The 5 Platonic Solids

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Chapter 1

Introduction

There are several way to study Platonic solids. Here we present a construction of them starting at a plannar set of regular polygons which as an origami game will bend to form the Platonic solid. We find that there are angles that will allow us to construct a regular polyhedron for the case of the 5 Platonic solids.

A different way to construct Platonic solids is a tesselation on a sphere. We show that way in my notes on spherical geometry.

Chapter 2

Construction of the 5 Platonic Solids

The subject of Platonic solids is better understood under the topic of spherical geometry [1] , where the sphere is tessellated using spherical polygons such as squares with 120° degrees on the internal angles (the tetrahedron), equilateral triangles with internal angles of 90° (the octahedron), squares of 120° internal angles (the cube or hexahedron), pentagons of 120° internal angles and finally triangles of 72° internal angles. However we will use a different approach.

We showed in my graph theory notes that there cannot be more than 5 Platonic solids. Here we show how to build the 5 Platonic solids using pure geometrical arguments. We show that the only possible polygons used to build Platonic solids could be triangles, squares, and pentagons. We list the possible Platonic solids and then prove that each of this is realizable. The list below assumes that we start with a given number of polygons sitting initially in a flat plane.

(i) Three triangles (the tetrahedron).

(ii) Four triangles (the octahedron).

(iii) Five triangles (the icosahedron).

(iv) Three squares (the hexahedron or cube).

[1] http://euler.slu.edu/escher/index.php/Spherical_Geometry

(v) Three pentagons (the dodecahedron).

We construct each one of them in order of the number of faces, but before we deal with the construction of the polyhedra we want to exploit a few facts from the geometry of the pentagon.

2.1 Facts on the Pentagon

We show that the ratio between a diagonal and the side of the pentagon is given by the Golden Ratio [2] ϕ.

With this we find some exact trigonometrical values of angles for some of the polyhedra constructed below.

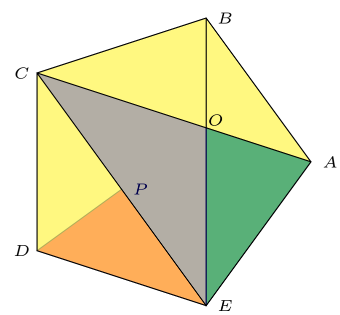

Figure 2.1: Pentagon used to extract values of trigonometrical functions in terms of the Golden ratio.

In the next set of computations we use the Figure 2.11.

We find the following attributes of the pentagon.

- The diagonals $\overline{AC}$, $\overline{BE}$, and $\overline{CE}$ are congruent (same length) since they are the largest side of congruent isosceles triangles. Observe that the three triangles have congruent sides along the regular pentagon.
- The triangles $\triangle BAE$ and $\triangle ABE$ are congruent since they both are isosceles (sides equal to the length of the side of the pentagon) and have congruent angles at A and B (internal angle of the regular pentagon equal to $3\pi/5$.

[2] http://mathworld.wolfram.com/GoldenRatio.html

- Angles $\angle BAC$, and $\angle ABC$ are congruent since they correspond to congruent triangles.
- Segments $\overline{AB}$ and $\overline{EC}$ are parallel due to the alternate interior congruent angles.
- Segments $\overline{OB}$ and $\overline{OA}$ are congruent since they correspond to congruent triangles. Hence segments $\overline{OE}$ and $\overline{OC}$ are congruent and the quadrilateral $OEDC$ is a rhombus with all sides congruent with the side of the pentagon.
- We now turn to the two isosceles triangles $\triangle EAC$ (gray and green) and $\triangle EAO$ (green). They have share the same base angle (look at the vertex A). So they are similar. Then we have the ratios

  $$\frac{OA}{AE} = \frac{AE}{CE},$$

  That is, calling ℓ the length of the side of the pentagon, and d the length of the diagonal,

  $$\frac{d-\ell}{\ell} = \frac{\ell}{d},$$

  from which $d^2 - d\ell - \ell^2 = 0$, with the two solutions

  $$d = \ell\frac{1 \pm \sqrt{5}}{2}$$

  where we discard the negative solution and find that the ratio between the diagonal and the side of the of the pentagon is

  $$\frac{d}{\ell} = \frac{1+\sqrt{5}}{2} = \phi,$$

  where $\phi \approx 1.618033988749895$ is the Golden ratio.

The importance of these results is that we can get infinite precision in the computations that follow.

Let us find the measure a of the angle at A (that is $\angle EAC$). The measure of the angle at B is $b = 3\pi/5$, and we know that

$$2a + b = \pi,$$

from which

$$a = \frac{\pi - 3\pi/5}{2} = \frac{\pi}{5}.$$

Then from the cosine law

$$BC^2 = BA^2 - 2(BA)(AC)\cos a + AC^2$$

from which (since $BA = BC$, and with $AC = d$, and $BA = \ell$)

$$\cos a = \frac{AC}{2BA} = \frac{\phi}{2}$$

So

$$\cos\frac{\pi}{5} = \frac{\phi}{2} = \frac{1+\sqrt{5}}{10} \qquad \sin\frac{\pi}{5} = \frac{5-\sqrt{5}}{2^{3/2}}. \tag{2.1}$$

In the triangle $\triangle ECA$, if we drop a height h from C to the base AE we divide the triangle into two right triangles. The green angle at A is

$$\frac{3\pi}{5} - \frac{\pi}{5} = \frac{2\pi}{5}$$

then

$$\cos\frac{2\pi}{5} = \frac{\ell/2}{d} = \frac{1}{2\phi} = \frac{1}{1+\sqrt{5}}.$$

Now $2\pi/5 - \pi/2 = -\pi/10$ or in degrees $72^\circ - 90^\circ = -18^\circ$, then

$$\begin{aligned}
\sin(-18^\circ) &= \sin(2\pi/5 - \pi/2) = -\cos(2\pi/5) = -\frac{1}{2\phi} = -\frac{1}{\sqrt{5}+1} \\
\cos(-18^\circ) &= \sqrt{1-\sin^2(-18^\circ)} = \sqrt{1-1/4\phi^2} = \frac{\sqrt{4\phi^2-1}}{2\phi} = \frac{\sqrt{2\sqrt{5}+5}}{\sqrt{5}+1}.
\end{aligned} \tag{2.2}$$

We now apply the cosine law to the yellow triangle $\triangle CBA$.

$$d^2 = 2\ell^2 - 2\ell\cos 3\pi/5,$$

that is

$$\cos 3\pi/5 = \frac{2\ell^2 - d^2}{2\ell} = \frac{2-\phi^2}{2} = \frac{\sqrt{5}-1}{4}.$$

Now let us focus in the orange right triangle $\triangle DPE$. The angle at D, that is $\angle PDE$ is half of the interior angle of the pentagon. That is it has a measure of $3\pi/10$. We have then that

$$\sin\left(\frac{3\pi}{10}\right) = \frac{PE}{DE} = \frac{d/2}{\ell} = \frac{\phi}{2} = \frac{1+\sqrt{5}}{4}.$$

and

$$\cos\left(\frac{3\pi}{10}\right) = \sqrt{1-\sin^2\phi} = \sqrt{1-\phi^2/4} = \frac{\sqrt{4-\phi^2}}{2} = \frac{5-\sqrt{5}}{2^{3/2}}.$$

This result should not come with surprise since from equations 2.1 , $\cos\pi/5 = \phi/2$, and $\sin\pi/5 = (5-\sqrt{5})/2^{3/2}$.

Now, from the previous two expressions:

$$\tan\left(\frac{3\pi}{10}\right) = \frac{\sin(3\pi/10)}{\cos(3\pi/10)} = \frac{\sqrt{2}\sqrt{5}+\sqrt{2}}{2\sqrt{5-\sqrt{5}}}. \tag{2.3}$$

So we found, based on the regular pentagon, a collection of closed analytical values for trigonometric functions of angles such as $\pi/5, 2\pi/5, 3\pi/10$ and $3\pi/10$ which will be useful in the construction of the Platonic solids.

2.2 The tetrahedron

We consider three equilateral triangles sitting in a horizontal plane and uniformly distributed along the azimuthal angle. The three triangles have a total angle of 180 degrees in the center, the gaps between them is 60 degrees between triangle. The left frame of Figure 2.2 sketches this situation.

We want to rotate the triangular face $\triangle OWV$ by a certain angle with respect to the Y axes Similarly we want to rotate the triangular face with vertex at V' by the same angle and show that the vertex V' and V after rotation will coincide at the same point V_f. In the rotation of the triangular face $\triangle OWV$ the vertex V will draw a circle with center at the projection of

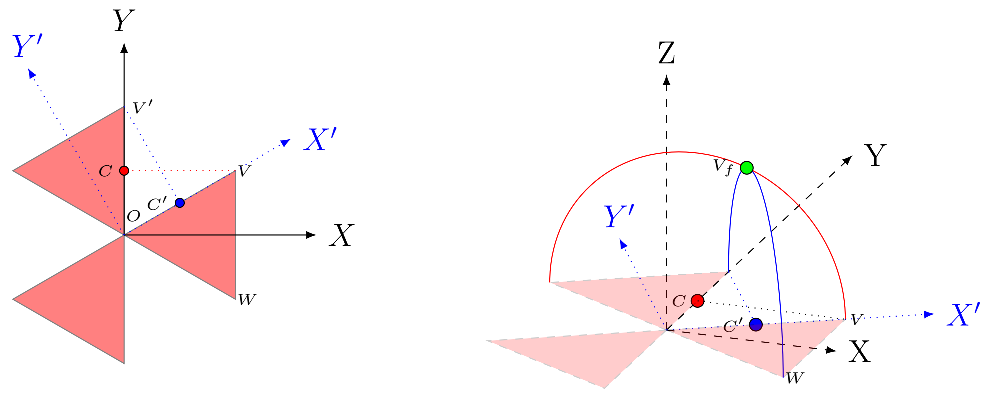

Figure 2.2: Initial stage for the construction of a tetrahedron

the vertex V into the Y axis. This projection is given by $\ell \cos \pi/6 = \ell\sqrt{3}/2$. For simplicity we will use the length of the each side of the triangles as $\ell = 1$. We can call this center of rotation $C = (0, 1/2, 0)$. The radius of rotation is $r = \sqrt{3}/2$ which is the length of the perpendicular from V to the axis Y. Since the circumference is in the vertical plane $Y = 1/2$, we can just write an equation in terms of x and z. This is

$$x^2 + z^2 = r^2 \tag{2.4}$$

This circumference is represented by the red arc in the right frame of Figure 2.2.

The rotation of the face with vertex V' is a bit harder to do. Let us now rotate the coordinates 30° in the counter-clockwise direction using Z as the rotation axis. Then new rotated Y' axes will bisect the angle at O of the triangle with vertex at V'. Figure 2.2 shows in dotted blue lines the new (X', Y') coordinates. In this case the new circumference is in the (X', Y', Z) coordinate system with center at $C' = (1/2, 0, 0)$, with the same radius r. In this coordinate system the new X' axis does not change along the circumference, so we can write the equation for the circle as

$$y'^2 + z^2 = r^2. \tag{2.5}$$

To find Y' we recall that we rotated the axes of coordinates by an angle $\varphi = \pi/6$, with the rotation matrix with axes in the Z direction

$$R_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so

$$\begin{aligned} x' &= x\cos\varphi + y\sin\varphi \\ y' &= -x\sin\varphi + y\cos\varphi \end{aligned}$$

Given that $y = y'$ is known since it does not change under rotation about the y axis, we have the couple of equations in x and z,

$$\begin{aligned} x^2 + z^2 &= r^2 \\ (-x\sin\varphi + y\cos\varphi)^2 + z^2 &= r^2. \end{aligned} \tag{2.6}$$

From here it is easy to eliminate z to find that

$$x^2 = (x\sin\varphi - y\cos\varphi)^2$$

from which we find

$$x = \pm(x\sin\varphi - y\cos\varphi).$$

we choose the "$-$" to solve the equation on the first quadrant sign and write

$$x(1 + \sin\varphi) = y\cos\varphi$$

so

$$x = \frac{y\cos\varphi}{1 + \sin\varphi}.$$

Now from equation 2.6

$$z = \sqrt{r^2 - x^2} = \sqrt{r^2 - \frac{y^2\cos^2\varphi}{(1+\sin\varphi)^2}} = \frac{\sqrt{r^2 + 2r^2\sin\varphi + r^2\sin\varphi - y^2\cos^2\varphi}}{1 + \sin\varphi}$$

and since $y^2 = 1 - r^2$,

$$z = \sqrt{r^2 - x^2} = \frac{\sqrt{2r^2 + 2r^2 \sin\varphi - \cos^2\varphi}}{1 + \sin\varphi} = \sqrt{\frac{2r^2 + \sin\varphi - 1}{1 + \sin\varphi}}.$$

Then we find the three coordinates

$$\begin{aligned} x &= \frac{y\cos\varphi}{1+\sin\varphi} \\ y &= \sqrt{1-r^2} \\ z &= \sqrt{\frac{2r^2+\sin\varphi-1}{1+\sin\varphi}} \end{aligned} \tag{2.7}$$

The dip angle θ satisfies the equation

$$\tan\theta = \frac{z}{x} = \frac{\sqrt{2r^2 + 2r^2\sin\varphi - \cos^2\varphi}}{\cos\varphi\sqrt{1-r^2}}. \tag{2.8}$$

Of course since $x^2 + y^2 + z^1 = \ell^2 = 1$, we can always find z from x and y using

$$z = \sqrt{1 - x^2 - y^2}.$$

Since here $\varphi = \pi/6$, and $y = 1/2$ we find

$$x = \frac{(\sqrt{3}/2)(1/2)}{3/2} = \frac{\sqrt{3}}{6}. \tag{2.9}$$

Also $y = \sqrt{1-r^2} = \sqrt{1-3/4} = 1/2$, and since

$$\begin{aligned} 2r^2 + 2r^2\sin\varphi - \cos^2\varphi &= 2(3/4)(1+1/2) - 3/4 = 3/2 \\ 1 + \sin\varphi &= 3/2 \\ \cos\varphi = \frac{\sqrt{3}}{2} \end{aligned}$$

we finally find that

$$\begin{aligned} x &= \frac{\sqrt{3}}{6} \\ y &= \frac{1}{2} \\ z &= \sqrt{\frac{2}{3}}, \end{aligned}$$

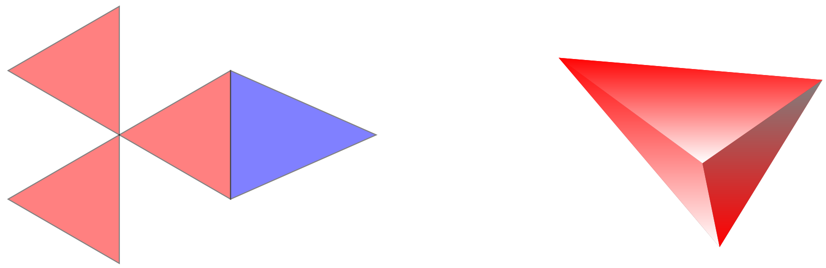

Figure 2.3: On the left frame we see a flat origami model from which the bending along the edges we can build the tetrahedron. On the right we see the tetrahedron after lifting the faces. According to the way you focus you can only see either two lateral faces and the top, or three lateral faces out of the four faces. The blue triangle on the left will fold to fill up the gap between the red base triangles.

and

$$\theta = \arctan(z/x) = \arctan\left(\frac{6\sqrt{2}}{3}\right) = \arctan 2\sqrt{2} \approx 70.53^\circ.$$

Since we can solve the system composed by equations for the two circles, then we can say that the two vertices V and V' tie well after rotation into the single point $V_f = (x, y, z)$. The right frame of Figure 2.2 shows the intersection of the two circumferences on the green point.

Figure 2.3 shows the tetrahedron after been lifted from the plane projection in Figure 2.2. On the left of we see the plane tessellation from which we can build the tetrahedron by folding it, on the right we see the tetrahedron itself after the correct folding.

At this moment we have only two vertices. The vertex at $(0, 0, 0)$ and the vertex found by intersecting the two circumferences. We need to find three other vertices. Since all vertices are at located at the same horizontal plane $z = \sqrt{2/3}$, we need to find the x and y coordinates by knowing that they are all in a circle of radius r such that

$$r^2 = x^2 + y^2 = 1 - z^2 = 1 - 2/3 = 1/3,$$

so $r = \frac{\sqrt{3}}{3}$. The initial phase (angle) φ is that of the found solution (x, y, z),

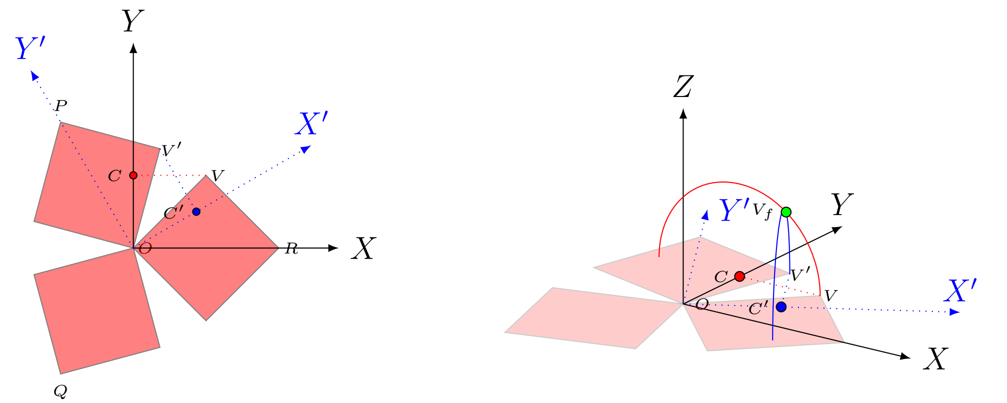

Figure 2.4: The lower part of a construction of a hexahedron.

that is we have

$$\tan\varphi = \frac{y}{x} = \frac{3}{\sqrt{3}}$$

from which $\varphi = 60°$. The other points are located by knowing that their distribution is uniform, so that 3 points for $360°$ are spread each by $\delta = 120°$. The phases of the other points are then $\varphi, \varphi+\delta$, and $\varphi + 2\delta$.

2.3 The hexahedron (cube)

While it is clear that the hexahedron can be build with 6 squares at right angles, we follow the same procedure used in the building of the tetrahedron and which will be used for the other Platonic solids.

We build the hexahedron in two faces.

(i) **The bottom:** To build the bottom of the hexahedron we start with three squares in the plane $z = 0$. Each square contributes to an angle of $90°$ and so there is a $90°$ to divide in three for the gaps between the squares. The left frame of Figure 2.4 sketches the three squares that make the lateral faces of the bottom of the cube.

The three squares share a common vertex at the origin $O = (0, 0, 0)$. We label a square with a vertex in V with coordinates $(\cos\pi/4, \sin\pi/4) =$

$(\sqrt{2}/2, \sqrt{2}/2)$. The opposite vertex has coordinates $(\sqrt{2}/2, -\sqrt{2}/2)$. The vertex opposite to the origin O has coordinates $(\sqrt{2}, 0)$. Each edge has length $\ell = 1$. As in the case of the tetrahedron we need to tilt the face with vertex at V, an unknown angle having Y as the axes of rotation. The center of rotation is the projection of the vertex at V into the Y axes. This is the red dot in the left frame of Figure 2.4 labeled as C and represented numerically as

$$C = (0, \sqrt{2}/2).$$

We should also rotate the square with vertex at V', but before we do this, we should find a new (rotated) coordinate system such that the new Y' axes goes through the diagonal of the square with vertex at V'. The new coordinates (X', Y') (in blue in Figure 2.4) are found after a rotation by $\varphi = 30°$ of the old coordinates axis. Again we will build another circle with center at the projection of the vertex V' into the new coordinate axes X'. This center represented by the blue dot in the left frame of Figure 2.4. Given two unit vectors v_1 and v_2 , the projection of v_1 into v_2 is given by $(v_1 \cdot v_2)v_2$. Then in this particular case $v_1 = V' = (\cos 75°, \sin 75°) \approx (0.259, 0.966)$, and $v_2 = V = \cos 30°, \sin 30°) = (1/2, \sqrt{3}/2) \approx (0.5, 0.867)$. so

$$C' \approx [(0.259, 0.966) \cdot (0.707, 0.707)](0.707, 0.707) \approx (0.483, 0.483).$$

The rotation of the face with vertex at V, with respect to the Y axes, traces a circumference with center at C and radius $r = \sqrt{2}\ell/2 = \sqrt{2}/2$ (recall that we use $\ell = 1$). Since the Y axes is not changing under this rotation the $y = \sqrt{2}/2$ value remains fixed, and we can write the equation of the circle as

$$x^2 + z^2 = r^2.$$

In the rotated axes the new coordinate X' remains constant since the rotation uses this new X' direction as the axes of rotation. We then have

$$y'^2 + z^2 = r^2$$

We recognize that the two equations above correspond exactly with the equations 2.4 and 2.5 derived for the case of the tetrahedron. This will be the case for all other Platonic solids that we construct below. The difference is only the rotation angle φ and the radius of the circle r. We can then copy the solutions from equations 2.7 and 2.12 above.

$$\begin{aligned} x &= \frac{y\cos\varphi}{1+\sin\varphi} \\ y &= \sqrt{1-r^2} \\ z &= \frac{\sqrt{2r^2+2r^2\sin\varphi-\cos^2\varphi}}{1+\sin\varphi} \end{aligned}$$

The dip angle θ satisfies the equation

$$\tan\theta = \frac{z}{x} = \frac{\sqrt{2r^2+2r^2\sin\varphi-\cos^2\varphi}}{\cos\varphi\sqrt{1-r^2}}.$$

In this particular case $\varphi = 30°$ $(\pi/6 \text{ rad})$, $r = \sqrt{2}/2$, and $y = \sqrt{2}/2$. Then

$$\begin{aligned} x &= \frac{(\sqrt{2}/2)(\sqrt{3}/2)}{3/2} = \frac{\sqrt{6}}{6} \\ y &= \frac{\sqrt{2}}{2} \\ z &= \frac{\sqrt{1+1/2-3/4}}{3/2} = \frac{\sqrt{3}}{3}. \end{aligned}$$

which is represented by the green point in the right frame of Figure 2.4. The tilt angle is found by solving

$$\tan\theta = \frac{\sqrt{3}/3}{\sqrt{6}/6} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

That is,

$$\theta \approx 54.74° \tag{2.10}$$

At this point we have two vertices of the cube. The bottom (origin O), and the vertex $V_f = (\sqrt{6}/6, \sqrt{2}/2, \sqrt{3}/3)$. We see that together with the vertex V_f where the two vertices V and V' merged after tilting the faces. This could be done exactly with the other couple of vertices in the 30° apertures, due to the invariance of rotation which maps any face to another. Then all vertices, after tilting the faces are at the same height $z = \sqrt{3}/3$. All the three vertices in this plane are uniformly distributed in a circle of radius $r = \sqrt{6/36 + 2/4} = \sqrt{2/3}$. The phase α of the first vertex is at the angle $\alpha = \arctan(y/x) = \arctan(\sqrt{2}/2) \approx 35.264°$. Since the three vertices are uniformly distributed in the circle the angle between them is $\gamma = 120°$. Then the three points are represented by

$$v_i = (r\cos(\alpha + i\gamma), r\sin(\alpha + i\gamma), z) \quad , \quad i = 0, 1, 2.$$

There are three more vertices that are attached to the three faces on the bottom of the cube. These vertices are labeled as P, Q, and R in the left frame of the Figure 2.4. There are several way to find those vertices. We use some simple equations. That is, the vector sum of two adjacent vertices provides the third vertex (the parallelogram law of adding vectors justifies this method). Then we can call $w_1 = v_0 + v_1$, $w_2 = v_1 + v_2$, and $w_3 = v_2 + v_0$. Figure i sketches the 7 vertices found so far and the faces corresponding to those vertices.

(ii) **The top:** At this point we have three faces and 7 vertices constructed. We just need to add one more vertex and join the faces in a proper way to finish the cube.

We can use again the parallelogram law to construct the missing vertex. Take the vector represented by the difference $P - V_f$, and the vector represented by the difference $R - V_f$. They both are in the same plane of the missing vertex and their vector addition (up to a translation by V_f should provide the missing vertex N. That is

$$N = V_f + (R - V_f) + (P - V_f) = R + P - V_f.$$

This vertex N is shown in Figure i as an empty circle. By building the three faces that merge at N we finish the cube.

2.4 The octahedron

As in the hexahedron we build the octahedron in two faces:

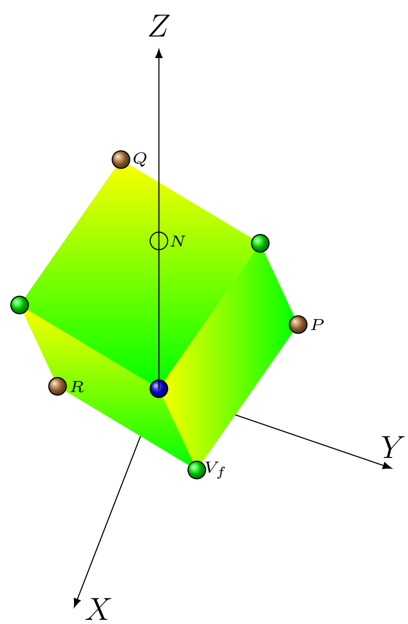

Figure 2.5: Bottom part of the construction of a hexahedron (cube). We can see three faces and 7 vertices. The bottom vertex is blue, then comes a second layer of green vertices and on the upper layer we have three brown vertices. The upper vertices are labeled B_i, $i = 1, \cdots, 6$.

1. **The bottom:** The octahedron is built starting at 4 equilateral triangles in a plane with a common vertex at the origin O. Since each triangle has $60°$ degrees at the center we are left with $360° - 240° = 120°$ which divided by 4 gives us a 30 degree gap between triangles. Figure 2.7 shows an sketch of the four starting triangles.

   The folding of this plane diagram is easier than in all the other Platonic solids, since in the first case we use the Y axes as the axes of rotation and in the other the X axes. While lifting the triangle with vertex at V, this vertex draws a circular trajectory with center at C, and radius $r = 1/2$. The center is the projection of the vertex V into the Y axes. That is at the point

   $$C = (0, 1/2, 0).$$

   Since the rotation is with respect to the Y axes, this does not change

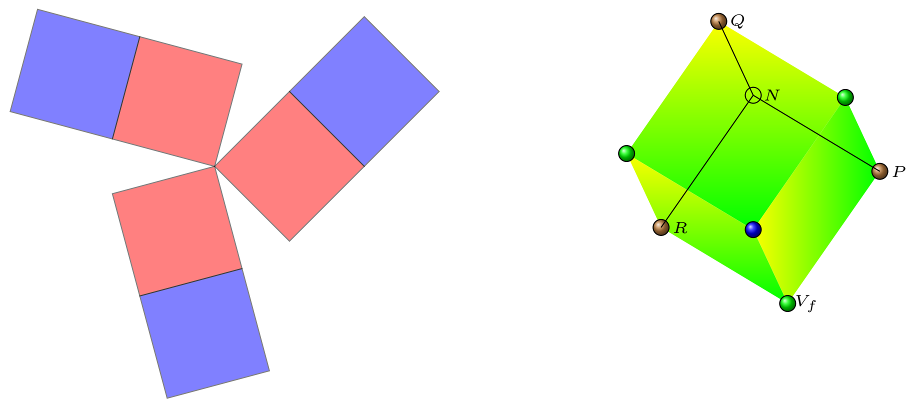

Figure 2.6: The hexahedron (cube). On the left is an origami cut indicating the lines of folding. The red squares are initially folded by an angle θ found in equation 2.10. Then the after that the blue squares are folded 90 degrees. On the right is the complete cube. The three faces attached to the vertex N are the top faces found in the last step.

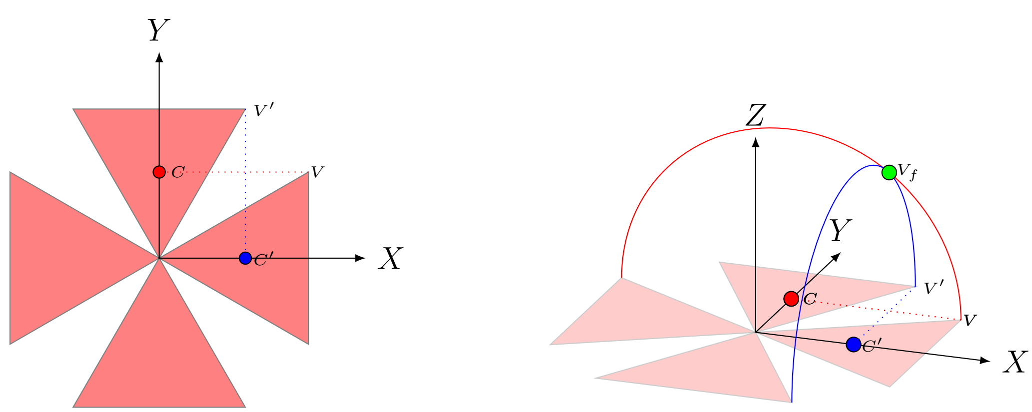

Figure 2.7: On the left frame we see the four triangles used to start building the octahedron. On the right we see two arcs of circumference intersecting at a green point V_f which is the first vertex for this layer.

and for simplification we can just write the equation

$$x^2 + z^2 = r^2 \tag{2.11}$$

for the circular trajectory of the vertex V. The circular trajectory is shown on the right frame of Figure 2.7. Similarly the rotation of the face with vertex V' with respect to the X axis draws a circle of radius r and center C' shown in blue on the right frame of Figure 2.7. Along this circle the X does not change and we can write

$$y^2 + z^2 = r^2.$$

In the rotation of face with vertex in V we have $y = 1/2$ fixed since the rotation is with respect to the Y axis. Then from the last equation above,

$$z = \sqrt{r^2 - y^2}$$

and from this in equation 2.11

$$x = \sqrt{r^2 - z^2} = \sqrt{r^2 - r^2 + y^2} = y$$

So the two circles intersect at $V_f = (x, y, z) = (1/2, 1/2, \sqrt{2}/2)$ marked with a green circle in the right frame of Figure 2.7.

At this moment we have two vertices. The bottom vertex at the origin O and the vertex V_f found by solving the system of two circle equations. At the second layer we find that at the same depth $z = \sqrt{2}/2$, there will be four vertices, all of the equally spaced by an angle $\gamma = 90°$ along a circle of radius $r = \sqrt{x^2 + y^2} = \sqrt{2}/2$ The first vertex has phase $\alpha = \arctan(y/x) = \arctan(1) = 45°$. All the vertices in this layer are found using the equation

$$v_i = (r\cos(\alpha + i\gamma), r\sin(\alpha + i\gamma), z) \quad , \quad i = 0, 1, 2, 3.$$

Figure 1 shows the four triangular faces tilted to their correct location. The blue vertex is the bottom vertex, then come 4 vertices at $z = \sqrt{2}/2$, labeled with green color.

2. **The Top :** There is only one vertex to be found. The top vertex. The symmetry of the volume is such that this vertex is as the location

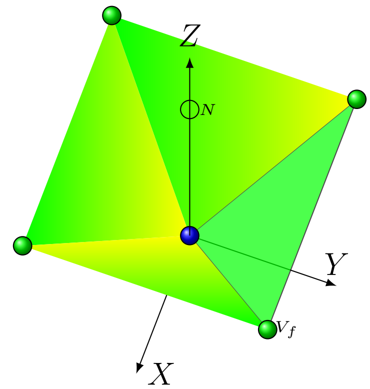

Figure 2.8: Bottom part of the construction of an octahedron. We can see four faces and 5 vertices. The bottom vertex is blue, then comes a second layer of green vertices.

$(0, 0, 2z) = (0, 0, \sqrt{2})$. This vertex is drawn as an empty circle in Figure 1. The completion of the octahedron is now trivial. We need to connect the triangles in the top face

Figure 2 shows a “paper cut map” to start an origami that builds the octahedron on the left and on the right the figure of the octahedron.

2.5 The dodecahedron

We build the dodecahedron in three steps using pentagons.

(i) **bottom :** We start with 5 pentagons in a flat plane attached to a vertex $O = (0, 0, 0)$. Each pentagon has an internal angle of $\theta_i = 3\pi/5 = 108°$. The gap left by the three pentagons around the origin is of $360 - 3(108)° = 36°$. So the gap between pentagons is of $12°$. The length of a diagonal of the pentagon is $d = \ell\sqrt{2 - 2\cos\theta_i}$

Figure 2.10 shows a representation of the three pentagons.

As in the previous cases we need to rotate the pentagon with vertex at V an angle with respect to the axes Y. This vertex V will draw an arc of circumference, and since the Y axes does not change we can write

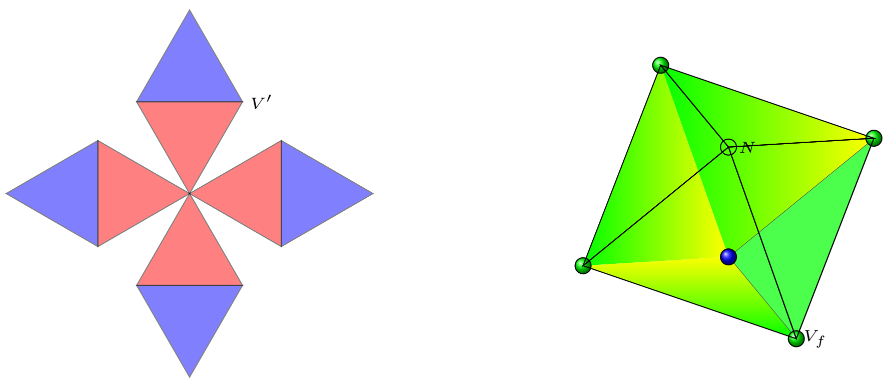

Figure 2.9: The octahedron . On the left is an origami cut indicating the lines of folding. The red triangles are initially folded by an angle $\theta = 45°$. Then the after that the blue squares are folded toward the upper vertex until they merge. On the right is the complete octahedron. The four faces attached to the vertex S are the top faces found in the last step.

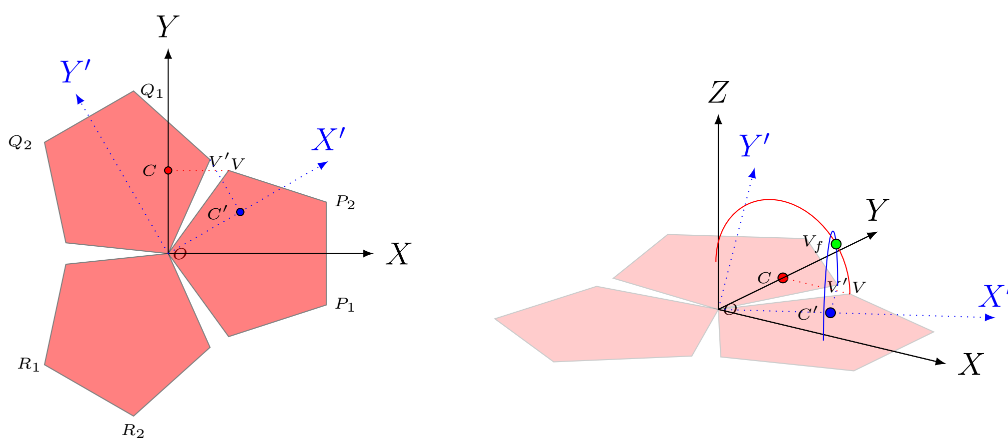

Figure 2.10: The lower part of a construction of a dodecahedron.

this arc as described by the equation

$$x^2 + z^2 = r^2.$$

The center of curvature is the projection of V into the Y axes. The foot of the projection is at the coordinate $X = 0$ and $Y = \ell \sin(\theta_i/2) \approx 0.809$, using $\ell = 1$, and the radius r is given by $r = \cos(\theta_i/2) \approx 0.588$. In Figure 2.10 , this center is represented by the red dot and labeled as C. The rotation of the face F should be done with respect to the axis X' which is a rotated version of the axes X. We need to figure out the rotation to convert the coordinates (X, Y, Z) into (X', Y', Z). We need the Y axis to bisect the second face. If $\theta_i = 108°$ is the internal angle of the pentagon, $\delta = 12°$ is the angle between two consecutive pentangos, then the bisector of the second pentagon would have an angle of $\theta_i/2+\delta+\theta_i/2 = \theta_i+\delta = 120°$. The new rotated axis (X', Y', Z) are shown in blue in Figure 2.10. To find the center of rotation for the second circle we project the vector $V' = (\cos 66°, \sin 66°, 0)$ into the new X' axis with unit vector of $u = (\cos \pi/6, \sin \pi/6, 0) = (\sqrt{3}/2, 1/2, 0)$. This provides the point

$$C' = V' \cdot u \approx (0.701, 0.405).$$

The equation for this circle in the new coordinate system is

$$y'^2 + z^2 = r^2,$$

since nor z, neither r change. The new y' is computed from the rotation matrix as $y' = x \sin \varphi - y \cos \varphi$, where $\varphi = 30°$ is the rotation angle. As in the solution of the first two Platonic solids we solve the system of two circles, for the two unknowns x and z, (recall that we already know y since it does not change in the first circle) to find (copy equation 2.7)

$$\begin{aligned} x &= \frac{y \cos \varphi}{1 + \sin \varphi} \\ y &= \sqrt{1 - r^2} = \ell \sin(\theta_i/2) \\ z &= \sqrt{\frac{2r^2 + \sin \varphi - 1}{1 + \sin \varphi}}. \end{aligned}$$

The dip angle θ satisfies the equation

$$\tan\theta = \frac{z}{x} = \frac{\sqrt{2r^2 + 2r^2\sin\varphi - \cos^2\varphi}}{\cos\varphi\sqrt{1-r^2}}. \tag{2.12}$$

The numerical values are

$$\begin{aligned} x &\approx .4670861794813579 \\ y &\approx \cos(\theta_i/2) \approx 0.588 \\ z &\approx 0.3568220897730898 \end{aligned}$$

Once we found a vertex at the plane $z \approx 0.357$, we find its argument and then build the other two vertices at the same height. The argument (phase) of the vertex V_f is given by

$$\alpha = \arctan(y/x) = 60^\circ$$

Then since there are three vertices at this height separated each from the other by an angle of $\gamma = 120^\circ$ and with a radius of $r = \sqrt{x^2+y^2} \approx 0.934$, we can find the three vertices at this level as

$$v_i = r(\cos(\alpha + i\gamma), \sin(\alpha + i\gamma), z) \quad , \quad i = 0, 1, 2.$$

This takes care of the three vertices at $z \approx 0.357$, however there are 6 more vertices that are moved when we tilt these faces. They are the vertices labeled as P_1, P_2, Q_1, Q_2, R_1, and R_2 in the left frame of Figure 2.10. Since the figures are rigid, these new vertices move preserving all the distances to all points in the pentagons, so we can find them using vector operations.

To design a vector operation to find the vertices P_i, Q_i and R_i, $i = 1, 2$ we label the three vertices at the plane $z \approx 0.3575$ as $A0, B0$ and C, and want to find linear combinations of them two find the new vertices. We ask: what are the coefficients required to finish a regular pentagon given that two edges are known? These coefficients are two invariant of a pentagon. That is, no matter its size or orientation the coefficients are the same.

If two edges are adjacent we can, without loss of generality, consider that they have $O = (0, 0, 0)$ as their common vertex. Even more, since the coefficients that we are looking for are invariant we can consider the problem in the plane $\mathbb{R}^2$.

The five points of a pentagon are easily found by solving the equation $x^5 + 1 = 0$ These solutions are:

$$
\begin{aligned}
x_1 &= 1 \quad , \quad y_1 = 0 \\
x_2 &= \cos\varphi \quad , \quad y_2 = \sin\varphi \\
x_3 &= \cos 2\varphi \quad , \quad y_3 = \sin 2\varphi \\
x_4 &= \cos 3\varphi \quad , \quad y_4 = \sin 3\varphi \\
x_5 &= \cos 4\varphi \quad , \quad y_5 = \sin 4\varphi
\end{aligned}
$$

where $\varphi = 2\pi/5 = 72°$.

We want to push this solutions so that one of the vertices is 0. At the moment the solutions are centered at 0. We can do this by subtracting the vertex (x_3, y_3) to all other vertices and getting the coordinates

The new vertices (vectors in $\mathbb{R}^2$) are

$$
\begin{aligned}
A &= (1 - \cos 3\varphi \ , \ -\sin 3\varphi) \\
B &= (\cos\varphi - \cos 3\varphi \ , \ \sin\varphi - \sin 3\varphi) \\
C &= (\cos 2\varphi - \cos 3\varphi \ , \ \sin 2\varphi - \sin 3\varphi) \\
D &= (0, 0) \\
E &= (\cos 4\varphi - \cos 3\varphi \ , \ \sin 4\varphi - \sin 3\varphi)
\end{aligned}
$$

The Figure 2.11 shows the pentagon (on the left) and before being translated, and on the right after being pushed up to have the origin $O = (0, 0)$ as one its vertices.

We now set the equations that we want to solve. These are

$$\alpha E + \beta C = A.$$

This is a linear system of two equations and two unknowns. Its solutions are:

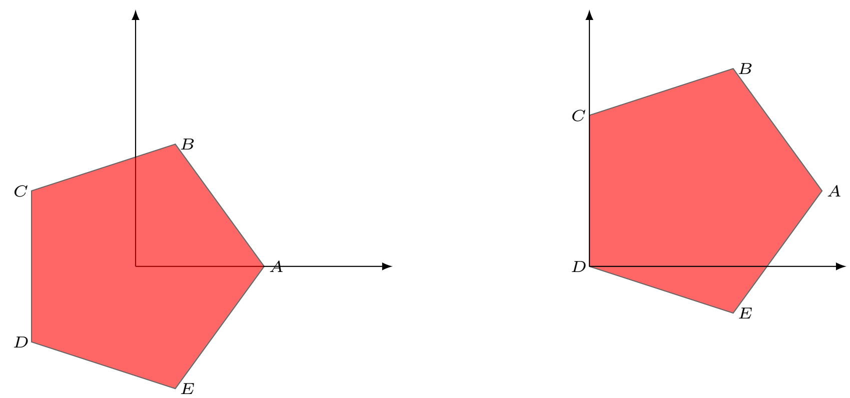

Figure 2.11: On the left is a pentagon on the unit circle with a vertex at $(1, 0)$. On the right is the pentagon moved so that vertex D is now at the origin $O = (0, 0)$.

$$\begin{aligned} \alpha &= 2\cos\varphi - 1 \\ \beta &= 4\cos^2\varphi + 2\cos\varphi. \end{aligned}$$

Since for the pentagon $\varphi = 2\pi/5 = 72°$ the numerical evaluation this is

$$\begin{aligned} \alpha &\approx 1.618033988749895 \\ \beta &= 1 \end{aligned}$$

With this we can then find the new 6 vertices corresponding to P_1, P_2, Q_1, Q_2, R_1, and R_2, that we rename for convenience. Let us name the vertices A_1, A_2, A_3, as the three vertices found so far. Based on these three vertices we create six new vertices B_1, B_2, B_3, B_4, B_5, and B_6 using the linear combination of the known vectors A_1, A_2, and A_3 with the coefficients $\alpha = \phi$, and $\beta = 1$. These 6 new vertices are derived in

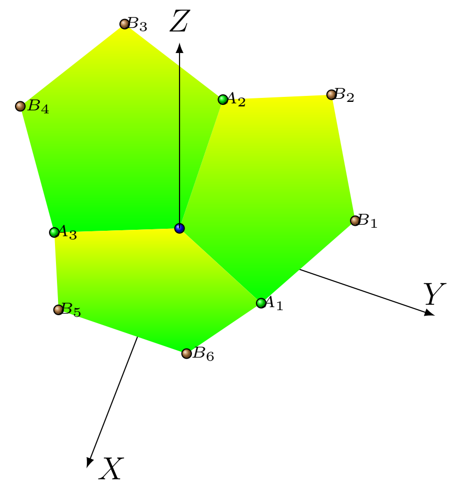

Figure 2.12: Bottom part of the construction of a dodecahedron. We can see three faces and 7 vertices. The bottom vertex is blue, then comes a second layer of green vertices and on the upper layer we have three brown vertices. The upper vertices are labeled P,Q, and R.

the following equation:

$$
\begin{aligned}
B_1 &= \phi A_1 + A_2 \\
B_2 &= A_1 + \phi A_2 \\
B_3 &= \phi A_2 + A_3 \\
B_4 &= A_2 + \phi A_3 \\
B_5 &= \phi A_3 + A_1 \\
B_6 &= A_3 + \phi A_1
\end{aligned}
$$

Figure iii shows the three faces after being rotated properly. and the new vertices. The blue vertex is the origin $O = (0, 0, 0)$, the green vertices are the bottom layer vertices A_1, A_2, and A_3, and the newly found vertices are B_1, B_2, B_3, B_4, B_5, and B_6 with with brown color in the figure.

It is interesting to see that α seems to be the Golden Ratio. [3] This

[3]http://mathworld.wolfram.com/GoldenRatio.html

suggest a pure geometrical proof: In the first figure right frame (in red) figure, the segment that joints A_3 to A_1 is the vector $A_1 - A_3$, and it is parallel to the vector B_3. Then we know that the ratio between a diagonal and the side of a pentagon is the Golden ratio ϕ. That is

$$\phi = \frac{|A_1 - A_3|}{|B_3|}. \tag{2.13}$$

So the scalar required to get to A_1 through B_3 is exactly the Golden ratio ϕ. o Then $A_1 = \phi B_3 + A_3$, from which the coefficients become $\alpha = \phi = (1+\sqrt{5})/2 \approx 1.61896243291592$ and $\beta = 1$. So, indeed α is the Golden ratio and we found an exact representation for $\cos 2\pi/3$. That is

$$\cos 2\pi/3 = \phi = \frac{1+\sqrt{5}}{2}$$

(ii) **middle :** We use the same idea of adding edges as vectors to find other edges. If we look back into Figure iii we see that at the vertices A_1, A_2 and A_3 we can tessellate three new faces. This faces are found from the following equations:

$$\begin{aligned}
C_1 &= A_1 + (B_6 - A_1) + \phi(B_1 - A_1) = B_6 + \phi(B_1 - A_1)\\
C_2 &= A_1 + \phi(B_6 - A_1) + (B_1 - A_1) = B_1 + \phi(B_6 - A_1)\\
C_3 &= A_2 + (B_2 - A_2) + \phi(B_3 - A_2) = B_2 + \phi(B_3 - A_2)\\
C_4 &= A_2 + \phi(B_2 - A_2) + (B_3 - A_2) = B_3 + \phi(B_2 - A_2)\\
C_5 &= A_3 + (B_4 - A_3) + \phi(B_5 - A_3) = B_4 + \phi(B_5 - A_3)\\
C_6 &= A_3 + \phi(B_4 - A_3) + (B_5 - A_3) = B_5 + \phi(B_4 - A_3)
\end{aligned}$$

Figure ii illustrates the addition of the new vertices, edges, and faces to the dodecahedron.

(iii) **top :** At the moment we have the following vertices computed

- The vertex at the origin $O = (0, 0, 0)$ (blue vertex) in Figure ii.

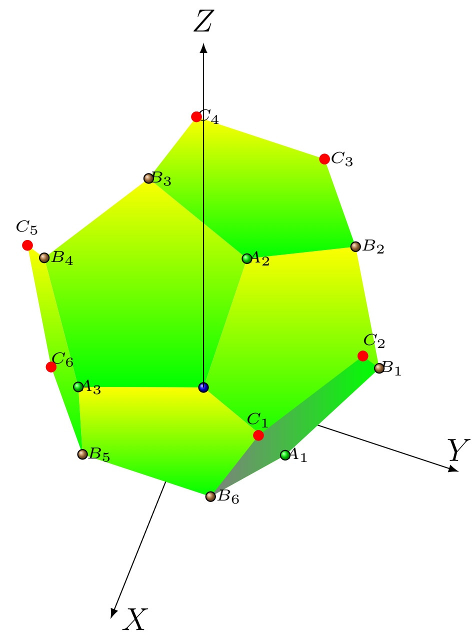

Figure 2.13: Middle part of the construction of a dodecahedron. We can see six faces and 13 vertices. The bottom vertex is blue, then comes a second layer of green vertices, then a middle layer we have three brown vertices. Finally the new found vertices are located at the top and painted red.

- The first three vertices corresponding to the bottom layer. The vertex A found as the intersection of two circles and then B y C by two 120° rotations of A in the same horizontal plane. Green vertices in Figure ii.
- 6 vertices A_i, $i = 1, \cdots, 6$, found by using the vector rule that two vertices of a pentagon can be found by using a vectorial sum $E_1 + \phi E_2$ and $E_2 + \phi E_1$, where E_1, and E_2 are two segments with a common vertex assumed to be at 0. Brown vertices in Figure ii. A total of 10 vertices has been found at this time. This is half of the dodecahedron. The other half is exactly symmetric to this half.
- 6 more vertices using the previous vertices using the same vectorial rule above. See the red vertices in Figure ii.

We need to find 3 and 1 vertices. That is, the sequence count by layers of equal hight is $1, 3, 6, 6, 3, 1$. Wind three vertices and in the final stage the top vertex.

We now use the existing points to find three more vertices and complete three more faces. For example starting at C_2 we can find the vertex that is between C_2 and C_3. By using the equation

$$D_1 = B_1 + B_2 - B_1 + \phi(C_2 - B_1) = B_2 + \phi(C_2 - B_1)$$

Likewise

$$\begin{aligned} D_2 &= B_3 + (B_4 - B_3) + \phi(C_4 - B_3) = B_4 + \phi(C_4 - B_3) \\ D_3 &= B_5 + (C_6 - B_5) + \phi(B_6 - B_5) = C_6 + \phi(B_6 - B_5) \end{aligned}$$

We are now ready for the last point of the dodecahedron. This point is in the top of the dodecahedron along the Z axis.

This point completes three pentagons. To show that the dodecahedron has a perfect match, we should show that any possible path that completes the three upper pentagons leads to this apex. Note that the three pentagons involved have four vertices from which we should construct the five vertices. Four of the vertices for each pentagon are,

$$\begin{array}{c} C_1C_2D_1D_3 \\ C_3C_4D_2D_1 \\ C_5C_6D_3D_2 \end{array}$$

Let us take the first three points on each of the sequence above, since a plane is uniquely determined by two vectors. We write equations for three planes which should intersect on a point (non of the planes are parallel to each other. No proof of that here). The planes are given by

$$\begin{array}{c} C_2 + \alpha_1(D_1 - C_2) + \beta_1(C_1 - C_2) \\ C_4 + \alpha_2(D_2 - C_4) + \beta_2(D_2 - C_4) \\ C_6 + \alpha_3(D_3 - C_6) + \beta_3(C_5 - C_6) \end{array} \tag{2.14}$$

We show that their intersection point exists and satisfies the conditions of the apex E. System 2.14 is a system of 9 equations with 9 unknowns.

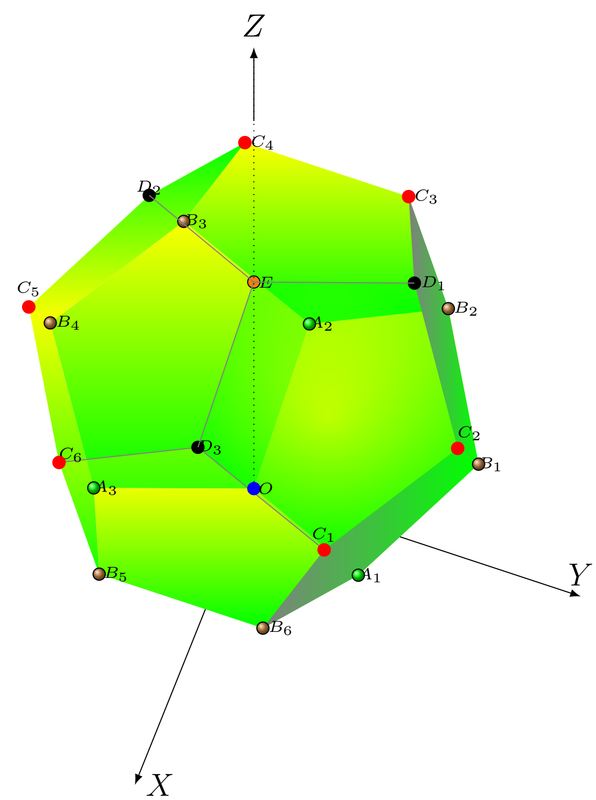

Figure 2.14: Complete construction of dodecahedron.

The scalars α_i, β_i, $i = 1, 2, 3$, and the point $E = (x, y, z)$, which goes in the right hand side of system 2.14. This system is linear and solvable, so there is a unique solution for the α_i, β_i, and (x, y, z), $i = 1, 2, 3$. We will exhibit a solution

Given three vertices of a pentagon we can find the other two. Then we have the following equations: following three following equations

$$
\begin{aligned}
E_1 &= C_2 + (C_1 - C_2) + \phi(D_1 - C_2) = C_1 + \phi(D_1 - C_2) \\
E_2 &= C_4 + (C_3 - C_4) + \phi(D_2 - C_4) = C_3 + \phi(D_2 - C_4) \\
E_3 &= C_6 + (C_5 - C_6) + \phi(D_3 - C_6) = C_5 + \phi(D_2 - C_5)
\end{aligned}
$$

If we show that $E_1 = E_2 = E_3$ we are done since the dodecahedron clicks perfectly at each vertex and we built a total of $12 = 3+3+3+3$, pentagonal faces with $1 + 3 + 6 + 6 + 3 + 1 = 20$ vertices.

Clearly $\alpha_1 = \alpha_2 = \alpha_3 = \phi$, $\beta_1 = \beta_2 = \beta_3 = 1$, and $E = E_1 = E_2 = E_3$, satisfies the requirement of the intersection of the three planes in the point E. This proofs that the iterative construction of the dodecahedron, started at point O in the bottom on the Z axis and finishes in the top E along the Z axis as shown in Figure iii

The construction of an origami map of the dodecahedron is a bit more complicated that for the other Platonic solids. We start by showing the bottom, middle, and top layers of one of the three chains that created the whole model. That is, from the pentagon with vertex V in Figure 2.10, we add two more for the middle layer and one for the top layer. Given that we know the vertices of this first pentagon, and we call them A_i, $i = 1, \cdots, 5$,

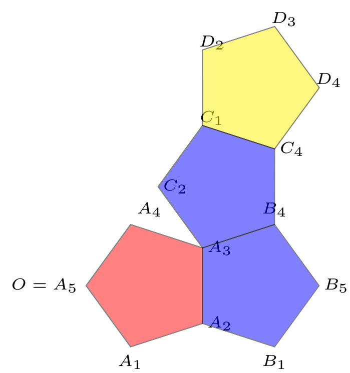

Figure 2.15: One third of the origami paper cut of a dodecahedron in a plane.

Let us call the coordinates of $A_i = (x_i, y_i)$, $i = 1, \cdots, 5$. Given that the vertex A_5 is at the origin, we want to create new vertices B_1, B_4 and B_5 such that the line $y = x_2 = x_3$ is a mirror for those coordinates. Then $B_4 = (x_3, 0)+(x_3-x_4, y_4) = (2x_3-x_4, y_4)$, $B_1 = (x_2, 0)+(x_2-x_1, y_1) = (2x_2 - x_1, y_1)$, $B_5 = A_5 + (x_3, 0) = (x_5 + x_3, y_5)$. Figure iii shows the A_i and B_i vertices.

Regular pentagons do not tessellate, so we can not say that we can perfectly match a pentagon between the vertices A_4, A_3, and B_4. The best we can do is use vectorial operations to build the pentagon that is sharing the edge $A_3 - B_4$ with the blue pentagon in the right. We call the new pentagon vertices with the letter C and use sub-indices that correspond with the blue constructed pentagon thinking that the segment $A_3 - B_4$ is a mirror. Then The mirror image of A_2 is $C_2 = A_3 + (B_4 - B_5)$,

Now with C_2 we can construct C_1 and C_4 by using the $\alpha = 1$, $\beta = \phi$ coefficients. That is $C_1 = B_4 + \phi(C_2 - A_3)$ That is $C_4 = C_2 + \phi(B_4 - A_3)$

The top layer (face) can be located adjacent to the edge $C_4 - C_4$. To using this segment as a mirror. We label this layer with the letter D. We find the new vertices as we did in the previous case. The vertex D_2 is found as $D_2 = C_1 + (C_4 - B_4)$, and now with two edges we use the coefficients $\alpha = 1$, and $\beta = \phi$ to find the other two vertices. They are $D_3 = C_4 + \phi(D_2 - C_1)$ and $D_4 = D_2 + \phi(C_4 - C_1)$. The construction of one third of the paper cut origami for the dodecahedron construction is shown in Figure iii. The whole paper cut is shown in Figure iii.

2.6 The icosahedron

5 triangles: As already indicated in the main text, the most number of triangles that can be used is 5. We show how to build an icosahedron starting with 5 equilateral triangles. We do this in three parts.

1. **Bottom layer :** We can start with 5 triangles located in a flat surface in a homogeneous distribution all with a common vertex (the origin) and a constant angle gap of $[360 - 5(60)]/5 = 12$ degrees (the deficiency angle at each edge). The idea is to bend them from the flat surface until all edges merge (we will prove that they merge). Now, we have five segments sitting in the same horizontal plane. To each of those we can sit a triangle such that the edges on the horizontal plane coincide with the edges of the new 5 triangles. This provides half of the icosahedron. The other half is totally symmetric to this half with a little twist. On between each of the top (at the moment) triangles there is perfect whole that can exactly be filled with a triangle making the figure into 15 faces figure. On top of those 5 (inverted, that is with the base above) triangles we can sit other 5 triangles which can be tied to the same vertex above them forming the icosahedron.

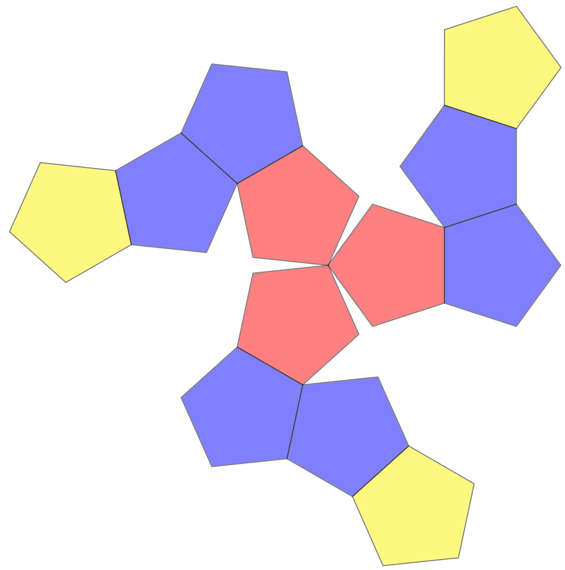

Figure 2.16: The origami paper cut of a dodecahedron in a plane. The base is formed by the three red pentagons. Then on top of them we can build 6 blue pentagons, and finally int the top the three yellow pentagons. Perfect simmetry $3, 6, 3$ from bottom to top. Here 5 planes of vertices with number of vertices $1, 3, 6, 6, 3, 1$.

Let us start with the 5 equilateral triangles sitting in a flat plane as shown in Figure 2.17

Assume that we see the plane view in the $X - Y$ coordinates and the coordinate Z is toward the reader. We want to rotate the triangle bisected by the X axis in the $X - Z$ plane, toward the reader, and likewise with the next triangle intersecting the positive Y axis. On these rotations the points V and V' will trace two circumferences, which intersect at two points. One above the $z = 0$ plane and the other below. Either one could be useful to build the icosahedron but we will choose the intersection above $z = 0$. Figure 2.17 on the right frame shows arcs on the two circumferences. We show that indeed the points V and V' from the flat surface on the left frame coincide in the intersection of the circumference at some point $V_f = (x, y, z)$ which becomes the new vertex for the icosahedron. By symmetry there will be other 4 points just like this at the same horizontal plane spread around a pentagon

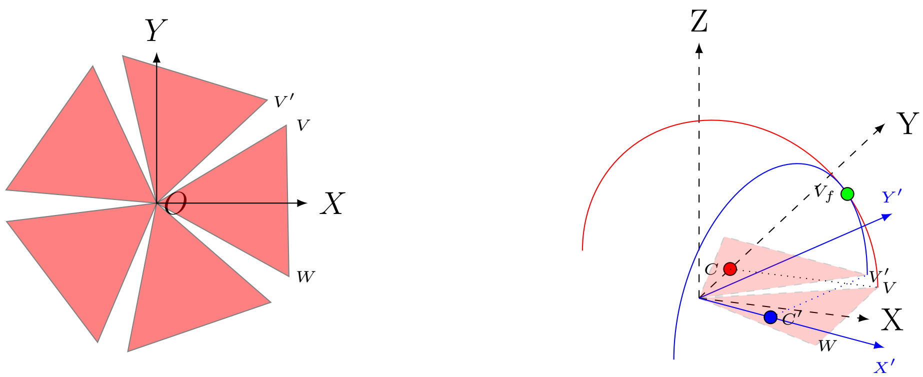

Figure 2.17: The icosahedron construction. The bottom layer.

with side length ℓ. We choose $\ell = 1$ for simplicity.

Let us first find the circumferences of rotation through the vertices V and V'. The argument (phase) of the point V is $\alpha = \pi/6$, and the argument of the point V' is $\beta = \alpha + \pi/15$. [4]

The coordinates of the point V are

$$
\begin{aligned}
V &= (v_1, v_2, v_3) = (\cos\alpha, \sin\alpha, 0) \\
V' &= (v_1', v_2', v_3') = (\cos\beta, \sin\beta, 0)
\end{aligned}
$$

When rotating the face $\triangle OWV$ with respect to the Y axis, the point V will generate a circle around the point $V_0 = (0, v_2, 0)$, with radius $r = v_1 = \cos\pi/6$. Since the circle is wholly contained in the vertical plane $y = v_2$ we can ignore the y coordinates and write

$$x^2 + z^2 = r^2. \tag{2.15}$$

The circle is shown in Figure 2.17 in red where its center (the red point) is labeled as C. The second circumference is a bit harder to find. To make things easier we want to use a different coordinate system. Let us imagine a coordinate system that shares the same z axis, but the

[4] recall that the aperture angle between V and V' is $12°$ which is $\pi/15$.

new $X'Y'$ axes are rotated clockwise such that the new axes Y' bisects the triangle with vertex on V'. The required angle of rotation is -90 plus the angle that we need to rotate X to bisect the second triangle. This last angle is $30 + 12 + 30$. Then the rotation angle is $\varphi = -18°$.

In the figure these new $X'Y'$ axes are in blue. The lifting of the triangle with vertex in V' is done with a rotation in the center $(0, v_2', 0)$, with radius $v_1' = v_1 = \cos\pi/6$, that is $r = \sqrt{3}/2$. The center is also at $(0, v_2', 0) = (0, \sqrt{3}/2, 0)$ in the new coordinate system. In the new coordinate system we have an equation such as that in 2.15. That is

$$(y')^2 + z^2 = r^2. \tag{2.16}$$

Note that we did not change z since the axis of rotation is z. We did not change r either since the radius of rotation is invariant.

We recognize that the system given by equations 2.15 and 2.16 was already solve by the equations 2.7 that we rewrite here.

$$\begin{aligned}
x &= \frac{y\cos\varphi}{1+\sin\varphi} \\
y &= \sqrt{1-r^2} \\
z &= \sqrt{\frac{2r^2+\sin\varphi-1}{1+\sin\varphi}}
\end{aligned} \tag{2.17}$$

We can write this equation in terms of the Golden ratio ϕ by using equations 2.2.

$$\begin{aligned}
x &= \frac{y\cos\varphi}{1+\sin\varphi} = \frac{y\sqrt{4\phi^2-1}/2\phi}{(1-1/2\phi)} = \frac{y\sqrt{4\phi^2-1}}{2\phi-1} \\
y &= \sqrt{1-r^2} \\
z &= \sqrt{\frac{2r^2-(1/2\phi)-1}{1-1/2\phi}} = \sqrt{\frac{4r^2\phi-1-2\phi}{2\phi-1}}
\end{aligned}$$

In terms of radicals we have

$$\begin{aligned} x &= \frac{\sqrt{2\sqrt{5}+5}}{2\sqrt{5}} = \frac{\sqrt{25+10\sqrt{5}}}{10} \\ y &= 1/2 \\ z &= \frac{\sqrt{\sqrt{5}-1}}{\sqrt{2}\,5^{1/4}} = \frac{\sqrt{50-10\sqrt{5}}}{10}. \end{aligned} \tag{2.18}$$

Since we can solve the system composed by equations for the two circles, then we can say that the two vertices V and V' tie well after rotation into the single point $V_f = (x, y, z)$. The rotation angle of the faces can be found from the equation

$$\tan\theta = \frac{z}{x} = \frac{\sqrt{2}\,5^{\frac{1}{4}}\,\sqrt{\sqrt{5}-1}}{\sqrt{2\,\sqrt{5}+5}} = 3 - \sqrt{5}$$

The numeric evaluation of these equations produces:

$$\begin{aligned} x &= 0.6881909602355868 \\ y &= 0.5 \\ z &= 0.5257311121191336 \end{aligned}$$

the lift angle is $\arctan(z/x) \approx 37.3773681406497°$.

On the right frame of Figure 2.18 we see the lifting of the triangle $\triangle OWV$ into a triangle $\triangle OW_fV_f$. We will show that this lifting is exactly the need lifting so that the vertices V and V' originally at different faces will merge into the vertex V_f.

We find the amount of rotation needed to go from the flat surface to the icosahedron shape. Let us assume that the length of each edge is fixed to ℓ. W_f and V_f (in the figure) are part of a pentagon with side length ℓ. We will work in the yellow triangle in Figure 2.17.

The tilt angle of the triangle $\triangle OW_fV_f$ is given by the angle $\theta = \angle OMP$. To compute this angle we compute the length of the segment OM and PM, since

$$\cos\theta = \frac{PM}{OM}.$$

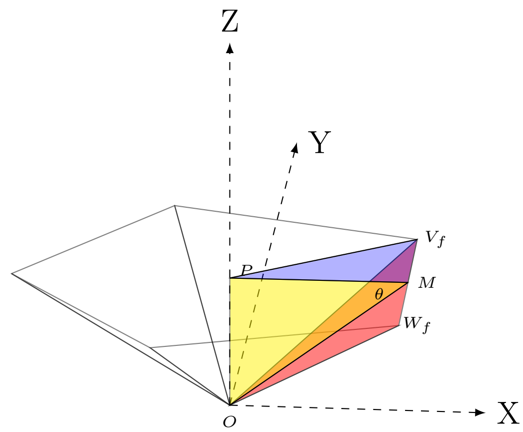

Figure 2.18: The icosahedron construction. An independent way to find the inclination angle.

Now OM is the height of the equilateral triangle $\triangle OW_fV_f$, so it is given by

$$OM = \ell \sin \pi/3 = \frac{\sqrt{3}}{2}\ell.$$

On the other hand $PM = a$ is the apothem of the pentagon with side ℓ. In the blue triangle $\triangle PMV_f$ we find

$$\tan\varphi = \frac{a}{\ell/2},$$

where $\varphi = \angle PV_fM$. For a pentagon each internal angle has the dimension $3\pi/5$, so $\varphi = 3\pi/10$. Then

$$a = \frac{\ell\tan\varphi}{2},$$

and the inclination angle is given by

$$\cos\theta = \frac{\cancel{\ell}\tan\varphi/\cancel{2}}{\sqrt{3}\cancel{\ell/2}} = \frac{\tan\varphi}{\sqrt{3}}$$

From equation 2.12 we know that

$$\tan\varphi = \frac{\sqrt{2}\sqrt{5}+\sqrt{2}}{2\sqrt{5-\sqrt{5}}}.$$

and so

$$\cos\theta = \tan\varphi = \frac{\sqrt{2}\sqrt{5}+\sqrt{2}}{2\sqrt{3}\sqrt{5-\sqrt{5}}} = \sqrt{\frac{5+2\sqrt{5}}{15}}.$$

Then we find

$$\theta \approx 0.6523581397843682$$

or in degrees $\theta = 37.3773681406497°$. This angle matches the angle found above.

Since we found a vertex $V_f = (x, y, z)$ at a height z, the other 4 vertices are at the same *height* and uniformly distributed along a circumference of radius $r = \sqrt{x^2 + y^2}$. Provided an initial angle $\alpha = \arctan(y/x)$, and the increment angle $\gamma = 2\pi/5$, we find the vertices using the equation

$$A_{i+1} = (r\cos(\alpha + i\gamma), r\sin(\alpha + i\gamma), z) \quad , \quad i = 0, \cdots, 4 \qquad (2.19)$$

Figure 2.18 shows the five vertices, and five faces (triangles) which form the base of the icosahedron. We are ready to start building the middle part of the icosahedron.

2. **Middle layer :** In this analytical tessellation of the icosahedron, we use a similar technique to find the points for the vertices of the middle layer of the icosahedron. These layer has 10 vertices. We find one vertex and with that we will be ready to find the other 9 vertices for this layer.

   The 10 vertices come from the following construction. On top of each triangular face we position another triangular face sharing a common edge. For example the edge $V_f - W_f$ in Figure 2.18. We find the upper vertex of the triangular face (the bottom vertices are already known). This will provide 5 vertices. The other 5 vertices come from triangles which will be inserted in the voids between the triangles on the second layer.

   To find a vertex, we use again the reference triangle, $\triangle OW_fV_F$ shown above in Figure 2.17. We already know that the vertex will align with the vertex at the origin O in the same azimuthal plane. That is, the second component is $y = 0$. We still need to find the other two components x, z.

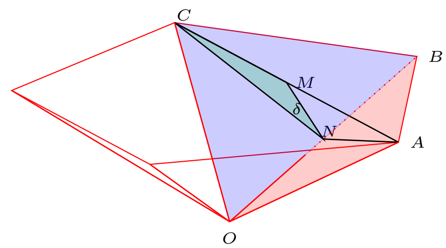

Figure 2.19: The icosahedron construction. Finding the dihedral angle.

One alternative is to find the dihedral angle (the angle between two faces) since we need to shoot from the bottom triangle to the next triangle on top of it with the right direction.

We use elementary trigonometry to find the dihedral angle as we did for the lift angle. This time we use the Figure 2.19

When using the notation AB we refer to the segment that connects point A to B or to the length of that segment. The context will indicate to which AB we are referring to.

In the figure we have the bottom part of the icosahedron. We want to focus in the dihedral angle between the faces $\triangle OAB$ (red) and $\triangle OBC$ (blue). Half of that dihedral angle $\angle CNA$, is given by the angle δ ($\angle CNM$) in the figure, where AC is the diagonal of the pentagon that joints the vertices A and C, and M is the middle point over the diagonal. We drop a perpendicular line to the edge OB at the point N. Our problem is now in the green triangle $\triangle CNM$. This is a right triangle with the $90°$ angle at the vertex M, since M is in the middle of the line AC and $NA = NC$, so the segment NM is a height of $\triangle CNA$.

We want to find δ such that

$$\sin \delta = \frac{CM}{CN}.$$

The length CM is half of the diagonal of a pentagon. We already know that the diagonal of a pentagon is $\phi\ell$, where ϕ is the golden ratio. Now, the length CN is the altitude of an equilateral triangle that is

$CN = \ell\sqrt{3}/2$. So

$$\sin\delta = \frac{\phi\ell/2}{\sqrt{3}\ell/2} = \frac{\phi}{\sqrt{3}}.$$

The numerical computation is

$$\delta \approx 1.205932498681414 \text{ rad} \approx 69.09484255211072°$$

and since the dihedral angle is 2δ we found that the dihedral angle of the icosahedron is approximately 138.1896851042214°.

Given that we chose two arbitrary (adjacent) faces of the bottom layer, we can assure that the dihedral angle is the same between the other faces since they present the same geometry (sides and angles relations).

We are ready to shoot up to the next layer of vertices. Here we need to find one vertex for the top of each triangle sitting on the pentagon in Figure 2.18. The next vertex is located then in the plane of the yellow triangle $\triangle OPM$. That is, we know that the y coordinate of this vertex is 0. In the $X-Z$ plane we have a segment starting at the middle point M and ending up some point Q above, making an angle $\pi-(2\delta-\theta)$ with the horizontal plane. This means that the height increment is $\Delta z = \ell(\sqrt{3}/2)\sin(2\delta-\theta))$, the Δx is $-\ell(\sqrt{3}/2)\cos(2\delta-\theta))$, where the $\sqrt{3}/2\ell$ is the length of the altitude of the equilateral triangle of side ℓ. We have

$$\Delta z = \ell(\sqrt{3}/2)(\sin 2\delta\cos\theta - \sin\theta\cos 2\delta),$$

with

$$\begin{aligned}
\sin 2\delta &= 2\sin\delta\cos\delta = \frac{2\phi}{\sqrt{3}}\sqrt{1-\phi^2/3} \\
\cos 2\delta &= \cos^2\delta - \sin^2\delta = 1 - 2\sin^2\delta = 1 - 2\phi^2/3 \\
\cos\theta &= \tan\varphi \\
\sin\theta &= \sqrt{1-\tan^2\varphi}.
\end{aligned}$$

We write exact expressions in terms of radicals and fractions and simplify to find

$$\begin{aligned}
\Delta z &= \ell\frac{\cancel{\sqrt{3}}}{2}\,\frac{\sqrt{3-\sqrt{5}}(\sqrt{5}+1)}{\cancel{\sqrt{3}}\sqrt{5-\sqrt{5}}} = \ell\frac{\sqrt{3-\sqrt{5}}(\sqrt{5}+1)}{2\sqrt{5-\sqrt{5}}} \\
\Delta x &= \ell\frac{\cancel{\sqrt{3}}}{2}\,\frac{\sqrt{11\sqrt{5}-25}}{\cancel{\sqrt{3}}\sqrt{5}\sqrt{3-\sqrt{5}}} = \ell\frac{\sqrt{11\sqrt{5}-25}}{2\sqrt{5}\sqrt{\sqrt{5}-3}}
\end{aligned}$$

We can also simplify these equations as That is, (for $\ell = 1$)

$$\begin{aligned}
\Delta z &= \frac{1}{\sqrt{10}}\sqrt{5+\sqrt{5}} \\
\Delta x &= \frac{1}{2}\sqrt{1-\frac{2}{\sqrt{5}}}.
\end{aligned} \tag{2.20}$$

The numerical evaluation produces

$$\begin{aligned}
\Delta z &\approx 0.85065080835204 \\
\Delta x &\approx 0.1624598481164529
\end{aligned}$$

So from $M = (m_x, m_y, z) = (x, 0, z)$ this first vertex on the second layer is

$$\begin{aligned}
B_1 &= (m_x + \Delta x, m_y, z + \Delta z) \\
&= (x + \Delta x, 0, z + \Delta z)).
\end{aligned}$$

The numerical coordinates for B_1 are:

$$\begin{aligned}
x &\approx 0.8506508083520397 \\
y &= 0 \\
z &\approx 1.376381920471174
\end{aligned}$$

With this point we complete the other 4 tips of the triangles using the formula

$$B_{i+1} = r(\cos\alpha + i\gamma), \sin(\alpha + i\gamma), z) \quad , \quad i = 0, \cdots, 4$$

with $\alpha = \arctan(y/x) = 0$, and $\gamma = 2\pi/5$. These new 5 B_i vertices together with the A_i vertices tessellate 10 triangular faces.

Figure 2.20 on the left shows the first two 15 bottom layers constructed so far.

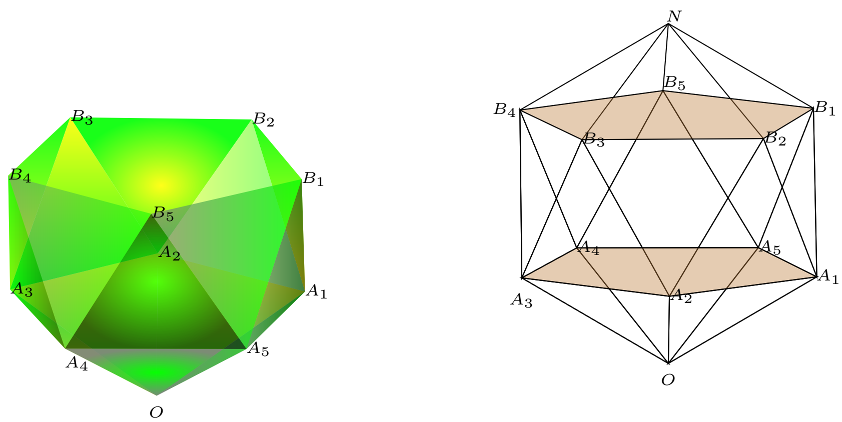

Figure 2.20: First 15 bottom faces of the icosahedron. The left frame shows the volume from the front direction in a semitransparent coloring to observe all the faces so far constructed. The right frame shows the skeleton seen from back direction, with two horizontal congruent pentagons painted in brown, and the last vertex on the top N added with its corresponding faces.

3. **Top layer :**

   The pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$, painted brown on the right frame of Figure **??** are congruent. They are regular pentagons with equal side length $\ell = 1$. However each is a rotated by $2\pi/10 = \pi/5$ version of the other. By pure symmetry corresponding to the point O below the first pentagon, there is a point N above the second pentagon such that the connection from each B_i to N creates the last 5 faces of the icosahedron. That point N has coordinates $x = 0, y = 0$. We need to find the z coordinate such that all triangles attached to this $(0, 0, z)$ coordinate are equilateral and the dihedral angles are all equal.

   We do not need to solve new equations to find z. Since the pentagon with the B_i vertices is congruent to the pentagon with the A_i vertices and both are parallel to the $X - Y$ plane (by construction), the added

Δz to the vertices B_i to reach N is the same initial z for the vertices A_i. That is the new z for the top vertex N is:

$$z_{total} = 2z + \Delta z,$$

where z is the value in equation 2.18 and Δz is found in equation 2.20. That is

$$z_{total} = \frac{\sqrt{50 - 10\sqrt{5}}}{5} + \frac{1}{\sqrt{10}}\sqrt{5 + \sqrt{5}} = \sqrt{\frac{5 + \sqrt{5}}{2}}. \tag{2.21}$$

Please observe that z_{total} in equation 2.21 is the diameter of the icosahedron. This diameter can be easily computed by observing the right frame of Figure 2.20. The diameter is the distance between B_4 and A_1 (due to the symmetry this is the same as the distance between O and N, or B_1 and A_3. Under each of the six vertices of the outside hexagon is sitting a pentagonal pyramid with height $\sqrt{50 - 10\sqrt{5}}/5$ found previously). The triangle formed by B_4, B_1, A_1 is a right triangle with the 90° angle at B_1. From Pythagoras theorem

$$(A_1B_4)^2 = (B_1B_4)^2 + (A_1B_1)^2.$$

Now, A_1B_4 is a diagonal of the pentagon given by $\phi\ell$, and A_1B_1 is a side of the triangle given by ℓ then the diameter of the icosahedron d is given by

$$d = \ell\sqrt{\phi^2 + 1} = \sqrt{\frac{5 + \sqrt{5}}{2}}$$

which agrees with the diameter found in equation 2.21. Then we showed that the last vertex found "clicks" perfectly and the construction of the icosahedron is finished.

Figure 2.21 show all the steps indicated to construct the icosahedron.

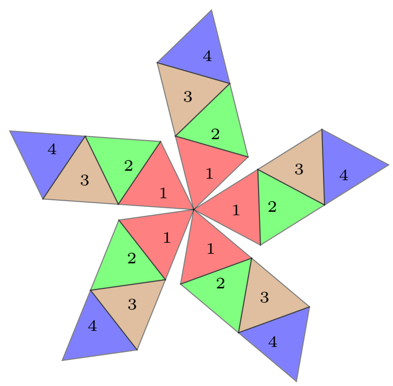

Figure 2.21: The construction of an icosahedron. First step marked as “1” with the red inner ring of faces. The 5 gaps of 12 degrees make an angle deficiency of 60 degrees at each vertex.

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