Abstract

Here the abstract

1 Introduction

These notes try to explain, using the method of stationary phase, why the Huygens Principle makes sense. I start with the Fresnel–Kirchoff integral representation of the principle (equation 4) in [these notes][1]. Then, using the method of stationary phase, prove that the energy in a point in the wavefront can be seen as a summation of contributions originated from sources on a previous wavefront. This is reverse order. I started with a sophisticated mathematical integral and find a principle (from which such an integral was derived). It is a bit silly, but help me on the understanding. It verifies the mathematical model.

2 The principle in words

In words it says that each point of a wavefront, is a center of a new disturbance; the advancing wave as a whole may be regarded as the sum of all the secondary waves with sources in a previous wavefront.

This might sound confusing, but if you are reading this is because you already have an idea about what is the Huygens principle.

3 The Integral Formulation

As I said, I take the integral formulation as equation (4) from http://www.dtic.mil/cgi-bin/GetTRDoc?AD=ada429355. There are many integral formulations evolving from Huygens and going through Kirchhoff, Fresnel, Rayleigh and Sommerfeld among many others. The History and derivation of those equations is found in the classical book of Goodman[2].


This equation is

\[ U_{FK}(x_{Q}, y_{Q}, z_{2}) = -\frac{i}{\lambda} \int_{Area} U_0(x, y, 0) \frac{\exp(ikr)}{r} \frac{1}{2} (\cos \xi + \cos \chi) \, dxdy. \] (3.1)

I explain the symbols in this equation. Starting with Figure 1. The figure is a cross-section of the actual experiment. The y axis points out of the visual plane. The z axis is along the horizontal and the x axis is along the vertical. We assume a point source \( P \) and a wavefront hitting a barrier, with a small aperture. The purpose of this barrier is to isolate a little piece of wavefront. The area \( Area \) is the piece of spherical surface (we assume constant velocity for simplicity) drawn by the aperture in the \( xy \) plane. The integral has as support the \( Area \) surface. The point \( Q \) is the output point. That is \( Q = (x', y', z = z_{2}) \). We show the rays from the source \( P \) to the aperture part of the wavefront, and the rays from the aperture to the output point \( Q \). The point \( O \) is to reference one typical ray and the associated radius \( r = OQ \), the distance between the \( O \) and \( P \) points, and the two obliquity angles \( \xi \) and \( \chi \). The blue arrow is drawn only for illustrating the angles. We picked the last ray to avoid clustering of symbols, but the illustration should be understood for any ray on the package beam. The distance \( r_P = P0 \) is not shown in the formula explicitly but is part of it. We assume that the wavefront is “flat” due to the large value of the radius \( r_P \), so that \( z_{1} \) is constant, this is a high frequency approximation. A curve front will require a change of coordinates (from cartesian to polar) in our integral equation I am lazy to go in that direction. Then, we
can think of that wavefront as an approaching plane wave. $\lambda$ and $k$ are related by the formula

$$\lambda = \frac{2\pi}{k}.$$ 

Finally, the field $U_0(x, y, 0)$ represents the energy along the wavefront through the point $(x, y)$. For a perfect impulsive source (a Dirac delta) this is given by

$$U_0(x, y, 0) = \exp(i k r_P) \frac{1}{r_P}$$

Hence we can write equation 3.2 as

$$U_{FK}(x_Q, y_Q, z_2) = -\frac{i}{\lambda} \int_{\text{Area}} \exp[i k (r + r_P)] 1 \frac{1}{2} (\cos \xi + \cos \chi) dxdy.$$  (3.2)

We want to evaluate this formula using the method of stationary phase. The formula for stationary phase is as follows.

Given

$$I(k) = \int_S A(x, y) \exp[i k \phi(x, y)] dxdy,$$

Where $k$ is large (I will not explain what large means. Bleistein claims that $K$ s large if $\lambda = 2\pi/k > 3L$, where $L$ is the “natural” scale of the problem...).

Then $I(k)$ can be approximated using the stationary phase formula (1)

$$I(k) \approx \frac{2\pi}{k \sqrt{|H(x_0, y_0)|}} A(x_0, y_0) \exp[i k \phi(x_0, y_0) + i \text{sgn}H\pi/4].$$  (3.3)

with

$$H(x_0, y_0) = \det \phi_{xy} = \phi_{xx} \phi_{yy} - \phi_{xy}^2.$$  

is the Hessian for the matrix of second partial derivatives of the form

$$\phi_{xy} = \frac{\partial^2 \phi}{\partial x \partial y},$$

$$\phi_{xx} = \frac{\partial^2 \phi}{\partial x^2},$$

$$\phi_{yy} = \frac{\partial^2 \phi}{\partial y^2}.$$

All of this expressions are evaluated at the point $(x_0, y_0)$ such tht

$$\nabla \phi(x, y) = 0.$$ 

If such a point does not exist, then the formula can be evaluated by integration by parts.

We evaluate the formula in pieces.
• **Find the stationary phase points.** The phase function is given by

\[ \phi(x, y) = r_P + r. \]

Now \( P = (x_P, y_P, 0) \) is the source location so

\[ r_P = \sqrt{(x - x_P)^2 + (y - y_P)^2 + z_1^2}, \quad (3.4) \]

similarly

\[ r = \sqrt{(x - x_Q)^2 + (y - y_Q)^2 + (z_2 - z_1)^2}, \quad (3.5) \]

with \( Q = (x_Q, y_Q, z_2) \).

The partial derivatives of the distances are:

\[
\begin{align*}
\frac{\partial r}{\partial x} &= \frac{x - x_Q}{r} - \frac{x - x_P}{r_P} = \frac{r_P(x - x_Q) + r(x - x_P)}{r r_P} \\
\frac{\partial r}{\partial y} &= \frac{y - y_Q}{r} - \frac{y - y_P}{r_P} = \frac{r_P(y - y_Q) + r(y - y_P)}{r r_P} \\
\end{align*}
\]

so

\[
\begin{align*}
\phi_x &= \frac{\partial \phi}{\partial x} = \frac{x - x_Q}{r} + \frac{x - x_P}{r_P} = \frac{r_P(x - x_Q) + r(x - x_P)}{r r_P} \\
\phi_y &= \frac{\partial \phi}{\partial y} = \frac{y - y_Q}{r} + \frac{y - y_P}{r_P} = \frac{r_P(y - y_Q) + r(y - y_P)}{r r_P} \\
\end{align*}
\]

and from

\[ \nabla \phi(x, y) = 0 \]

We find:

\[ r_P(x - x_Q) + r(x - x_P) = r_P(y - y_Q) + r(y - y_P) = 0 \]

That is

\[ \frac{x - x_Q}{x - x_P} = \frac{y - y_Q}{y - y_P} = -\frac{r}{r_P} \quad (3.7) \]

Now, from (3.7) into (3.4)

\[ r_P = \sqrt{(x - x_Q)^2 \frac{r_P^2}{r^2} + (y - y_Q)^2 \frac{r_P^2}{r^2} + z_1^2} \]
so
\[
r = \sqrt{(x - x_Q)^2 + (y - y_Q)^2 + z_1^2 \frac{r^2}{r_P^2}}
\]
and from 3.5 we find that
\[
z_1^2 \frac{r^2}{r_P^2} = (z_1 - z_1)^2.
\]
That is
\[
\frac{r}{r_P} = \pm \frac{z_2 - z_1}{z_1}.
\]
We pick the “-” sign (see figure) and
\[
\frac{z_1 - z_2}{z_1} = \frac{r}{r_P}.
\]
Then from this equation and equation 3.7 we find
\[
\frac{y - y_Q}{x - x_Q} = \frac{y - y_P}{x - x_P} = \frac{z_1 - z_2}{z_1} = \frac{r}{r_P}.
\]
The direction of the two rays is the same. The only ray pair, with that direction is the red pair drawn in red in figure 1. That is a straight path between $P$ and $Q$. This is in fact a restatement of Fermat’s principle, given that the velocity is constant in both medios (before and after the barrier).

The specular ray is the direct ray. Other rays have lower order contributions. If there is an \((x, y)\) in the aperture such that this happens, that is the stationary phase point. Otherwise all rays bend and this is the classical case of diffraction as “going around a corner”. In that case since there is no \((x, y)\) such that \(\nabla \phi(x, y) = 0\), the integral can be evaluated by integration by parts and the leading order will indicate the dominant (asymptotic) wavefield for large wavenumbers.

Let us then assume that there is an \((x, y)\) such that the ray can travel straight from \(P\) to \(Q\) along the aperture and call it \((x_0, y_0)\) (the intersection of the wavefront with the red ray in the picture).

We will not find the explicit solution in algebraic form of this point, since it is not needed for our purpose.

- **The Hessian.** The Hessian is always the most cumbersome computation for stationary phase evaluation. The Hessian is defined by equation 3.4 which I rewrite here.
\[ H(x_0, y_0) = \det \Phi = \phi_{xx} \phi_{yy} - \phi_{xy}^2. \] (3.8)

In general, let us now compute the crossing term \( \phi_{xy} = \phi_{yx}. \)

From equation (3.6) we find

\[ \phi_{xy} = \frac{r r_P [(x - x_Q) (r_P)_y + (r)_y (x - x_P)] - [r_P (x - x_Q) + r (x - x_P)] [(r)_y r_P + r (r_P)_y]}{(r r_P)^2} \]

From the definition of \( r \) and \( r_P \) (see formulas 3.5 and 3.4) we find \( (r)_y \) and \( (r_P)_x. \)

\[ (r)_y = \frac{y - y_Q}{r} \quad (r_P)_y = \frac{y - y_P}{r_P} \]

So the numerator for \( \phi_{xy} \) expands to

\[ r r_P [(x - x_Q) (y - y_P)/r_P + (x - x_P) (y - y_Q)/r] - T \]
\[ = r (x - x_Q) (y - y_P) + r_P (x - x_P) (y - y_Q) - T \]
\[ = r (x - x_Q) (y - y_P) + r_P (x - x_Q) (y - y_P) - T \]
\[ = (x - x_Q) (y - y_P) (r + r_P) - T \]

where

\[ T = [r_P (x - x_Q) + r (x - x_P)] \left[ \frac{(y - y_Q) r_P}{r} + \frac{r (y - y_P)}{r_P} \right] \]
\[ = \frac{r^2}{r} (x - x_Q) (y - y_Q) + r_P (x - x_P) (y - y_Q) + r (x - x_Q) (y - y_P) + \frac{r^2}{r} (x - x_P) (y - y_P) \]
\[ = 0 \]

The cancelations are done by using \( x - x_P = -(r_P / r) (x - x_Q). \)

So

\[ \phi_{xy} = \frac{(x - x_Q) (y - y_P) (r + r_P)}{(r r_P)^2}. \] (3.9)

Let us find \( \phi_{xx}. \) This would be given by

\[ \phi_{xx} = \frac{r r_P [(x - x_Q) (r_P)_x + r_P + (r)_x (x - x_P) + r] - [r_P (x - x_Q) + r (x - x_P)] [(r)_x r_P + r (r_P)_x]}{(r r_P)^2} \]

As done before, with the help of

\[ (r)_x = \frac{x - x_Q}{r} \quad (r_P)_x = \frac{x - x_P}{r_P} \]

the numerator is
The cancelations are done after using $x - x_P = -(r_P/r)(x - x_Q)$. Then

$$ \phi_{xx} = \frac{(r + r_P)(x - x_Q)(x - x_P) + r_P(r_P + r)}{(rr_P)^2} $$

$$ = \frac{(r + r_P)((x - x_Q)(x - x_P) + r_P)}{(rr_P)^2} \quad (3.10) $$

Similarly

$$ \phi_{yy} = \frac{(r + r_P)((y - y_Q)(y - y_P) + r_P)}{(rr_P)^2} \quad (3.11) $$

From equations $3.10$ and $3.11$

$$ \phi_{xx}\phi_{yy} = \frac{(r + r_P)((x - x_Q)(x - x_P) + r_P)}{(rr_P)^2} \frac{(r + r_P)((y - y_Q)(y - y_P) + r_P)}{(rr_P)^2} $$

$$ = \frac{(r + r_P)^2(x - x_Q)(x - x_P)(y - y_Q)(y - y_P)}{(rr_P)^4} + \frac{(r + r_P)^2r_P(x - x_Q)(x - x_P)}{(rr_P)^4} $$

$$ + \frac{(r + r_P)^2r_P(y - y_Q)(y - y_P)}{(rr_P)^4} + \frac{(r + r_P)^2}{(rr_P)^2} $$

$$ = \frac{(r + r_P)^2(x - x_Q)^2(y - y_P)^2}{(rr_P)^4} - \frac{(r + r_P)^2r_P^2(x - x_Q)^2}{(rr_P)^4} $$

$$ - \frac{(r + r_P)^2r_P^2(y - y_Q)^2}{(rr_P)^4} + \frac{(r + r_P)^2}{(rr_P)^2}. \quad (3.12) $$

and from equation $3.9$

$$ \phi_{xy}^2 = \frac{(x - x_Q)^2(y - y_P)^2}{(rr_P)^4} \circledast \quad (3.14) $$
Figure 2: This picture corresponds to the experiment that predicts the correct interpretation of the Huygens principle.

Then this equation and \(3.13\)

\[
H(x_0, y_0) = \frac{(r + r_P)^2}{(rr_P)^2} - \frac{(r + r_P)^2 r_P^2 (x - x_Q)^2}{(r r_P)^4} - \frac{(r + r_P)^2 r_P^2 (y - y_Q)^2}{(r r_P)^4}
\]

\[
= \frac{(r + r_P)^2}{(r r_P)^2} \left( 1 - \frac{r_P^2}{(r r_P)^2} [(x - x_Q)^2 + (y - y_Q)^2] \right)
\]

I can not ignore this :(. I only can predict the appropriate Green’s function if \(x = x_Q\) and \(y = y_Q\) which corresponds to figure 2.

\[
\sqrt{|H(x_0, y_0)|} = \sqrt{\phi_{xx} \phi_{yy} - \phi_{xy}^2} = \frac{r + r_P}{r r_P}
\]  \(3.15\)

with signature of the Hessian is 2.
So, since the stationary point is such that the rays connecting $P$ with $Q$ across the aperture are colinear, then

$$\xi = \chi = 0.$$ 

so, the amplitude factor is

$$A(x_0, y_0) = \frac{-i}{\lambda} \frac{1}{rr_P} \frac{1}{2}(1 + 1) = \frac{-i}{\lambda} \frac{1}{rr_P}$$

- **We apply the stationary phase formula** \[3.3\]. We find, since $\lambda = 2\pi/k$,

$$I(k) \approx -\frac{2\pi}{i} \frac{L_{PP}}{k} \frac{i}{X_{PP}} \exp(ik(r + r_P)) \exp(i\pi/2)$$

$$= -ii \frac{1}{r + r_P} \exp(ik(r + r_P))$$

$$= \frac{1}{r + r_P} \exp(ik(r + r_P))$$

Which predicts the Green’s function obtained as if there is no barrier, and the wave would travel directly from the source at $P$ to the receiver at $Q$.

**References**
