The Variable Density Acoustic Wave Equation is not Self–adjoint

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Abstract
According to Bleistein et. al., [3] the acoustic wave equation for variable density is not self–adjoint. We verify that. For an on–line link to Bleistein et. al., statement click [here].

1 The problem
Show that the variable density Helmholtz operator
\[
\rho(x) \nabla \cdot \left[ \frac{1}{\rho(x)} \nabla g(x, x_2, \omega) \right] + \frac{\omega^2}{c^2(x)} g(x, x_3, \omega)
\]
is not self–adjoint.

2 The Proof
The part of the operator
\[
\frac{\omega^2}{c^2(x)} g(x, x_3, \omega)
\]
is not interesting since it is a scalar. That is the adjoint of this part is
\[
\frac{\omega^2}{c^2(x)} g^*(x, x_3, \omega)
\]
Table 1: Some common operators and its adjoint pairs

<table>
<thead>
<tr>
<th>Operator</th>
<th>Expression 1</th>
<th>Expression 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gradient</td>
<td>$a(x) \nabla u(x)$</td>
<td>$-\nabla \cdot [a(x)G(x)]$</td>
</tr>
<tr>
<td>Divergence</td>
<td>$a(x) \nabla \cdot U(x)$</td>
<td>$-\nabla[a(x)g(x)]$</td>
</tr>
<tr>
<td>Curl</td>
<td>$a(x) \nabla \times U(x)$</td>
<td>$\nabla \times [a(x)G(x)]$</td>
</tr>
<tr>
<td>Laplacian</td>
<td>$a(x) \nabla^2 u(x)$</td>
<td>$\nabla^2[a(x)g(x)]$</td>
</tr>
<tr>
<td>Mixed</td>
<td>$\nabla \cdot a(x) \nabla u(x)$</td>
<td>$\nabla \cdot a(x) \nabla g(x)$</td>
</tr>
</tbody>
</table>

but $g$ (the Green’s function) is self–adjoint. We will show that the operator

$$L(x) = \rho(x) \nabla \cdot \left( \frac{1}{\rho(x)} \right) \nabla$$

(2.1)
is not self–adjoint. Pick a test function $\phi = \phi(x)$ Then

$$\langle L[g(x)] , \phi(x) \rangle = \int_V \left[ \rho(x) \nabla \cdot \left( \frac{1}{\rho(x)} \right) \nabla g(x) \right] \phi(x) dV$$

$$= \int_V \left[ \nabla \cdot \left( \frac{1}{\rho(x)} \right) \nabla g(x) \right] \rho(x)\phi(x) dV$$

Now, from the last raw of table 1 we observe that the operator

$$M = \nabla \cdot \left( \frac{1}{\rho(x)} \right) \nabla g(x)$$
is self–adjoint. Hence

$$\langle L[g(x)] , \phi(x) \rangle = \int_V g(x) \left[ \nabla \cdot \left( \frac{1}{\rho(x)} \right) \nabla \rho(x)\phi(x) \right] dV$$

So the adjoint of $L$ is

$$L^* = \nabla \cdot \left( \frac{1}{\rho(x)} \right) \nabla \rho(x)$$
and indeed the adjoint of the acoustic wave equation can be written as shown in Bleistein et. al:

\[
L^* + \frac{\omega^2}{c^2(x)} g^*(x, x_s, \omega) = \nabla \cdot \left[ \frac{1}{\rho(x)} \nabla \rho(x) g^*(x, x_2, \omega) \right] + \frac{\omega^2 g^*(x, x_s, \omega)}{c^2(x)}
\]

3 Discussion

Self-adjointness is a desired operation because for example, a code to model forward propagation on a given operator could be recycled to simulate backward propagation without changing its core design. Phil Anno, [2] shows how to avoid the non self-adjoint issue on the variable density acoustic wave equation by relating the acoustic variable density wave equation with the electro–magnetic Klein–Gordon equation which is self-adjoint, after a simple change of a dependent variable. Another way to avoid the non self-adjointness is to use, for example Tarantola’s equation which inverts for variable density and compressibility.

Tarantola’s example on the acoustic wave equation is

\[
Lu = \left[ \frac{1}{K(x)} \frac{\partial^2}{\partial t^2} - \nabla \cdot \left( \frac{1}{\rho(x)} \nabla \right) \right] u(x, t) = s(x, t).
\] (3.2)

From the last row of table this acoustic wave equation self-adjoint.

That equation is self-adjoint (from the last row of table).

References

