Appendix of Tarantola, 1984 Paper

Herman Jaramillo

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Abstract

Tarantola [1] developed an algorithm to invert for acoustic parameters compressibility $K$, density $\rho$ and source signature $S$ from the wave equation

$$Lp = \left[ \frac{1}{K(x)} \frac{\partial^2}{\partial t^2} - \nabla \left( \frac{1}{\rho(x)} \nabla \right) \right] p(x, t) = S(x, t)$$

I follow here Tarantola’s appendix for cross-checking with my own derivations.

A Appendix

In the appendix, Tarantola examines the derivatives of the pressure field and show the action of their transposes in an arbitrary vector.

The equation numbers here should match those of Tarantola’s appendix. He starts with modeling solution (pressure field)

$$p = f(K, \rho, S),$$

where the scalar functions $K$, $\rho$ and $S$ are the compressibility, density and source signature respectively. He introduces the derivative through a perturbation theory argument. That is,

$$\delta p = U\delta K + V\delta \rho + T\delta S + O(\delta K, \delta \rho, \delta S)^2$$  \hspace{1cm} (A.1)

with

$$\delta p = f(K + \delta K, \rho + \delta \rho, S + \delta S) - f(K, \rho, S).$$
and \( U, V \) and \( T \) are the partial derivatives

\[
U = \frac{\partial p}{\partial K} \quad V = \frac{\partial p}{\partial \rho} \quad T = \frac{\partial p}{\partial S}. \tag{A.3}
\]

In the 10–th dimensional space \((x_s, x_g, x, t)\) Tarantola identify the material properties, such as for example density \( \rho = \rho(x) \) and the wavefields such as for example \( p = p(x_g, t; x_s|x) \) (I use \( x \), Tarantola uses \( r \)) to emphasize that the wavefield depends on the source \( x_s \) location, the receiver location \( x_g \), time \( t \) and the inside scatter points \( x \).

In equation (A.1) we see the products as matrix products where for example \( U \) is a row vector and \( \delta K \) a column vector. In general we can write this as inner products with the integral symbol as

\[
\delta p(x_g, t; x_s) = \int U(x_g, t; x_s|x) \delta K(x) dx
+ \int V(x_g, t; x_s|x) \delta \rho(x) dx
+ \int W(x_g, t; x_s|x) \delta S(x, t) dx
+ \mathcal{O}(\delta K, \delta \rho, \delta S)^2. \tag{A.2}
\]

Tarantola’s next step is to compute the differential on the wave equation operator. That is, he computes

\[
\delta[(Lp) - S] = L\delta p + (\delta L)p - \delta S \equiv 0.
\]

These symbols that work for functions of real variable should work as operators in more general spaces as Tarantola shows. He writes

\[
\delta(Lp) = (L + \delta L)(p + \delta p) - Lp = L(\delta p) + (\delta L)p + L\delta \rho + \delta L\delta \rho - L\delta \rho
\]

Or writing differently \( L_{u+\delta u}(p + \delta p) - L_u(p) \) where \( u = (K, \rho) \) is the vector of parameters. That is

\[
L_u(p) = \left[ \frac{1}{K(x)} \frac{\partial^2}{\partial t^2} - \nabla \cdot \left( \frac{1}{\rho(x)} \nabla \right) \right] p(x, t)
\]

and

\[
L_{u+\delta u}(p + \delta p) = \left[ \frac{1}{(K + \delta K)(x)} \frac{\partial^2}{\partial t^2} - \nabla \cdot \left( \frac{1}{(\rho + \delta \rho)(x)} \nabla \right) \right] (p + \delta p)(x, t)
\]
thinking along the way of

\[
\lim_{\delta u \to 0} \frac{L(p)(u + \delta u) - L(p)(u)}{\delta u}
\]

which simulates the derivative of the operator \(L\), we can write, by taking this derivative as if \(K\), \(\rho\) and \(S\) would be real variables (they are not, they are scalar functions)

\[
\begin{align*}
\frac{\partial L}{\partial K} &= -\frac{1}{K^2} \frac{\partial^2}{\partial t^2} \\
\frac{\partial L}{\partial \rho} &= -\nabla \cdot \left( \frac{1}{\rho^2(x)} \nabla \right)
\end{align*}
\]

Tarantola, thinking that it is not right to assume that we can do this is doing the job by “brute force” (I think he is doing the right thing).

This is how we take the derivative of \(g(h) = 1/h\) in calculus

\[
\lim_{\delta h \to 0} \frac{1/(h + \delta h) - 1/h}{\delta h} = \lim_{\delta h \to 0} \frac{h - h - \delta h}{h\delta h(h + \delta h)} = \lim_{\delta h \to 0} \frac{-1}{h(h + \delta h)} = -\frac{1}{h^2}
\]

Tarantola instead writes that

\[
\frac{1}{(h + \delta h)} - \frac{1}{h} = \frac{K - K - \delta h}{h(h + \delta h)} = -\frac{\delta h}{h^2} \frac{1}{1 + \delta h/h} = -\frac{\delta h}{h^2} \left( 1 - \frac{\delta h}{h} + \mathcal{O}(\delta h^2) \right).
\]

Or

\[
\frac{1}{(h + \delta h)} = \frac{\delta h}{h^2} + \mathcal{O}(\delta h^2)
\]

We can then call the differential of \(L\) as

\[
\delta L = -\frac{\delta K}{K^2} \frac{\partial^2}{\partial t^2} - \nabla \cdot \left[ \frac{\delta \rho}{\rho^2} \nabla \right]
\]

He applies this for \(h\) being equal to \(K\) and then to \(\rho\). Then after taking the differential on the operator \(Lp = S\), and moving to the right, all the terms that do not match \(L\) he finds (in symbols) first

\[
L_{u+\delta u}(p + \delta p) - L_u(p) - \delta S = L(\delta p) + (\delta L)p - \delta S = 0,
\]
and from here

\[ L(\delta p) = dS - (\delta L)p = \Delta s \]

where \( \Delta s \) acts here as a source term for the new wave equation, with solution \( \delta p \).

and explicitly

\[
\left[ \frac{1}{K(x)} \frac{\partial^2}{\partial t^2} - \nabla \cdot \left( \frac{1}{\rho(x)} \nabla \right) \right] \delta p(x, t) = \delta S + \delta K \frac{\partial^2 p}{K^2 \partial t^2} - \nabla \cdot \left( \frac{\delta \rho}{\rho^2} \nabla p \right) + \mathcal{O}(\delta K, \delta \rho, \delta S)^2 \quad (A.4)
\]

The next step in Tarantola’s appendix is to apply the Green’s function theorem, and reciprocity. That is,

\[
\delta p(x_g, t; r_s) = \int dx \ g(x, t; x_g, 0) * \Delta s(x, t; r_s).
\]

We see the new source term \( \Delta s \) as expanded in three base vectors \( \delta S \), \( \delta K \), and \( \delta \rho \). That is let us rewrite

\[
\Delta s(x_g, t; x_s) = 1(\delta S) + \left( \frac{1}{K^2} \frac{\partial^2 p}{\partial t^2} \right) \delta K - \left[ \nabla \cdot \left( \frac{1}{\rho^2} \nabla p \right) \right] \delta \rho + \mathcal{O}(\delta K, \delta \rho, \delta S)^2
\]

with

\[
S(x_s, x, t) = \delta(x - x_s)S(t),
\]

abusing the notation of \( S \) in both sides for different meaning. Since the base vectors are considered linearly independent, we can match coefficients and find three corresponding equations, from using equation [A.1] and this Green’s function representation.

\[
[U \delta K](x_g, t; x_s) = \int d\mathbf{x} \ g(\mathbf{x}, t; \mathbf{x}_g, 0) * \left[ \frac{\delta K(\mathbf{x})}{K^2(\mathbf{x})} \frac{\partial^2 p}{\partial t^2}(\mathbf{x}, t, x_s) \right] \quad (A.5a)
\]

\[
[V \delta \rho](x_g, t; x_s) = \int d\mathbf{x} \ g(\mathbf{x}, t; \mathbf{x}_g, 0) * \left\{ -\nabla \cdot \left[ \frac{\delta \rho(\mathbf{x})}{\rho^2(\mathbf{x})} \nabla p(\mathbf{x}, t; x_s) \right] \right\} \quad (A.5b)
\]

and

\[
[T \delta S](x_g, t; x_s) = g(\mathbf{r}_g, t; \mathbf{x}_s, 0) * \delta S(t). \quad (A.5c)
\]
B Conclusions

References