Notes About 3x3 Matrix Roations

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Chapter 1

Introduction

Rigid body rotations is a large topic and the reader can find extensive discussions in the four following Wikipedia pages:

Euler angles [1] rotation matrix [2] Rigid body [3] and Orientation [4]

In this document we only consider some fundamental about rotations in a three dimensional space. The document in orthogonal matrices [5] is of great importance for this problem, as well as the 3x3 rotation matrix notes [6] on Dr. Hanno Coetzer webpage.

The most complete document on rotation of 3x3 matrices that I have found is A review of useful theorems [7] involving proper orthogonal matries referenced to three dimensional physical space.

In this appendix:

(i) We review the cross product definition and provide a representation as a matrix multiplication which is useful for the derivation of the Rodrigues formula. As a matrix we find some important properties of the cross product matrix.

(ii) Provide a summary elementary (around the main coordinate axes) rotation matrices, and how they can be used to find the rotation of the

[1]https://en.wikipedia.org/wiki/Euler_angles
[2]https://en.wikipedia.org/wiki/Rotation_matrix
[3]https://en.wikipedia.org/wiki/Rigid_body
[4]https://en.wikipedia.org/wiki/Orientation_%28geometry%29
[5]https://www2.bc.edu/ reederma/Linalg17.pdf
[6]http://dip.sun.ac.za/ stefan/TWB264/notas/engels/12_rot_3x3_eng.pdf
[7]http://www.mech.utah.edu/ brannon/public/rotation.pdf

tetrahedron from a base parallel to the XY axis to fit the altenating cube representation.

(iii) Given a rotation matrix we show how to determine the axis of rotation and the angle of rotation.

(iv) Given an axis and angle of rotation we find the rotation matrix. This is the inverse problem of the the previous item (iii). The formula is known as the Rodrigues rotation formula [8]

(v) Given a rotation matrix, provide its decomposition as a product of elementary rotation matrices.

(vi) Illustrate the use of the theory developed with an example of the rotation of a tetrahedron from its base parallel to the XY plane to a tetrahedron in the alternating cube.

[8] https://en.wikipedia.org/wiki/Rodrigues%27_rotation_formula

Chapter 2

Rigid Body Rotations

Rotation matrices are a set of orthonormal transformations that preserve lengths, areas, volumnes, and angles (isometric). They all have determinant ± 1. It can be shown that all 3D matrices with determinant 1 are rotation matrices. The idea for this to set up a rotation as a continuous operator A_t from the interval $t \in [0, \theta]$ to the set of rotation matrices with determinant 1, such that $A_0 = I$, and $A_\theta = A$. Since the determinant is continuous and A_0 is a rotation then A_θ is a rotation, otherwise would be a reflection with determinant -1 which can not happen.

1 Cross Product

We review the concept of cross product which is fundamental when considering rotations in a three dimensional space.

Given two vectors a and b in $\mathbb{R}^3$ we define their product as follows,

$$\text{If} a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad , b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad \text{then} \quad a \times b = b = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 . \end{pmatrix} \quad \text{then}$$

The cross product has the following properties:

- $a \times b = -b \times a$.
- $a^T (a \times b) = 0$ and $b^T (a \times b) = 0$. Then the cross product is perpendicular to both of their participating vectors.

- $\|a \times b\| = \|a\|\|b\| \sin\theta$, where θ is the angle between a and b. Then $a \times a = 0$.

We can associate the cross product with a matrix as follows. For any vector b,

$$a \times b = Kb,$$

where

$$K = \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix} \tag{2.1}$$

This is verified by direct multiplication. This cross product matrix has the following properties:

(i) **anti-symmetry** $K^T = -K$.

(ii) **singular** $\det(K) = 0$**.**

(iii) **square symmetric** $K^2 = (K^2)^T$**.**

(iv) $K^3 = -\|a\|^2 K,$ and if $\|a\| = 1$, then $K^3 = -K$.

(v)

$$\text{Tr}(K^2) - 2\|k\|^2 \tag{2.2}$$

These properties are directly verifiable, however we verify (iii), (iv), and (v).

$$\begin{aligned} K^2 &= \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -k_2^2 - k_3^2 & k_1 k_2 & k_1 k_3 \\ -k_1 k_2 & -k_1^2 - k_3^2 & k_2 k_3 \\ k_1 k_3 & k_2 k_3 & -k_1^2 - k_2^2 \end{pmatrix} \end{aligned}$$

which is obviously symmetric.

Let us now compute K^3.

$$\begin{aligned}
K^3 &= K^2K = \begin{pmatrix} -k_2^2-k_3^2 & k_1k_2 & k_1k_3 \\ -k_1k_2 & -k_1^2-k_3^2 & k_2k_3 \\ k_1k_3 & k_2k_3 & -k_1^2-k_2^2 \end{pmatrix} \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & k_3^3+k_3k_2^2+k_1^2k_2 & -k_2^3-k_2k_3^2-k_1^2k_2 \\ -k_1^2k_3-k_3^2-k_2^2k_3 & 0 & k_1k_2^2-k_1^3+k_1k_3^2 \\ k_2^2k_3^2+k_2^3+k_1^2k_2 & k_1k_3^2+k_1^3-k_1k_2^2 & 0 \end{pmatrix} \\
&= (k_1^2+k_2^2+k_3^2)\begin{pmatrix} 0 & k_3 & -k_2 \\ -k_3 & 0 & -k_1 \\ k_2 & -k_1 & 0 \end{pmatrix}.
\end{aligned}$$

So indeed $K^3 = -\|k\|^2K$, and if $\|k\| = 1$ then $K^3 = -K$. Finally,

$$\mathrm{Tr}(K^2) = -2k_1^3 - 2k_2^2 - 2k_3^2 = -2\|k\|^2,$$

which verify all the statements above.

2 Rotation Matrices

The fundamental theorem of rigid body rotations is that it is only required a maximum of three elementary rotations to take a body into any arbitrary rotation. The three elementary rotations are defined as

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is a rotation about the Z axis by an azimuthal angle ϕ.

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

which is a rotation about the Y axis performed by moving the north pole a polar rotation angle θ. These two rotations can be found by using the spherical coordinates shown in Figure **??** and equations **??**. Finally a rotation about the X axis by an angle α.

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

The history of these rotations started with the work of Leonhard Eulger who formulated the fundamental theorem for body rotations. There is no consensus on the name of the angles. They are sometimes called α, β, γ, or ϕ and θ are interchanged (between physicists and mathematicians).

From the Wikipedia site; there are twelve posibilities of three sequences of rotations with respec to the x, y, and z axis. Six of them are called the and the other six the tait-bryan angles . The last set is is sometimes called as nautical angles or Cardan angles. The nautical angles are sometimes called heading, elevation, and bank or yaw, pitch, and roll. Here

- yaw is the $R_z(\phi)$ matrix,
- pitch is the $R_y(\theta)$ matrix,
- roll is the $R_x(\alpha)$ matrix.

These terms are most used in aeronautic navigation.

We want to find rotations matrices such that the tetrahedron is sent into the altenating cube.

Two solutions presented: First is an analytical derivation, the second is a solution found by using computer software (symbolic "Maxima" and graph "TiKz") by trial and error.

1.For simplification let us call the three vertices in the base of the tetrahedron

$$\begin{aligned} T_1 &= a(\sqrt{3}/3, 0, -\sqrt{6}/12) \\ T_2 &= a(-\sqrt{3}/6, 1/2, -\sqrt{6}/12) \\ T_3 &= a(-\sqrt{3}/6, -1/2, -\sqrt{6}/12) \end{aligned}$$

and the apex $a(0, 0, \sqrt{(6)}/4)$. We choose $a = 2\sqrt{2}$ so that each vector cas norm $\sqrt{3}$.

$$\begin{aligned}
T_1 &= (2\sqrt{6}/3, 0, -\sqrt{3}/3) \\
T_2 &= (-\sqrt{6}/3, \sqrt{2}, -\sqrt{3}/3) \\
T_3 &= a(-\sqrt{6}/3, -\sqrt{2}, -\sqrt{3}/3) \\
A &= (0, 0, \sqrt{3})
\end{aligned}$$

We want to set up equations to find the azimuth and polar angles in spherical coordinates. We want to make a solid rotation. Starting at the north pole A we should rotate this vector as to align with one of the vectors in the cube. The closest point to A in the cube with coordinates at ± 1 is $A' = (-1, -1, 1)$. So we want to make a solid rotation from A to A'.

We find first the polar and azimuthal angles for these two points, with the radius $r = \sqrt{3}$.

$$\begin{pmatrix} 0 \\ 0 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} \sqrt{3}\sin\phi_A \cos\theta_A \\ \sqrt{3}\sin\phi_A \sin\theta_A \\ \sqrt{3}\cos\phi_A \end{pmatrix}$$

From the last equation we find

$$\cos\phi_A = 1 \quad , \quad \sin\phi_A = 0 \tag{2.3}$$

At this moment we can not determine the azimuthal angle θ_A. We can assume $\theta_A = 0$ and this does not violate any of the equations above, and the pair

$$\sin\theta_A = 0 \quad \cos\theta_A = \pm 1$$

We choose the positive sign $\cos\theta = 1$.

Now, if we want to map this point into $(-1, -1, 1)$ we need to set up the equation

$$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}\sin\phi_{A'} \cos\theta_{A'} \\ \sqrt{3}\sin\phi_{A'} \sin\theta_{A'} \\ \sqrt{3}\cos\phi_{A'} \end{pmatrix}$$

We find from the third equation,

$$\cos\phi_{A'} = \frac{\sqrt{3}}{3} \quad , \quad \sin\phi_{A'} = \pm\sqrt{1 - 1/3} = \pm\frac{\sqrt{6}}{3}$$

From the first equation

$$-1 = \left(\sqrt{3}\right)\left(\pm\frac{\sqrt{6}}{3}\right)\cos\theta_{A'} = \pm\sqrt{2}\cos\theta_{A'}$$

If we choose the "−" sign then $\sin\phi_{A'}$ should (because the first component is negative) be positive. So let us pick

$$\cos\theta_{A'} = \frac{\sqrt{2}}{2} = \sin\theta_{A'} \quad , \quad \sin\phi_{A'} = -\frac{\sqrt{6}}{3}$$

We found the following convenient equations

$$\begin{aligned}
\cos\phi_A &= 1 \quad , \quad \sin\phi_A = 0 \\
\cos\phi_{A'} &= \frac{\sqrt{3}}{3} \quad , \quad \sin\phi_{A'} = -\frac{\sqrt{6}}{3} \\
\cos\theta_A &= 1 \quad , \quad \sin\theta_A = 0 \\
\cos\theta_{A'} &= \frac{\sqrt{2}}{2} \quad , \quad \sin\theta_{A'} = \frac{\sqrt{2}}{2}
\end{aligned}$$

which we use to find the following required trigonometric equations for the rotation.

The rotation needed to go from A to $(-1,-1,1)$ in the polar direction has the two following basic trigonometric functions:

$$\begin{aligned}
\cos(\theta_{A'} - \theta_A) &= \cos\theta_{A'}\cos\theta_A + \sin\theta_{A'}\sin\theta_A = \frac{\sqrt{2}}{2} + 0 = \frac{\sqrt{2}}{2} \\
\sin(\theta_{A'} - \theta_A) &= \sin\theta'_A\cos\theta_A - \sin\theta_A\cos\theta_{A'} = \frac{\sqrt{2}}{2} - 0 = \frac{\sqrt{2}}{2}.
\end{aligned}$$

We call $\theta = \theta_{A'} - \theta_A$ the total rotation polar angle between the point T_1 and $(1,-1,-1)$.

Now for the azimuthal rotation.

$$\cos(\phi_{A'} - \phi_A) = \cos\phi_{A'}\cos\phi_A + \sin\phi_{A'}\sin\phi_A = \frac{\sqrt{3}}{3}$$

and

$$\sin(\phi_{A'} - \phi_A) = \sin\phi_{A'}\cos\phi_A - \sin\phi_A\cos\phi_{A'} = -\frac{\sqrt{6}}{3} + 0 = -\frac{\sqrt{6}}{3}$$

from which

$$\sin(\phi_{A'} - \phi_A) = -\frac{\sqrt{6}}{3}$$

We call $\phi = \phi_{A'} - \phi_A$.

We are ready to make the rotations needed. The azimuthal rotation is done with respect to the z axis. This is

$$Az = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.4}$$

Which we identify as a $\pi/4 = 45°$ rotation of the XY plane with respect to the Z axis.

The polar rotation with respect to the Y axis in the counter-clockwise direction is given by

$$P = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{6}}{3} \\ 0 & 1 & 0 \\ \frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \end{pmatrix} \tag{2.5}$$

The matrices found above produce the proper rotation.

2. Most of this exercise here was done by trial and error.

First we want to bring all vectors to the same size ($\sqrt{3}$). This is done by choosing $a = 2\sqrt{2}$ (the side length) .

Then we call the matrix that rotates $\pi/4$ around the z axis, Y (for yaw):

$$Y = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.6}$$

Then the matrix which rotates around the y axis: P (for pitch) with an angle of $\theta = \arccos(\sqrt{3}/3)$ which is about $54°$: This is:

$$P = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{\sqrt{3}}{3} \end{pmatrix} \tag{2.7}$$

Then we can verify by matrix multiplication that if we call the base vectors

$$
\begin{aligned}
T_1 &= a(\sqrt{3}/3, 0, -\sqrt{6}/12) && (2.8)\\
T_2 &= a(-\sqrt{3}/6, 1/2, -\sqrt{6}/12) && (2.9)\\
T_3 &= a(-\sqrt{3}/6, -1/2, -\sqrt{6}/12) && (2.10)
\end{aligned}
$$

Then: $YPT_1 = (1, 1, 1)$, $YPT_2 = (-1, 1, -1)$, $YPT_3 = (1, -1, -1)$, and finally for the apex $A = (0, 0, a\sqrt{(}6)/4)$, $YPA = (-1, -1, 1)$.

I used [Maxima][1] to validate my computations. I also used TiKZ to draw and test the transformations. I did 4 plots, three of the plots where the projections into the XY, XZ, and YZ axis (because the optical illusions from seeing a 3D figure in a 2D screen). The other figure was a perspective in 3D which is the only one that I include here.

[![Rotation of a tetrahedron][2]][2]

The figure on the right has the rotated (blue) into (brown) on top of the (red) diagonals of the cube. The blue dots in the second figure are smaller to see how the overlay is doing the job.

3 Given a rotation matrix find the axis of rotation and angle of rotation

Let us assume that $Q \in \mathbb{R}^3 \times \mathbb{R}^3$ is a rotation matrix. We show the following things:

1. 1 is an eigenvalue of Q.

2. The other two eigenvalues are $e^{\pm i\theta}$. We interprete θ.

If Q is a rotation matrix $\det(Q) = 1$. This is a well known fact and it means that there is no change of volume under rotation.

We have the following sequence of equalities

$$\begin{aligned}\det(I-Q) &= \det(Q)\det(I-Q)\\ &= \det(Q^T)\det(I-Q)\\ &= \det(Q^T-I)\\ &= \det(Q-I)\\ &= -\det(I-Q) \quad \text{, since 3 is odd,}\end{aligned}$$

So $\det(I-Q)=0$ and $\lambda_1 = 1$ is an eigenvalue of Q.

Now, let u_1 the unit eigenvector of λ_1, so $Qu_1 = u_1$. We show that the matrix Q is a rotation of an angle θ around this axis u_1. Let us form a new coordinate system using $u_1, u_2, u_1 \times u_2$, where u_2 is a vector orthogonal to u_1, so the new system is right handed (has determinant $= 1$). The transformation between the old system $\{e_1, e_2, e_3\}$ and the new system is given by a matrix B with column vectors u_1, u_2, and u_3. So we have that $Be_i = u_i$. Let us call the coordinate transformation (similarity) matrix $C = B^{-1}Q$. Then

$$Ce_1 = B^{-1}QBe_1 = B^{-1}Qu_1 = B^{-1}u_1 = e_1.$$

The fact that $Ce_1 = e_1$ means two things:

(i) The first column of C is $(1,0,0)^T$.

(ii) The first raw of C is $(1,0,0)$.

Then C is a matrix of the type

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

Since Q is orthogonal C is orthogonal and so the vectors $(a,c)^T$ and $(b,d)^T$ are orthogonal and since

$$1 = \theta Q = \det C = ad - bc$$

we have that the minor matrix with entries a, b, c, d is a rotation (orthogonal with determinat 1). Rotations in 2D are of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

then

$$C = B^{-1}QB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

But since the eigenvalues of $B^{-1}Q$ are the same as those of Q we find the eigenvalues of this matrix and those would be the eigenvalues of Q.

For the eigenvalues of this matrix we have that

$$\det(C - \lambda I) = 0 \implies (\cos\theta - \lambda)^2 + \sin^2\theta = 0.$$

That is,

$$1 - 2\lambda\cos\theta + \lambda^2 = 0$$

Then

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} \cos\theta \pm i\sin\theta = \mathrm{e}^{\pm i\theta}.$$

Now, to provide the interpretation of the rotating vector , we see that using the matrix C above,

$$\begin{aligned} Qu_2 &= (BB^{-1})(QBe_2) = B(B^{-1}QB)e_2 = Ce_2 = au_2 + bu_3 \\ Qu_3 &= (BB^{-1})(QBe_3) = B(B^{-1}QB)e_3 = Ce_3 = -bu_2 + au_3 \end{aligned}$$

That is we have that

$$\begin{aligned} Qu_1 &= u_1 \\ Qu_2 &= au_2 + bu_3 \\ Qu_3 &= -bu_2 + au_3 \end{aligned}$$

So Q has the effect of rotating the plane spanned by u_2 and u_3 around u_1 in a counter-clockwise direction. This is no news since the matrix $B^{-1}Q$

represents the transformation in the new coordinate system given by the three vectors u_1, u_2, and u_3.

Since the trace of the matrix is found as the negative of linear coefficient in the characteristic polynomial and the trace does not change under similarity transformations, we can find the angle from the trace as follows: The characteristic polynomial is given by

$$\chi_Q(x) = (x-1)(x-\mathrm{e}^{i\theta})(x-\mathrm{e}^{-i\theta}) = (x-1)(x^2 - 2\cos\theta + 1).$$

hence Since the trace of a matrix is the negative coefficient of the linear term, we have that

$$\mathrm{tr}(Q) = 2\cos\theta + 1$$

and

$$\cos\theta = \frac{\mathrm{tr}(Q)-1}{2}.$$

As an application of the theory on this section let us consider the matrix Q that sends the tetrahedron with horizontal base to the tetrahedron in the altenating cube. That is, the matrix

$$\begin{aligned} Q = YP &= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{3} & 0 & -\sqrt{\frac{2}{3}} \\ 0 & 1 & 0 \\ \sqrt{\frac{2}{3}} & 0 & \frac{\sqrt{3}}{3} \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} \sqrt{6} & -3\sqrt{2} & -2\sqrt{3} \\ \sqrt{6} & 3\sqrt{2} & -2\sqrt{3} \\ 2\sqrt{6} & 0 & 2\sqrt{3} \end{pmatrix} \end{aligned} \tag{2.11}$$

It is easy to verity that Q is orthonormal. To find the angle of rotation we see that

$$\mathrm{tr}(Q) = \frac{1}{6}(\sqrt{6} + 3\sqrt{2} + 2\sqrt{3}) = 2\cos\theta - 1$$

from which

$$\cos\theta =$$

4 Given an angle θ and axis of rotation a find the rotation matrix

This is in a way the inverse problem of the problem in the previous section. Here we assume that we know the new axis or rotation and the angle that needs to be rotated then find the matrix that would do the rotation job. We now refer to A as defined in equation 2.13.

We build a triple a, b, and $a \times b = Ab$ which is a right hand side orthonormal basis. We want to find a matrix Q, that rotates the plane spanned by b and c around the axis a, and angle θ. Figure 2.1 shows an illustration of the variables involved in this problem.

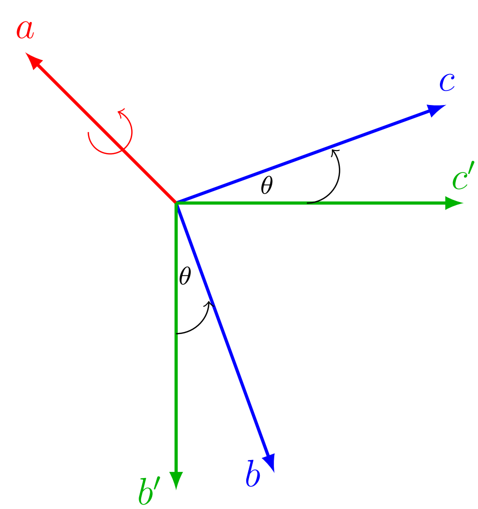

Figure 2.1: We assume that a in red is the axis of rotation. The vector b is built orthogonal to a and $c = a \times b$. We want to find the matrix that rotates the b, c plane by an angle θ counterclockwise.

We then know the following actions from Q to the three vectors a, b, and c:

(i) $Qa = a$ (the axis is invariant).

(ii) $Qb = b\cos\theta + c\sin\theta = b'$.

(iii) $Qc = b\cos\theta - b\sin\theta = c'$.

since the matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

provides a counter-clockwise rotation in the plane spanned by b and c by an angle θ. Now, since $c = a \times b = Ab$, then

$$\begin{aligned} Qb &= b\cos\theta + Ab\sin\theta = (I\cos\theta + A\sin\theta)b \\ Qc &= QAb = Ab\cos\theta - b\sin\theta = (A\cos\theta - I\sin\theta)b \end{aligned} \quad (2.12)$$

We now differentiate the first of these equations with respect to θ and find

$$\frac{dQ}{d\theta}b = (-I\sin\theta + A\cos\theta)b.$$

so that, using equation 2.12 we find

$$\frac{dQ}{d\theta}b = QAb.$$

Since $a \times c = -b$, then we have that $Ac = -b$, so $b = -Ac$. Let us now differentiate equation 2.12 with respect to θ. That is,

$$\begin{aligned} \frac{dQ}{d\theta}c &= (-I\cos\theta - A\sin\theta)b \\ &= -I\cos\theta - \sin\theta c \\ &= -Qb \\ &= -Q(-Ac) \\ &= QAc. \end{aligned}$$

From $Qa = a$ and by differentiation we find that $dQa/d\theta = 0$, and since $Aa = 0$, then $QAa = 0$, so we could write

$$\frac{dQ}{d\theta}a = QAa.$$

In summary we find the three equations

$$\begin{aligned}\frac{dQ}{d\theta}a &= QAa\\ \frac{dQ}{d\theta}b &= QAb\\ \frac{dQ}{d\theta}c &= QAc\end{aligned}$$

and since a, b, and c are linearly independent then we conclude that

$$\frac{dQ}{d\theta} = QA.$$

This is a matrix first order ordinary differential equations with initial condition

$$Q(0) = I$$

and solution

$$Q = \mathrm{e}^{\theta A} = I + \theta A + \frac{1}{2}\theta^2 A^2 + \cdots + \frac{1}{n!}\theta^n A^n + \cdots$$

We now use the equation $A^3 = -A$ derived above. Then we can see that all powers of A can be written in terms of powers of A up to the maximum power A^2 and there will be a pattern. For this let us write the following chain of powers:

$$\begin{aligned}&A^3 = -A\\ &A^4 = AA^3 = -A^2\\ &A^5 = AA^4 = -A^3 = A\\ &A^6 = AA^5 = -A^4 = A^2\\ &A^7 = AA^6 = A^3 = -A\\ &\vdots\end{aligned}$$

The pattern is cyclic with period 4, and the chain of powers goes like

$$\begin{aligned}
-A &= A^3 = A^7 = A^{11} = \cdots = A^{3+4i} \cdots \\
-A^2 &= A^4 = A^8 = A^{4+4i} = \cdots = \\
A &= A^5 = A^9 = A^{5+4i} = \cdots = \\
A^2 &= A^6 = A^{10} = A^{6+4i} = \cdots =
\end{aligned}$$

$$Q = I + \theta A + \frac{1}{2}\theta^2 A^2 - \frac{1}{3!}\theta^3 A - \cdots (-1)^{\lfloor \frac{n}{2} \rfloor} \frac{1}{n!}\theta^n A^{n \bmod 3}$$

We would like to sort the terms on different order to take advantage of the series representations of the $\cos\theta$ and $\sin\theta$ functions which are embedded on the series above. To be able to reorder the matrix we should show convergence. For any $n > 0$ we have the following inequality

$$0 \leq \left\| \frac{A^n}{n!} \right\| \leq \frac{\|A\|^n}{n!},$$

and we know that , for each real number x, the series $\sum_{n=0}^{+\infty} \frac{x^n}{n!}$ converges (it defines the exponential function). Therefore, for any A the series $\sum_{n=0}^{+\infty} \frac{A^n}{n!}$ converges. We also find here that

$$\|e^A\| \leq e^{\|A\|}$$

We now write Q by reordering all even powers and all odd powers of θ, and the I separated from all the other. That is

$$\begin{aligned}
Q &= I + \left(\frac{1}{2}\theta^2 A^2 - \frac{1}{4!}\theta^4 A^2 + \cdots + (-1)^{n/2}\theta^{2n} A^2 + \cdots \right) \\
&+ \left(\theta A - \frac{1}{3!}\theta^3 A + \cdots (-1)^n \theta^{2n+1} A \cdots \right)
\end{aligned}$$

and from the cosine and sine Taylor series expansion, and factoring out the A and A^2 factors we find that

$$Q = I + A\sin\theta + A^2(1-\cos\theta).$$

This formula is known as the **Rodrigues rotation formula** . The Wikipedia website [1] provides a derivation of this formula using pure vector operations without the need to do differential equations.

Finally we derived a couple of formulas that relate Q with A and θ. We can compute the trace of Q as follows:

$$\mathrm{Tr}(Q) = \mathrm{Tr}(I) + \sin\theta\mathrm{Tr}(A) + (1-\cos\theta)\mathrm{Tr}(A^2).$$

Now, from equation 2.2 we have that $\mathrm{Tr}(A^2) = -\|a\|^2 A$ and since $\mathrm{Tr}(A) = 0$,then

$$\mathrm{Tr}Q = 3 + (1-\cos\theta)(-2) = 1 + 2\cos\theta,$$

or

$$\cos\theta = \frac{1}{2}\left(\mathrm{Tr}(Q) - 1\right)$$

The final formula is derived as follows:

$$Q - Q^T = [I + A\sin\theta + A^2(1-\cos\theta)] - [I - A\sin\theta + A^2(1-\cos\theta)] = 2A\sin\theta,$$

from which

$$A = \frac{1}{2\sin\theta}(Q - Q^T).$$

5 Rotate the tetrahedron to the alternating cube

The tetrahedron with vertices as shown in equation **??** has its apex at the point $N = (0, 0, \sqrt{3})$. In the cube with coordinates $(\pm 1, \pm 1, \pm 1)$, the closest

[1] https://en.wikipedia.org/wiki/Rodrigues%27_rotation_formula

point to the north pole is the point $N' = (-1, -1, 1)$ (see Figure **??**). We want to rotate the whole body so the new north pole is sitting at the point $N' = (-1, -1, 1)$. We need to follow the simple evaluations:

(i) Normalize vectors.

(ii) Find the axes of rotation.

(iii) Find angle of rotation.

(iv) Use Rodrigues rotation formula.

Let us call the new normalized vectors:

$$a = (0, 0, 1) \quad , \quad b = \frac{1}{\sqrt{3}}(-1, -1, 1).$$

Now the cross product is

$$k = a \times b = \frac{1}{\sqrt{3}}(1, -1, 0).$$

This is the axis of rotation. With this we build the matrix A

$$A = \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\sqrt{3}/3 \\ 0 & 0 & -\sqrt{3}/3 \\ \sqrt{3}/3 & \sqrt{3}/3 & 0 \end{pmatrix} \tag{2.13}$$

For the angle of rotation

$$\cos\theta = a \cdot b = \frac{\sqrt{3}}{3}$$

and $\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{2/3}$.

$$Q = I + A\sin\theta + A^2(1 - \cos\theta) = \begin{pmatrix} \frac{\sqrt{3}+1}{2\sqrt{3}} & -\frac{\sqrt{3}-1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{\sqrt{3}-1}{2\sqrt{3}} & \frac{\sqrt{3}+1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

On the other hand the product of rotation matrices Y and P from 2.11 is

$$\frac{1}{6}\begin{pmatrix} \sqrt{6} & -3\sqrt{2} & -2\sqrt{3} \\ \sqrt{6} & 3\sqrt{2} & -2\sqrt{3} \\ 2\sqrt{6} & 0 & 2\sqrt{3} \end{pmatrix}$$