum_{i=0}^{\infty}\left(\frac{z}{p_2}\right)^i$$

At $N = 25$ the function is periodic. That is,

From this infinite series The term $i = 0, 25, 50, \cdots$ is the same. That is

$$\frac{A}{p_1} + \frac{B}{p_2} = \frac{A}{p_1}\left(\frac{z}{p_1}\right)^{25} + \frac{B}{p_2}\left(\frac{z}{p_2}\right)^{25} = \frac{A}{p_1}\left(\frac{z}{p_1}\right)^{50} + \frac{B}{p_2}\left(\frac{z}{p_2}\right)^{50} = \cdots$$

From which we can say that

$$\left(\frac{1}{p^1}\right)^{25} = 1 \quad \text{and} \quad \left(\frac{1}{p^2}\right)^{25} = 1$$

That is p_1 and p_2 satisfy

$$p_1^{25} - 1 = 0 \quad \text{and} \quad p_2^{25} - 1 = 0.$$

The filter is not stable because the $\|p_1\| = \|p_2\| = 1$, so the geometrical series does not converge. The poles are sitting in the unit circle.

The most enhanced frequency is that where there is a phase match. That is where p_1 and p_2 have the same phase. There are 25 roots of $z^{25} - 1 = 0$, The first (non-zero phase) root corresponds to the angle

$$\frac{2\pi}{25} = \omega\Delta t = 2\pi f\Delta t$$

from where

$$\frac{1}{25} = f\,\Delta t$$

and

$$f = \frac{1}{25\Delta t} Hz.$$