The Variable Density Acoustic Wave Equation is not Self–adjoint

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Abstract

Accoerding to Bleistein et. al., [3] the acoustic wave equation for variable density is not self–adjoint. We verify that. For an on–line link to Bleistein et. al., statement click here.

1 The problem

Show that the variable density Helmholtz operator

$$\rho(\boldsymbol{x})\nabla \cdot \left[\frac{1}{\rho(\boldsymbol{x})}\nabla g(\boldsymbol{x}, \boldsymbol{x}_2, \omega)\right] + \frac{\omega^2}{c^2(\boldsymbol{x})} g(\boldsymbol{x}, \boldsymbol{x}_s, \omega)$$

is not self–adjoint.

2 The Proof

The part of the operator

$$\frac{\omega^2}{c^2(\boldsymbol{x})} g(\boldsymbol{x}, \boldsymbol{x}_s, \omega)$$

is not interesting since it is a scalar. That is the adjoint of this part is

$$\frac{\omega^2}{c^2(\boldsymbol{x})} g^*(\boldsymbol{x}, \boldsymbol{x}_s, \omega)$$

Table 1: Some common operators and its adjoint pairs

Gradient	$a(\boldsymbol{x})\nabla u(\boldsymbol{x})$	$-\nabla\cdot[a(\boldsymbol{x})\boldsymbol{G}(\boldsymbol{x})]$
Divergence	$a(\boldsymbol{x})\nabla\cdot\boldsymbol{U}(\boldsymbol{x})$	$-\nabla[a(\boldsymbol{x})g(\boldsymbol{x})]$
Curl	$a(\boldsymbol{x})\,\nabla\times\boldsymbol{U}(\boldsymbol{x})$	$\nabla\times[a(\boldsymbol{x})\boldsymbol{G}(\boldsymbol{x})]$
Laplacian	$a(\boldsymbol{x})\nabla^2 u(\boldsymbol{x})$	$\nabla^2[a(\boldsymbol{x})g(\boldsymbol{x})]$
Mixed	$\nabla\cdot a(\boldsymbol{x})\nabla u(\boldsymbol{x})$	$\nabla\cdot a(\boldsymbol{x})\nabla g(\boldsymbol{x})$

but g (the Green's function) is self–adjoint. We will show that the opeartor

$$\boldsymbol{L}(\boldsymbol{x}) = \rho(\boldsymbol{x})\,\nabla\cdot\left(\frac{1}{\rho(\boldsymbol{x})}\right)\nabla \tag{2.1}$$

is not self–adjoint. Pick a test function $\phi = \phi(\boldsymbol{x})$ Then

$$\begin{aligned}\langle\, L[g(\boldsymbol{x})]\ ,\ \phi(\boldsymbol{x})\,\rangle &= \int_V \left[\rho(\boldsymbol{x})\,\nabla\cdot\left(\frac{1}{\rho(\boldsymbol{x})}\right)\nabla g(\boldsymbol{x})\right]\ \phi(\boldsymbol{x})dV \\ &= \int_V \left[\nabla\cdot\left(\frac{1}{\rho(\boldsymbol{x})}\right)\nabla g(\boldsymbol{x})\right]\ \rho(\boldsymbol{x})\phi(\boldsymbol{x})dV\end{aligned}$$

Now, from the last raw of table 1 we observe that the operator

$$M = \nabla\cdot\left(\frac{1}{\rho(\boldsymbol{x})}\right)\nabla g(\boldsymbol{x})$$

is self–adjoint. Hence

$$\langle\, L[g(\boldsymbol{x})]\ ,\ \phi(\boldsymbol{x})\,\rangle = \int_V g(\boldsymbol{x})\left[\nabla\cdot\left(\frac{1}{\rho(\boldsymbol{x})}\right)\nabla\ \rho(\boldsymbol{x})\phi(\boldsymbol{x})\right]dV$$

So the adjoint of L is

$$L^* = \nabla\cdot\left(\frac{1}{\rho(\boldsymbol{x})}\right)\nabla\ \rho(\boldsymbol{x})$$

and indeed the adjoint of the acoustic wave equation can be writen as shown in Bleistein et. al:

$$\left[L^* + \frac{\omega^2}{c^2(\boldsymbol{x})}\right] g^*(\boldsymbol{x}, \boldsymbol{x}_s, \omega) = \nabla \cdot \left[\frac{1}{\rho(\boldsymbol{x})}\nabla\rho(\boldsymbol{x})\ g^*(\boldsymbol{x}, \boldsymbol{x}_2, \omega)\right] + \frac{\omega^2 g^*(\boldsymbol{x}, \boldsymbol{x}_s, \omega)}{c^2(\boldsymbol{x})}$$

3 Discussion

Self–adjointness is a desired operation because for example, a code to model forward propagation on given operator could be recycled to simulate backward propagation without changing its core design. Phil Anno, [2] shows how to avoid the non self–adjoint issue on the variable density acoustic wave equation by relating the acoustic variable density wave equation with the electro–magnetic Klein–Gordon equation which is self–adjoint, after a simple change of a dependent variable. Another way to avoid the non self-adjoiness is to use, for example Tarantola's equation which inverts for variable density and compressibility.

Tarantola's [1] example on the acoustic wave equation is

$$Lu = \left[\frac{1}{K(\boldsymbol{x})}\frac{\partial^2}{\partial t^2} - \nabla.\left(\frac{1}{\rho(\boldsymbol{x})}\nabla\right)\right] u(\boldsymbol{x}, t) = s(\boldsymbol{x}, t). \tag{3.2}$$

From the last raw of table 1 this acoustic wave equation self–adjoint.

That equation is self–adjoint (from the last row of table 1).

References

[1] Tarantola A. Inversion of seismic reflection data in the acoustic approximation. *Geophysics*, 49:1259–1266, 1984.

[2] P. D. Anno. *A Klein–Gordon Acoustic Theory.* Ph.D. thesis, Colorado School of Mines, 1992.

[3] N. Bleistein, J.K. Cohen, and J. Stockwell. *Mathematics of multidimensional seismic Migration, Imaging and Inversion.* Springer–Verlag, 2000.