Solution to selected problems of Munkres Analysis on Manifolds Book

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Introduction

These notes show the solutions of a few selected problems from Munkres [1], book.

Chapter 4: Change of Variables

Section 16: Partitions of Unity

Problem 1.

Prove that the function f of Lemma 16.1 is of class C^∞ as follows: Given any integer $n \geq 0$, define $f_n : \mathbb{R} \to \mathbb{R}$ by the equation

$$f_n(x) = \begin{cases} (e^{-1/x}/x^n) & \text{for} \quad x > 0, \\ \\ 0 & \text{for} \quad x \leq 0. \end{cases}$$

(a) Show that f_n is continuous at 0. [*Hint:* Show that $a < e^a$ for all a. Then set $a = t/2n$ to conclude that

$$\frac{t^n}{e^t} < \frac{(2n)^n}{e^{t/2}} \tag{4.0.1}$$

Set $t = 1/x$ and let x approach 0 through positive values.]

sln. Let us first show that $a < e^a$. The function $f(a) = e^a - a$, has derivative $f'(a) = e^a - 1$. That is, $f'(a) > 0$, $\forall a > 0$. That is the function is monotonically increasing in the positive a axis and monotonically decreasing in the negative a axis. That is,

$$a < e^a \qquad \forall a > 0$$

If $a = 0$, from $0 < 1$, and if $a < 0$, from $a < 0 < e^a, \forall a < 0$, we see that in general $a < e^a$. In particular, if $a = t/2n$ we get

$$\frac{t}{2n} < e^{t/2n}$$

and by raising the power to the n in both sides

$$\frac{t^n}{(2n)^n} < e^{t/2} = e^t/e^{t/2}$$

from which equation 4.0.1 follows. Let us now call $x = 1/t$ so we rewrite 4.0.1 as

$$\frac{e^{-1/x}}{x^n} < \frac{(2n)^n}{e^{1/2x}}$$

Clearly as $x \to 0^+$, $e^{1/2x} \to \infty$ and so $(2n)^n/e^{1/2x} \to 0$, so by the squeeze theorem, $f_n(x) = \frac{e^{-1/x}}{x^n} \to 0$. That is $\lim_{x\to 0^+} f_n(x) = f_n(0)$. So f is continuous in 0.

(b) Show that f_n is differentiable at 0. The left derivative exists and is 0, since the function is identically zero on the left x axis. Let us check along the right x axis. By definition, and from $f_n(0) = 0$,

$$\begin{aligned} f_n'(0^+) &= \lim_{h\to 0^+} \frac{f_n(h) - f_n(0)}{h} \\ &= \lim_{h\to 0^+} \frac{f_n(h)}{h} \\ &= \lim_{h\to 0^+} \frac{e^{-1/h}}{h^{n+1}} \\ &< \frac{(2n)^n}{he^{1/2h}} \end{aligned}$$

The denominator $he^{1/2h}$ goes to ∞ as $h \to 0+$. To understand this, expand the denominator in Taylor series (which is fine for small h). Then

$$\begin{aligned} he^{1/2h} &= h(1 + 1/2h + (1/2h)^2/2! + \cdots + (1/2h)^k/k! + \cdots) \\ &= h + 1/2 + 1/8h + \mathcal{O}(1/h^2) \end{aligned}$$

which clearly goes to ∞ as $h \to 0$. So in the limit $f_n'(0^+) = 0$ and f_n is differentiable at 0.

(c) Show that $f_n'(x) = f_{n+1}(x) - nf_{n+1}(x)$ for all x. By direct evaluation, from Leibniz product rule

$$f_n'(x) = e^{-1/x}/x^{n+2} - ne^{-1/x}/x^{n+1} = f_{n+2}(x) - nf_{n+1}(x), \quad \forall x \; (4.0.2)$$

(d) Show that f_n is of class C^∞. The recursion formula 4.0.2 shows that the derivative of f_n can be written in terms of functions f_{n+1} and f_{n+2}, weighted with coefficients 1 and $-n$ respectively. To the derivative exists and its continuous for all n (an induction argument).

Problem 2.

Show that the functions defined in Example 1 form a partition of unity on $\mathbb{R}$. [*Hint:* Let

$$f_m(x) = f(x - m\pi), \quad \text{for all integers } m. \tag{4.0.3}$$

Show that

$$\sum f_{2m}(x) = (1 + \cos x)/2. \tag{4.0.4}$$

Then find $\sum f_{2m+1}(x)$].

sln. I believe the hint make the problem more complicated. Let us perform a direct evaluation of $\sum \phi_i(x)$.

Let us consider the case of $x \in [-\pi, \pi]$ where f has a support. here are two cases

- Even indices:

$$\phi_{2m}(x) = f(x + m\pi), \quad m \geq 1.$$

  If $-\pi \leq x \leq \pi$ and $-\pi \leq x + m\pi \leq \pi$ then

  Here the only possible value is $m = 1$, since a higher value will shift the argument from 0 by 2π or greater and the function f would evaluate to zero there, and it is necessary that $x \leq 0$. Hence, for $-\pi \leq x \leq 0$,

$$\sum \phi_{2m}(x) = f(x + \pi) = \frac{1 + \cos(x + \pi)}{2} = \frac{1 - \cos x}{2}, \tag{4.0.5}$$

- Odd indices:

$$\phi_{2m+1}(x) = f(x - m\pi), \quad m \geq 0.$$

If $-\pi \leq x \leq \pi$ and $-\pi \leq x - m\pi \leq \pi$ then we have two possibilities $m = 0, 1$. That is,

$$\begin{aligned} \phi_1(x) &= f(x) = (1+\cos x)/2,\ -\pi \leq x \leq \pi \\ \phi_3(x) &= f(x-\pi) = (1+\cos(x-\pi))/2 = (1-\cos x)/2,\ 0 \leq x \leq \pi \end{aligned}$$

so

$$\sum \phi_{2m+1}(x) = \begin{cases} (1+\cos x)/2 & -\pi \leq x \leq 0, \\ \\ 1 & 0 \leq x \leq \pi, \end{cases} \tag{4.0.6}$$

We now add all parts, from equations 4.0.5 and 4.0.6

$$\sum \phi_i(x) = \sum (\phi_{2m}(x) + \phi_{2m+1}(x)) = \begin{cases} 1 & -\pi \leq x \leq 0 \\ 1 & 0 \leq x \leq \pi \end{cases}$$

That is

$$\sum \phi_i(x) = 1.$$

in the interval $[-\pi, \pi]$.

Note that for the negative branches we added $(1-\cos x)/2 + (1+\cos x)/2 = 1$, and for the positive branches we added $(1+\cos x)/2 + (1-\cos x)/2 = 1$.

Section 19: Proof of Change of Variables Theorem

Problem *6.

Let $B^n(a)$ denote the closed ball of radius a in $\mathbb{R}^n$, centered at 0,

(a) Show that

$$v(B^n(a)) = \lambda_n a^n$$

for some constant λ_n. Then $\lambda_n = v(B^n(1))$.

Before defining the mapping let us find a "natural" [1] coordinate system for this problem.

Call $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{u}$ a unit vector aligned with $\boldsymbol{x}$. The unit vector has coordinates

$$\boldsymbol{u} = (u_1, \cdots, u_n),$$

where

$$u_i = \frac{\boldsymbol{x} \cdot \boldsymbol{e}_i}{\|x\|} = \frac{x_i}{a} = \cos\theta_i$$

with θ_i is the director angle between the vector $\boldsymbol{x}$ and the coordinate base vector $\boldsymbol{e}_i$.

Since

$$u_1^2 + \cdots + u_n^2 = 1,$$

all u_i coordinates are not independent and we can write the last coordinate as

$$u_n = \pm\sqrt{1 - (u_1^2 + \cdots + u_{n-1}^2)}.$$

We define the mapping

$$\begin{aligned} \beta : B &\rightarrow \mathbb{R}^n \\ (u_1, \cdots, u_{n-1}, r) &\mapsto (ru_1, \cdots, ru_n), \end{aligned} \tag{4.0.7}$$

It is clear that this mapping turns our coordinates into a ball of radius r, which for $0 \leq r \leq a$ is $B^n(a)$.

[1] I solved this problem using 4 other methods including generalized polar–spherical coordinates, recursion formulas and evaluation of Gaussian functions, in my notes on PDE.

The Jacobian of this transformation is given by:

$$D\beta = \begin{bmatrix} r & 0 & \cdots & 0 & u_1 \\ 0 & r & 0\cdots & 0 & u_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r & u_{n-1} \\ -\frac{ru_1}{u_n} & -\frac{ru_2}{u_n} & \cdots & -\frac{ru_{n-1}}{u_n} & u_n \end{bmatrix}$$

Since $u_n = \pm\sqrt{1 - \sum_{i=1}^{n-1} u_i^2}$, that is, u_n is multivalued and we need to consider two patches. Each patch (due to symmetry) has the same area and volume. We then only evaluate one patch and duplicate our result.

Let us evaluate $\det D\beta(\boldsymbol{u}, r)$. For this we perform Gaussian elimination to put zeroes in the last raw, except for the last entry of that raw. We find

$$\det D\beta(\boldsymbol{u},r) = \frac{u_n}{u_1}\det\begin{bmatrix} \frac{ru_1}{u_n} & 0 & \cdots & 0 & \frac{u_1^2}{u_n} \\ 0 & r & 0\cdots & 0 & u_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r & u_{n-1} \\ -\frac{ru_1}{u_n} & -\frac{ru_2}{u_n} & \cdots & -\frac{ru_{n-1}}{u_n} & u_n \end{bmatrix}$$

$$= \frac{u_n}{u_1}\det\begin{bmatrix} \frac{ru_1}{u_n} & 0 & \cdots & 0 & \frac{u_1^2}{u_n} \\ 0 & r & 0\cdots & 0 & u_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r & u_{n-1} \\ 0 & -\frac{ru_2}{u_n} & \cdots & -\frac{ru_{n-1}}{u_n} & \frac{u_1^2+u_n^2}{u_n} \end{bmatrix}$$

$$= \frac{u_n}{u_1}\frac{u_n}{u_2}\det\begin{bmatrix} \frac{ru_1}{u_n} & 0 & \cdots & 0 & \frac{u_1^2}{u_n} \\ 0 & \frac{ru_2}{u_n} & 0\cdots & 0 & \frac{u_2^2}{u_n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r & u_{n-1} \\ 0 & 0 & \cdots & -\frac{ru_{n-1}}{u_n} & \frac{u_1^2+u_2^2+u_n^2}{u_n} \end{bmatrix}$$

That is

$$\det D\beta(\boldsymbol{u},r) = \frac{u_n}{u_1}\frac{u_n}{u_2}\cdots\frac{u_n}{u_{n-1}}$$

$$\det\begin{bmatrix} \frac{ru_1}{u_n} & 0 & \cdots & 0 & \frac{u_1^2}{u_n} \\ 0 & \frac{ru_2}{u_n} & 0\cdots & 0 & \frac{u_2^2}{u_n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{ru_{n-1}}{u_n} & \frac{u_{n-1}^2}{u_n} \\ 0 & 0 & \cdots & 0 & \frac{\sum_{i=1}^{n-1}u_i^2+u_n^2}{u_n} = \frac{1}{u_n} \end{bmatrix}$$

from which

$$\det D\beta(\boldsymbol{u},r) = \frac{r^{n-1}}{u_n} \tag{4.0.8}$$

Hence, the volume of the $B^{n-1}(a)$ ball is given by (recall that there are two patches with equal volume)

$$\begin{aligned} v(B^n(a)) &= 2\int_{B_+^n(a)} \det D\beta(\boldsymbol{u},r) \\ &= 2\int_0^a r^{n-1}dr \int_{B_+^n(1)} \frac{1}{u_n}du_1\cdots du_{n-1} \\ &= \frac{2a^n}{n}\int_{B_+^n(1)} \frac{1}{u_n}du_1\cdots du_{n-1} \\ &= a^n\lambda_n \end{aligned} \tag{4.0.9}$$

with

$$\lambda_n = \frac{2}{n}\int_{B_+^n(1)}\frac{1}{u_n}du_1\cdots du_{n-1} \tag{4.0.10}$$

(b) Compute λ_1 and λ_2.

(c) Compute λ_n in terms of λ_{n-2}.

(d) Obtain a formula for λ_n. [*Hint:* Consider two cases, according as n is even or odd.]

sln. We found the formula

$$
\begin{aligned}
v(B^n(a)) &= 2\int_{B_+^n(a)} \det D\beta(\boldsymbol{u}, r) \\
&= 2\int_0^a r^{n-1}dr \int_{B_+^{n-1}(1)} \frac{1}{\sqrt{1-\sum_{i=1}^{n-1} u_i^2}} du_1 \cdots du_{n-1}. \\
&= \frac{2a^n}{n}\int_{B_+^{n-1}(1)} \frac{1}{\sqrt{1-\sum_{i=1}^{n-1} u_i^2}} du_1 \cdots du_{n-1}.
\end{aligned}
$$

Let us test this formula with a few simple cases

- If $n = 1$ (in the one–dimensional space, then

$$
v(B^0(a)) = \frac{2a}{1} = 2a,
$$

- If $n = 2$ (in the 2D space)

$$
v(B^1(a)) = \frac{2a^2}{2}\int_{-1}^{1} \frac{1}{\sqrt{1-u_1^2}} du_1 = \pi a^2.
$$

- If $n = 3$ (in the 3D space)

$$
\begin{aligned}
v(B^2(a)) &= \frac{2a^3}{3}\int_{-1}^{1} \frac{1}{\sqrt{1-u_1^2-u_2^2}} du_1 du_2 \\
&= \frac{2a^3}{3}\int_{-1}^{1} du_1 \int_{-1}^{1} du_2 \frac{1}{\sqrt{b^2-u_2^2}}
\end{aligned}
$$

where $b^2 = 1 - u_1^2$. The substitution $u_2 = b\sin\theta$, $du_2 = b\cos\theta$, where $\theta \in [-\pi/2, \pi/2]$ provides

$$
\begin{aligned}
v(B^2(a)) &= \frac{2a^3}{3}\int_{-1}^{1} du_1 \int_{-\pi/2}^{\pi/2} \frac{b\cos\theta d\theta}{b\cos\theta} \\
&= \frac{2a^3}{3}\int_{-1}^{1} du_1 \pi \\
&= \frac{4\pi a^3}{3}
\end{aligned}
$$

which corresponds to the three dimensional volume of a sphere.

In the last derivation we assumed that the variables u_1 and u_3 can be integrated iteratively. That is they are independent variables along orthogonal directions. This is not always true and we were lucky in obtaining the right answer with a weak assumption. In general we could have trouble.

In general, let us assume that we are considering the n–dimensional space. We will assume that the variable of integration r is decoupled from the rest of the variables (and this is right) there is not dependence between the radius and any of the polar/azimuthal directions. So we can write

$$v(B^n(a)) \quad = \quad \frac{2a^n}{n}\int_{B_+^{n-1}(1)} \frac{1}{\sqrt{1-\sum_{i=1}^{n-1} u_i^2}} du_1 \cdots du_{n-1}.$$

To solve this integral we assume that the denominator is bounded away from zero, so we can apply Fubini's rule, and then after we apply the rule we can take the limit as the denominator (u_n) goes to zero. Recall that the domain of integration $B_+^{n-1}(a)$ is the manifold of $n-1$–tuples $(u_1, \cdots, u_{n-1})$ under the mapping

$$u_1^2 + \cdots + u_{n-1}^2 = 1 - u_n^2,$$

and since $0 < u_n \leq 1$ then $1 - \sum_{i=1}^{n-1} u_i^2 > 0$.

Let us take the last coordinate u_{n-1} and let it be in the interval $u_{n-1} \in [-1+\epsilon, 1-\epsilon],\ \ 0 < \epsilon << 1$ and write

$$v(B^n(a)) \quad = \quad \frac{2a^n}{n}\int_{-1}^{1} du_{n-1} \int_{B_+^{n-1}(1)} \frac{1}{\sqrt{1-\sum_{i=1}^{n-1} u_i^2}} du_1 \cdots du_{n-2}.$$

with $B_+^{n-1} = B_+^{n-2} \times [-1, 1]$. B_+^{n-2} is the manifold defined by the $(n-2)$–tuples $(u_1, \cdots u_{n-2})$ such that

$$u_1^2 + \cdots u_{n-2}^2 \leq 1.$$

with $u_{n-1} \geq 0$. We rewrite the integral as

$$v(B^n(a)) = \lim_{\epsilon \to 0} \frac{2a^n}{n} \int_{B_+^{n-2}(1)} du_1 \cdots du_{n-2} \int_{-1+\epsilon}^{1-\epsilon} \frac{du_{n-1}}{\sqrt{b^2 - u_{n-1}^2}}.$$

with $b^2 = 1 - \sum_{i=1}^{n-2} u_i^2$ and make the change of variables $u_{n-1} = b \sin \theta$, $du = b \cos \theta$, so

$$\begin{aligned} v(B^n(a)) &= \frac{2a^n}{n} \int_{B_+^{n-2}(1)} du_1 \cdots du_{n-2} \int_{\arcsin(-1+\epsilon)}^{\arcsin(1-\epsilon)} d\theta \\ &= \frac{2a^n \pi}{n} \int_{B_+^{n-2}(1)} du_1 \cdots du_{n-2} \\ &= \frac{2a^n \pi}{n} v(B^{n-2})(1) \end{aligned} \tag{4.0.11}$$

This provides us with the following recursion formula. Starting at $n = 1$.

$$\begin{aligned} v(B^0(a)) &= 2a \\ v(B^2(a)) &= \frac{2a^3\pi}{3} v(B^0(1)) = \frac{4\pi a^3}{3} \\ v(B^4(a)) &= \frac{2\pi a^5}{5} v(B^2(1)) = \frac{8\pi^2 a^5}{15} \end{aligned}$$

on the other hand, starting at $n = 2$

$$\begin{aligned} v(B^1(a)) &= \pi a^2 \\ v(B^3(a)) &= \frac{2a^4\pi}{4} v(B^1(1)) = \frac{a^4\pi^2}{2} \\ v(B^5(a)) &= \frac{2a^6\pi}{6} v(B^3(1)) = \frac{a^6\pi^3}{6}. \end{aligned}$$

In general, by simple induction and the recursions above, it is easy to observe that the formula for the volume of the n–dimensional ball

$$v(B^n(a)) = \frac{\pi^{n/2} a^n}{\Gamma\left(\frac{n}{2} + 1\right)}. \tag{4.0.12}$$

From equation 4.0.9 we see that

$$\lambda_n = \frac{v(B^n(a))}{a^n} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} \tag{4.0.13}$$

Problem *7.

(a) Find the centroid of the upper half–ball

$$B^n_+(a) = \{\boldsymbol{x} | \boldsymbol{x} \in B^n(a) \quad \text{and} \quad x_n \geq 0\}$$

in terms of λ_n and λ_{n-1} and a, where $\lambda_n = v(B^n(1))$.

sln. From the definition in exercise 3, we have

$$c_k(B^n_+(a)) = \frac{1}{v(B^n_+(a))} \int_{B^n_+(a)} \pi_k = \frac{I}{v(B^n_+(a)}, \tag{4.0.14}$$

with

$$I = \int_{B^n_+(a)} \pi_k.$$

We know that half the sphere has half the volume (isn't this obvious?, since the density is constant :)) So

$$v(B^n_+(a)) = \frac{a^n \lambda_n}{2}, \quad \text{and so} \quad c_k(B^n_+(a)) = \frac{2I}{a^n \lambda_n}.$$

Next we evaluate the integral I. For this evaluation we use the change of variables defined by the function β in the mapping 4.0.7, for which we already know (see equation 4.0.8) that

$$V(D\beta) = \frac{r^{n-1}}{u_n}.$$

We should evaluate the integral

$$I = \int_{B^n_+(a)} \pi_k.$$

and since $\pi_k = x_k = r u_k$, then

$$\begin{aligned} I &= \int_{B^n_+(a)} r u_k \det D(\beta(\boldsymbol{u}, r)) \\ &= \int_0^a r^n dr \int_{B^n_+(1)} \frac{u_k}{u_n} du_1 \cdots du_{n-1} \\ &= \frac{a^{n+1}}{n+1} \int_{B^n_+(1)} \frac{u_k}{u_n} du_1 \cdots du_{n-1} \end{aligned} \tag{4.0.15}$$

Let us will consider two cases

(a) $k < n$. From equation 4.0.15

$$I = \frac{a^{n+1}}{n} \int_{B_+^n(1)} du_1 \cdots du_{k-1} du_{k+1} \cdots du_{n-1} \int_c^d du_k \frac{u_k}{u_n}$$

where the integration bounds c and d are to be defined. The question of Fubini's rule always should be addressed. I will assume that the k–integration can be moved to the end as I did here (and I could, but will not prove it).

From $\sum_i u_i^2 = 1$, $i = 1, \cdots, n$ we have

$$\begin{aligned} u_n &= \sqrt{1 - \sum_{i=1}^{n-1} u_i^2} \\ &= \sqrt{1 - \sum_{i=1, i \neq k}^{n-1} u_i^2 - u_k^2} \\ &= \sqrt{b^2 - u_k^2} \end{aligned}$$

where

$$b^2 = 1 - \sum_{i=1, i \neq k}^{n-1} u_i^2.$$

This defines the bounds of integration as $|u_k| \leq b$. That is $c = -b$ and $d = b$. Since, u_k is odd (recall $u_k = x_k/a$)

$$I = \frac{a^{n+1}}{n+1} \int_c^d du_k \frac{u_k}{u_n} = \frac{a^{n+1}}{n+1} \int_{-b}^b du_k \frac{u_k}{\sqrt{b^2 - u_k^2}} = 0 \quad (4.0.16)$$

(b) $k = n$. Initially, from 4.0.11 we have the recursion

$$\lambda_{n+1} = \frac{2\pi}{n+1} \lambda_{n-1}. \quad (4.0.17)$$

Now, equation 4.0.15 we find

$$I = \frac{a^{n+1}}{n+1} \int_{B_+^n(1)} du_1 \cdots du_{n-1} \quad = \quad \frac{a^{n+1}}{n+1}\, \lambda_{n-1} \qquad (4.0.18)$$

So from equations 4.0.14, 4.0.16 and 4.0.18 we find

$$c_k = \delta_{kn} \frac{2a}{n+1} \frac{\lambda_{n-1}}{\lambda_n} = \delta_{kn} \frac{a\lambda_{n+1}}{\pi\, \lambda_n}$$

where

$$\delta_{kn} = \begin{cases} 1 & \text{if} \quad i = n \\ 0 & \text{if} \quad i \neq n \end{cases}$$

is the Kronecker delta.
Two simple cases (we should start at $n \geq 2$).

- n=2 (semi–circle)

$$\begin{aligned} c_1 &= 0 \\ c_2 &= \frac{2a}{3}\frac{2}{\pi} = \frac{4a}{3\pi}. \end{aligned}$$

- n=3 (semi–sphere)

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= \frac{2a}{4}\frac{\pi}{4\pi/3} = \frac{3a}{8}. \end{aligned}$$

Finally let us express the formula of the centroid in terms of the Γ function using equation 4.0.13. That is, since

$$\lambda_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

then

$$c_k = \delta_{kn}\, \frac{2a}{n+1} \frac{\pi^{(n-1)/2}}{\pi^{n/2}} \frac{\Gamma((n-1)/2+1)}{\Gamma(n/2+1)} = \delta_{kn} \frac{2a}{(n+1)\pi^{1/2}} \frac{\Gamma(n/2+1/2)}{\Gamma(n/2+1)}.$$

(b) Express $c(B_+^n(a))$ in terms of $c(B_+^{n-2}(a))$.

sln. We will only compare the non–zero components.

$$\frac{c(B_+^n(a))}{c(B_+^{n-2}(a))} = \frac{a\lambda_{n+1}}{\pi\lambda_n}\frac{\pi\lambda_{n-2}}{a\lambda_{n-1}} = \frac{\lambda_{n+1}\lambda_{n-2}}{\lambda_n\lambda_{n-1}}$$

and by using equation 4.0.13

$$\begin{aligned}\frac{c(B_+^n(a))}{c(B_+^{n-2}(a))} &= \frac{\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2}+1)}\frac{\pi^{(n-2)/2}}{\Gamma(\frac{n-2}{2}+1)}\frac{\Gamma(\frac{n}{2}+1)}{\pi^{n/2}}\frac{\Gamma(\frac{n-1}{2}+1)}{\pi^{(n-1)/2}} \\ &= \frac{(n+2)/2}{(n+3)/2} \\ &= \frac{n+2}{n+3}\end{aligned}$$

So the relationship looked for is

$$c(B_+^n(a)) = \left(\frac{n+2}{n+3}\right)c(B_+^{n-2}(a)).$$

Chapter 5: Manifolds

Section 22: The Volume of a Parameterized Manifold

Problem 2.

Let A be open in $\mathbb{R}^k$; let $f : A \to \mathbb{R}$ be of class C^r; let Y be the graph of f in $\mathbb{R}^{k+1}$, parameterized by the function $\alpha : A \to \mathbb{R}^{k+1}$ given by $\alpha(\boldsymbol{x}) = (\boldsymbol{x}, f(\boldsymbol{x}))$. Express $v(Y_\alpha)$ as an integral.

sln. By definition

$$v(Y_\alpha) = \int_A V(D_\alpha).$$

Now D_α is a $(k+1) \times k$ matrix on the real numbers, defined as follows

$$\alpha_{i,j} = \frac{\partial \alpha_i}{\partial x_j} = \begin{cases} \delta_{ij} & \text{if} \quad j \leq k \\ f_{,i} & \text{if} \quad j = k+1 \end{cases}$$

with δ_{ij} being the Kronecker delta, that is the identity matrix up to dimension $k \times k$ and

$$f_{,i} = \frac{\partial f}{\partial x_i}.$$

Let us call

$$P = D_\alpha^{Tr} \cdot D_\alpha.$$

We should evaluate:

$$\det P.$$

That is we find the dot products between the different columns of D_α. In the diagonal entry P_{ii}, we dot multiply the i–th column of D_α by itself and find

$$P_{ii} = 1 + (f_{,i})^2.$$

Off the diagonal we find that the dot product only provides the last two entries of the column, mainly

$$P_{ij} = f_{,i} f_{,j}$$

In what follows of this problem I use repeated index notation (Einstein notation) and the Levi–Civita antisymmetric tensor. We write

$$\det P = \epsilon^{i_1 i_2 \cdots i_k} P_{1i_1} P_{2i_2} \cdots P_{ki_k}.$$

To unify notation let us call

$$P_{ij} = \delta_{ij} + f_{,i} f_{,j}$$

and so

$$\det P = \epsilon^{i_1 i_2 \cdots i_k} (\delta_{1i_1} + f_{,1} f_{,i_1})(\delta_{2i_2} + f_{,2} f_{,i_2}) \cdots (\delta_{ki_k} + f_{,k} f_{,i_k})$$

If we are to split the factors in monic terms, each term would have the form of

$$\epsilon^{i_1 i_2 \cdots i_k} \delta_{j_1 i_{j_1}} \delta_{j_2 i_{j_2}} \cdots \delta_{j_l i_{j_l}} (f_{,p_1} f_{,i_{p_1}})(f_{,p_2} f_{,i_{p_2}}) \cdots (f_{,p_m} f_{,i_{p_m}})$$

Here $j_1, j_2, \cdots, j_l, p_1, p_2, p_m$ are a permutation of the set $1, 2, \cdots, k$ I claim that if $m > 1$ then the contribution of the sum of the corresponding terms below is zero. To see why this should be true let us see a simple case. Let us say that $m = 2$, then there are $k - 2$ Kronecker deltas, and in those cases to avoid a zero we need $j_s = i_s$ for all $s = 1, k - 2$, and only two couple of f factors. That is

$$\epsilon^{i_1 i_2 \cdots i_k} (f_{,\lambda} f_{,i_\lambda})(f_{,\mu} f_{,i_\mu}) \tag{5.19}$$

$k-2$ superindices of the Levi–Civita symbol are fixed and two are variable. The two superindices which can change are i_λ and i_μ. For each (i_λ, i_μ) couple there is a (i_μ, i_λ) couple, and they give opposite signs in the sum (if one produces an odd permutation, the other produces an even permutation). Now, the interchange of λ by μ will not change the absolute value of equation 5.19. So all terms cancel pairwise.

The same applies if $m > 3$. We can always interchange by couples and see that the terms cancel pairwise.

The only remaining cases are $m = 0$ and $m = 1$. For $m = 0$ we have the expression

$$\epsilon^{i_1 i_2 \cdots i_k} \delta_{1 i_1} \delta_{2 i_{2_2}} \cdots \delta_{k i_k} = 1 \tag{5.20}$$

since the determinant of the identity matrix is 1, and if $m = 1$

$$\epsilon^{i_1 i_2 \cdots i_k} \delta_{j_1 i_{j_1}} \delta_{j_2 i_{j_2}} \cdots \delta_{j_{k-1} i_{j_{k-1}}} (f_{,j_k} f_{,i_{j_k}})$$

For a non–zero term we require $j_\mu = i_{j_\mu}$, $\mu = 1, 2, \cdots k-1$, and so $j_k = i_{j_k}$ by default, and the expansion of the Levi–Civita expression provides

$$(f_{,j_k} f_{,j_k}) \tag{5.21}$$

where j_k is the only free index running in the list $1, 2, \cdots k$. So, combining 5.20 and 5.21 we find

$$\det P = 1 + (f_{,j_k} f_{,j_k}).$$

Which in classical notation is

$$\det P = 1 + \sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_k} \right)^2 .$$

and the expression for the volume $v(Y_D)$ would then be

$$v(Y_D) = \int \sqrt{1 + \sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_k} \right)^2 } .$$

Problem 3.

Let A be open in $\mathbb{R}^k$; let $\alpha : A \to \mathbb{R}^n$ be of class C^r; let $Y = \alpha(A)$. The centroid $c(Y_\alpha)$ of the parameterized–manifold Y_α is the point $\mathbb{R}^n$ whose i^{th} coordinate is given by the equation

$$C_i(Y_\alpha) = \frac{1}{v(Y_\alpha)} \int_A \pi_i V, \tag{5.22}$$

where $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the i^{th} projection function.

(a) Find the centroid of the parameterized–curve

$$\alpha(t) = (a\cos t, a\sin t) \quad \text{with} \quad 0 < t < \pi$$

(b) Find the centroid of the hemisphere of radius a in $\mathbb{R}^3$. (See Example 4.)

sln.

(a) Let us first find

$$v(Y_\alpha) = \int_A V(D_\alpha)$$

where

$$D(\alpha) = \begin{bmatrix} -a\sin t \\ a\cos t \end{bmatrix}$$

and

$$V(D_\alpha) = \int_0^\pi \sqrt{a^2(\sin^2 t + \cos^2 t)} = \int_0^\pi a = a\pi.$$

(see Example 1, for the general form for a parameterized curve)

Now from the definition of centroid 5.22

$$C_1(Y_\alpha) = \frac{1}{v(Y_\alpha)} \int_A \pi_1 V = \frac{1}{\pi a} \int_0^\pi a^2 \cos t dt = 0$$
$$C_2(Y_\alpha) = \frac{1}{v(Y_\alpha)} \int_A \pi_2 V = \frac{1}{\pi a} \int_0^\pi a^2 \sin t dt = \frac{2\,a^2}{\pi a} = \frac{2a}{\pi}.$$

(b) From Example 4 we already know that

$$v(Y_\alpha) = 2\pi a^2.$$

Now, we can see from Example 4 also that

$$\begin{aligned} C_1(Y_\alpha) = \frac{1}{v(Y_\alpha)} \int_A \pi_1 V &= \frac{1}{2\pi a^2} \int_A \frac{ax}{(a^2 - x^2 - y^2)^{1/2}} dxdy \\ &= \frac{1}{2\pi a^2} \int_{-a}^{a} dy \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \frac{ax\, dx}{(a^2 - x^2 - y^2)^{1/2}} \\ &= -\frac{1}{2\pi a} \int_{-a}^{a} dy (a^2 - x^2 - y^2)^{1/2} \Big|_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} = 0. \end{aligned}$$

Along the same lines, by changing x by y (due to symmetry) we see that $C_2(Y_\alpha) = 0$. Let us now compute $C_3(Y(\alpha))$. That is

$$\begin{aligned} C_3(Y_\alpha) = \frac{1}{v(Y_\alpha)} \int_A \pi_3 V &= \frac{1}{2\pi a^2} \int_A \frac{az}{(a^2 - x^2 - y^2)^{1/2}} dxdy \\ &= \frac{1}{2\pi a^2} \int_A a\, dx\, dy \\ &= \frac{1}{2\pi a^2} a\pi a^2 \\ &= \frac{a}{2}. \end{aligned} \tag{5.23}$$

Problem 4.

The following exercise gives a strong plausibility argument justifying our definition of volume. We consider only the case of a surface in $\mathbb{R}^3$, but a similar result holds in general.

Given three points $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ in $\mathbb{R}^3$, let C be the matrix with columns $\boldsymbol{b} - \boldsymbol{a}$ and $\boldsymbol{c} - \boldsymbol{a}$. The transformation $h : \mathbb{R}^2 \to \mathbb{R}^3$ given by $h(\boldsymbol{x}) = C \cdot \boldsymbol{x} + \boldsymbol{a}$ carries $0, \boldsymbol{e}_1, \boldsymbol{e}_2$ to $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, respectively. The image Y under h of the set

$$A = \{(x, y) | x > 0 \quad \text{and} \quad y > 0 \quad \text{and} \quad x + y < 1\}$$

is called the (open) triangle in $\mathbb{R}^3$ with vertexes $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. See Figure 22.5. The area of the parameterized–surface Y_h is one–half the area of the parallelepiped with edges $\boldsymbol{b} - \boldsymbol{a}$ and $\boldsymbol{c} - \boldsymbol{a}$, as you can check.

Now let Q be a rectangle in $\mathbb{R}^2$ and let $\alpha : Q \to \mathbb{R}^3$; suppose α extends to a map of class C^r defined in an open set containing Q. Let P be a partition of Q. Let R be a sub–rectangle determined by P, say

$$R = [a, a+h] \times [b, b+k].$$

Consider the triangle $\Delta_1(R)$ having vertexes

$$\alpha(a,b), \alpha(a+h,b), \quad \text{and} \quad \alpha(a+h,b+k)$$

and the triangle $\Delta_2(R)$ having vertexes

$$\alpha(a,b), \alpha(a,b+k), \quad \text{and} \quad \alpha(a+h,b+k).$$

We consider these two triangles to be an approximation to the "curved rectangle" $\alpha(R)''$. See Figure 22.6. We then define

$$A(P) = \sum_R [v(\Delta_1(R)) + v(\Delta_2(R))],$$

where the sum extends over all sub–rectangles R determined by P. This number is the area of a polyhedral surface that approximations $\alpha(Q)$. Prove the following:

Theorem: *Let Q be a rectangle in $\mathbb{R}^2$ and let $\alpha : A \to \mathbb{R}^3$ be a map of class C^r defined in an open set containing Q. Given $\epsilon > 0$, there is a $\delta > 0$ such that for every partition P of Q of mesh less than δ,*

$$\left| A(P) - \int_Q V(D\alpha) \right| < \epsilon.$$

Proof.

(a) Given points $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_6$ of Q, let

$$\mathcal{D}\alpha(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_6) = \begin{bmatrix} D_1\alpha_1(\boldsymbol{x}_1) & D_2\alpha_1(\boldsymbol{x}_4) \\ D_1\alpha_2(\boldsymbol{x}_2) & D_2\alpha_2(\boldsymbol{x}_5) \\ D_1\alpha_3(\boldsymbol{x}_3) & D_2\alpha_3(\boldsymbol{x}_6) \end{bmatrix}$$

Then $\mathcal{D}\alpha$ is just the matrix $D\alpha$ with its entries evaluated at different points of Q. Show that if R is a sub–rectangle determined by P, then there are points $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_6$ of R such that

$$v(\Delta_1(R)) = \frac{1}{2} V(\mathcal{D}\alpha(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_6)) \cdot v(R).$$

Prove a similar result for $v(\Delta_2(R))$.

sln. The following solution was taking from Yan Zeng's document [2]. The reader needs Adobe Flash Player installed to be able to access the information there.

$$v(\Delta_1(R)) = \int_A V(D\alpha),$$

where A is the (open) triangle in $\mathbb{R}^2$ with vertexes $(a,b),(a+h,b)$ and $(a+h,b+k)$. $V(D\alpha)$ is a continuous function on the compact set $\overline{A}$, so it achieves its maximum M and minimum m on $\overline{A}$. Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \overline{A}$ such that $V(D\alpha(\boldsymbol{x}_1)) = M$ and $V(D_\alpha(\boldsymbol{x}_2)) = m$, respectively. Then

$$v(A)\cdot m \leq v(\Delta_1(R)) \leq v(A).M.$$

By the intermedian value theorem of a continuous function, and considering the segment connecting $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, we can find a point $\boldsymbol{\xi}_1 \in \overline{A}$ such that $V(D\alpha(\boldsymbol{\xi}_1))v(A) = \int_A V(D\alpha)$. This shows that there is a point $\boldsymbol{\xi}_1$ of R such that

$$v(\Delta_1(R)) = \int_A V(D\alpha) = V(D\alpha(\boldsymbol{\xi}_1))v(A) = \frac{1}{2}V(D\alpha(\boldsymbol{\xi}_1))\cdot v(R).$$

A similar result for $v(\Delta_2(R))$ can be proved similarly. Here we find a second point $\boldsymbol{\xi}_2$.

I like Yan Zeng's solution to the problem. I still do not get why Munkres require six different points in the rectangle R.

(b) Given $\epsilon > 0$, show one can choose $\delta > 0$ so that if $\boldsymbol{x}_i, \boldsymbol{y}_i \in Q$ with $|\boldsymbol{x}_i - \boldsymbol{y}_i| < \delta$ for $i = 1, \cdots, 6$, then

$$|V(\mathcal{D}\alpha(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_6)) - V(\mathcal{D}\alpha(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_6))| < \epsilon.$$

sln. $V(D\alpha)$ is a continuous function is uniformly continuous on the compact set Q.

(c) Prove the theorem.

[2] http://www.docin.com/p-306958076.html

sln.

$$\begin{aligned}\left|A(P) - \int_Q V(D\alpha)\right| &\leq \sum_R \left|v(\Delta_1(R)) + v(\Delta_2(R)) - \int_R V(D\alpha)\right| \\ &= \sum \left|\frac{1}{2}[V(D\alpha(\boldsymbol{\xi}_1(R))) + V(D\alpha(\boldsymbol{\xi}_2(R)))]v(R) - \int_R V(D\alpha)\right| \\ &\leq \sum_R \int_R \left|\frac{V(D\alpha(\boldsymbol{\xi}_1(R))) + V(D\alpha(\boldsymbol{\xi}_2(R)))}{2} - V(D\alpha)\right| \\ &\leq \frac{1}{2}\sum_R \int_R |V(D\alpha(\boldsymbol{\xi}_1(R))) - V(D\alpha)| + \\ &\qquad |V(D\alpha(\boldsymbol{\xi}_2(R))) - V(D\alpha)|\end{aligned}$$

Given $\epsilon > 0$, there exists a $\delta > 0$ such that if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in Q$ with $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < \delta$, we must have $V((D\alpha(\boldsymbol{x}_1)) - V(D\alpha(\boldsymbol{x}_2))| < \frac{\epsilon}{v(Q)}$. So for every partition P of Q of mesh less than δ,

$$\left|A(P) - \int_Q V(D\alpha)\right| < \sum_R \int_R \frac{\epsilon}{v(Q)} = \epsilon.$$

Section 23: Manifolds in $\mathbb{R}^3$

Problem 3.

(a) Show that the unit circle S^1 is a 1–manifold in $\mathbb{R}^2$.

sln. We can cover the circle with the following four patches, each half a circle and with an overlapping of a quarter of circle between patches.

$$\begin{aligned}\alpha_i : \mathbb{R} &\to \mathbb{R}^2 \\ t &\mapsto (\cos t, \sin t)\end{aligned}$$

where the domain U_i of α_i is defined as $(i\pi/2, (2+i)\pi/2)$, $i = 1 \cdots 4$, and the patch V_i is the circular arc corresponding to the angle swept by U_i. It is easy to verify the three conditions to be a manifold for each of the three patches. That is

(i) α_i is of class C^∞.

(ii) $\alpha_i^{-1} : V_i \to U_i$ is of class C^∞, and

(iii) $D\alpha(t) = (-\sin t, \cos t)$ has rank 1.

(b) The problem of letting the circle close (or in other words to let the domain interval be of length 2π), is that the point corresponding to 0 and 2π is the same, and so α^{-1} can not be continuous at that point, since the inverse image of a neighborhood of that point can be as closed to 2π as we want, or as closed to 0 as we want. Topologically we can see it like this. Take the open set $[0, 2\pi)$, in $\mathbb{H}^1$. The image under the mapping α is the whole circle which is a closed set in the relative topology of the set S^1 in $\mathbb{R}^2$. Hence, α^{-1} can not be continuous.

Problem 6.

(a) Show that $I = [0, 1]$ is a 1–manifold in $\mathbb{R}$.

sln. We show that I is a manifold with boundary.

Let $U = [0, 1)$ which is open in $\mathbb{H}^1$. We use two coordinate patches.

$$\begin{aligned} \alpha_1 : U &\to V_1 = [0, 1) \\ x &\mapsto x \end{aligned}$$

and

$$\begin{aligned} \alpha_2 : U &\to V_2 = (0, 1] \\ x &\mapsto 1 - x \end{aligned}$$

Clearly $I = V_1 \cup V_2$, and both α_1 and α_2 satisfy the manifold conditions. Note that the purpose of the second mapping α_2 is just to add the number 1 missing from V_1. The mapping could be defined in smaller set $[0, s)$ with $0 < s < 1$.

(b) Is $I \times I$ a 2–manifold in $\mathbb{R}^2$? Justify your answer. This problem really belongs to the next section. It is hard to prove without the definitions and properties of the next section.

sln. The answer is: "no". The reason is because of the corners. To better understand the reasons let me show first what type of unit squares are manifolds, and then show why the answer is "no".

- The open unit square is a manifold. The identity mapping would be the only coordinate patch needed.
- The square $A = (0,1) \times [0,1)$ is a manifold. Again, the identity mapping from $\mathbb{H}^1$ to itself satisfy the manifold definition.
- The square $B = (0,1) \times (0,1]$ is a manifold. Here the coordinate patch is given by

$$\begin{aligned} \alpha : (0,1) \times [0,1) &\rightarrow (0,1) \times (0,1] \\ (x,y) &\mapsto (x, 1-y). \end{aligned}$$

- The union $A \cup B$ is a manifold. The two patches in the previous two items cover this union.
- The square $C = [0,1) \times (0,1)$. The mapping

$$\begin{aligned} \alpha : (0,1) \times [0,1) &\rightarrow [0,1) \times (0,1) \\ (x,y) &\mapsto (y, x). \end{aligned}$$

  serves as the only coordinate patch.
- The square $D = (0,1] \times (0,1)$. The mapping

$$\begin{aligned} \alpha : (0,1) \times [0,1) &\rightarrow (0,1] \times (0,1) \\ (x,y) &\mapsto (1-y, x). \end{aligned}$$

  The union $C \cup D$ is also a manifold and the two previous patches serve to cover this union.

  Note that in all these sets no corner of the square is included. Any set with a corner, such as for example $[0,1) \times [0,1)$ can not be a manifold. If the manifold, is a manifold with boundary, then the boundary will come from the $y = 0$ line (the x–axis). Pick any point in the boundary which maps to a corner. If we approach that point from the left we will be approaching the corner from a given direction. If we change direction in the domain (say that we approach the point from the right), then we will approach the

Figure 5.1: Figure for problem 1

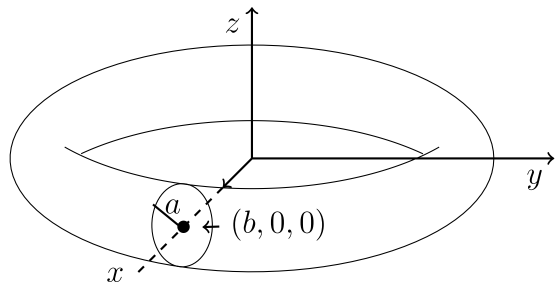


image under α from the other side of the corner. That is from a side which is 90 degrees out of phase with the first direction. A directional derivative can not depend on the sign where we approach the point to. So there is not differentiable function which satisfies a mapping with the manifold pre–requisites.

Section 24: The Boundary of a Manifold

Problem 1.

1. Show that the solid torus is a 3–manifold, and its boundary is the torus T. (See the exercises 17). [*Hint:* Write the equation for T in Cartesian coordinates and apply Theorem 24.4.]

Figure 5.1 shows an sketch of the torus. Referring to Problem 7 of section 17, the solid tours is the image under the cylindrical coordinate transformation

$$g(r, \theta, z) = (r\cos\theta, r\sin\theta, z),$$

for all (r, θ, z) satisfying

$$(r - b)^2 + z^2 \leq a \quad \text{and} \quad 0 \leq \theta \leq 2\pi.$$

The solid torus is generated by rotation the vertical circle centered at $(b, 0, 0)$ with radius a an angle of 2π. This is given, in Cartesian coordinates, by the points (x, y, z) such that $\sqrt{x^2 + y^2} - b)^2 + z^2 \leq a$.

We define f as follows:

$$\begin{array}{rcl} f:\mathbb{R}^3 & \to & \mathbb{R} \\ (x,y,z) & \mapsto & f(x,y,z) = a - (\sqrt{x^2+y^2} - b)^2 - z^2 \end{array}$$

So the points for which $f(x,y,z) = 0$ are the surface of the torus while those where $f(x,y,z) \geq 0$ are the volume of the torus. We see that f is of class C^∞ in the set points such that $f(x,y,z) = 0$ and $Df(x,y,z)$ has rank 1, since

$$\frac{\partial f}{\partial x} = -2(\sqrt{x^2+y^2} - b)\frac{x}{\sqrt{x^2+y^2}} = -2x + \frac{2xb}{\sqrt{x^2+y^2}}$$

similarly

$$\frac{\partial f}{\partial y} = -2y + \frac{2yb}{\sqrt{x^2+y^2}}$$

and $\partial f/\partial z = -2z$ so

$$Df(x,y,z) = \begin{bmatrix} -2x + \frac{2xb}{\sqrt{x^2+y^2}} \\ -2y + \frac{2yb}{\sqrt{x^2+y^2}} \\ -2z \end{bmatrix}$$

To show that Df is of rank one for (x,y,z) such that $f(x,y,z) = 0$, we should show that the tree components of Df can not be simultaneously 0, for which if this happens then

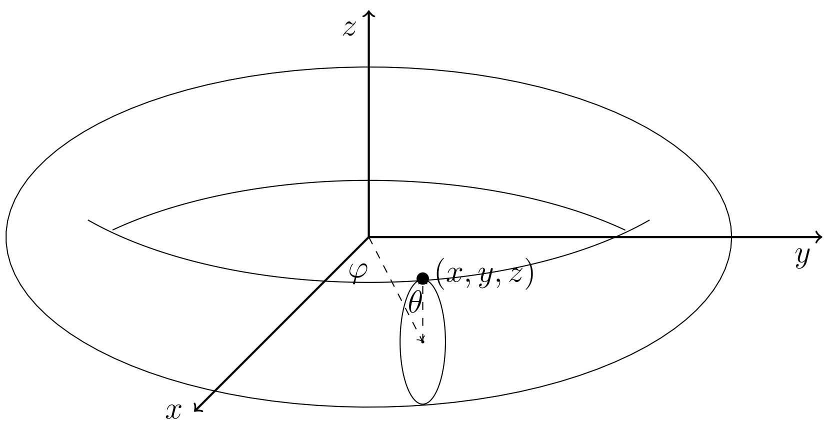
$$z = 0, \quad x = \frac{xb}{\sqrt{x^2+y^2}}, \quad \text{and} \quad y = \frac{yb}{\sqrt{x^2+y^2}}$$

so $x^2 + y^2 = b^2, z = 0$ and then from $f(x,y,z) = 0$ implies that $a = 0$ which is assumed to be strictly positive.

So the solid torus N is a manifold of rank 3 with boundary $T = \partial N$ a 2–manifold.

To appreciate the power of this theorem let us consider a different approach to prove that the surface and solid torus are manifolds. We can use to parameters to characterize the surface of a torus. These are the azimuth angle of a circular cross-section, and the angle along the circular cross–section. Figure 5.2 shows the representation of the two parameters. The azimuthal

Figure 5.2: Figure for problem 1: parameterization based on two angles. The azimuthal angle φ provides the location of the circular cross-section. The angle θ, locates the point within the circular cross–section.



parameter φ and the circular angle θ. The parameterization can be written as

$$(x, y, z) = (b + a\cos\theta)\cos\phi, (b + a\cos\theta)\sin\phi, a\sin\theta).$$

So formally we can write

$$\begin{aligned} \alpha : \mathbb{R}^2 &\to \mathbb{R}^3 & (5.24) \\ (\varphi, \theta) \in A &\mapsto ((b + a\cos\theta)\cos\phi, (b + a\cos\theta)\sin\phi, a\sin\theta) & (5.25) \end{aligned}$$

with $A = [0, 2\pi) \times [0, 2\pi)$. To cover the torus in a unique way we need several patches, since at the points 0 and 2π the inverse of the mapping function is not continuous. It is not hard to prove that this mapping is differentiable and the inverse is continuous for small (smaller than 2π length) intervals. Also that the differential is of rank 2. However, this representation does not tell us anything about the solid body, and without using the resources in Theorem 24.4 it seems hard to come up with an easy proof.

Problem 2.

2. Prove the following:

Theorem. *let* $f : \mathbb{R}^{m+k} \to \mathbb{R}^m$ [3] *be of class* C^r*. Let* M *be the set of all* $\boldsymbol{x}$ *such that* $f(\boldsymbol{x}) = 0$*. Assume that* M *is non–empty and that* $Df(\boldsymbol{x})$ *has rank* m *for* $\boldsymbol{x} \in M$*. Then* M *is a* k*–manifold without boundary in* $\mathbb{R}^{m+k}$*. Furthermore, if* N *is the set of all* $\boldsymbol{x}$ *for which*

$$f_1(\boldsymbol{x}) = \cdots = f_{m-1}(\boldsymbol{x}) = 0 \quad \text{and} \quad f_m(\boldsymbol{x}) \geq 0,$$

and if the matrix

$$\frac{\partial(f_1, \cdots, f_{m-1})}{\partial \boldsymbol{x}}$$

has rank $m-1$ *at each point of* N*, then* N *is a* $k+1$ *manifold, and* $\partial N = M$*.*

sln. From Yang Zeng's solution.

Lemma 1. *Let* $f : \mathbb{R}^{m+k} \to \mathbb{R}^m$ *be of class* C^r*. Assume* Df *has rank* n *at a point* $\boldsymbol{p}$*, then there is an open set* $W \subset \mathbb{R}^{m+k}$ *and a* C^r *function* $G : W \to \mathbb{R}^{m+k}$ *with* C^r *inverse such that* $G(W)$ *is an open neighborhood of* $\boldsymbol{p}$ *and* $f \circ G : W \to \mathbb{R}^n$ *is the projection mapping to the first* m *coordinates.*

proof. We write any point $\boldsymbol{x} \in \mathbb{R}^{m+k}$ as $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ with $\boldsymbol{x}_1 \in \mathbb{R}^m$, and $\boldsymbol{x}_2 \in \mathbb{R}^k$. We know from the hypothesis that Df has rank m. Without loss of generality we can assume that the independent columns of the matrix Df are the first columns, otherwise we could permute the columns until that happens. Since any permutation P has a determinant equal to $(-1)^{\circ(P)}$ where $\circ(P)$ is the order of the permutation, the non–zero hypothesis for the $D_{\boldsymbol{x}_1}$ is still valid for the composition of the permutation with the function f.

Define $F(\boldsymbol{x}) = (f(\boldsymbol{x}), \boldsymbol{x}_2)$, then

$$DF = \begin{bmatrix} D_{\boldsymbol{x}_1} f & D_{\boldsymbol{x}_2} f \\ 0 & I_k \end{bmatrix}.$$

So, $\det DF(\boldsymbol{p}) = \det D_{\boldsymbol{x}_1} f(\boldsymbol{p}) \neq 0$. By the inverse function theorem, there is an open set U of $\mathbb{R}^{m+k}$ containing $\boldsymbol{p}$ such that F carries U in a one–to–one

[3] I use "m" instead of "n''" because it is more convenient for the solution of Problem 4 below

fashion onto the open set W of $\mathbb{R}^{m+k}$ and its inverse is of class C^r. Denote the projection

$$\begin{aligned} \pi : \mathbb{R}^{m+k} &\to \mathbb{R}^m \\ \boldsymbol{x} &\mapsto \boldsymbol{x}_1 \end{aligned}$$

Then

$$f \circ G(\boldsymbol{x}) = (\pi \circ F) \circ G(\boldsymbol{x}) = \pi \circ (F \circ G)(\boldsymbol{x}) = I \circ \pi(\boldsymbol{x}) = \pi(\boldsymbol{x}).$$

on W.

With this we proceed to prove the theorem.

Pick $\boldsymbol{p} \in M$. By the preceding lemma there is a C^r diffeomorphism G between an open set W of $\mathbb{R}^{m+k}$ and an open set U of $\mathbb{R}^{m+k}$ containing $\boldsymbol{p}$, such that $f \circ G = \pi$ on W.

So, by calling, the m–coordinate vector

$$0_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

we have

$$\begin{aligned} U \cap M &= \{\boldsymbol{x} \in U : f(\boldsymbol{x}) = 0\} \\ &= G(W) \cap (f \circ G \circ G^{-1})^{-1}(\{0_m\}) \\ &= G(W) \cap (\pi \circ G^{-1})^{-1}(\{0_m\}) \\ &= G(W) \cap G \circ \pi^{-1}(\{0_m\}) \\ &= G(W \cap \pi^{-1}(\{0_m\})) \\ &= G(W \cap \{0_m\} \times \mathbb{R}^k) \end{aligned}$$

Define $\alpha(x_1, \cdots, x_k) := G(0_m, x_1, \cdots, x_k)$. So α is a k–dimensional coordinate patch on M about $\boldsymbol{p}$. Since $\boldsymbol{p}$ was arbitrarily chosen, we have proved that M is a k–manifold without boundary in $\mathbb{R}^{m+k}$.

Now, $\forall \boldsymbol{p} \in N = \{\boldsymbol{x} : f_1(\boldsymbol{x}) = \cdots f_{m-1}(\boldsymbol{x}) = 0, f_m(\boldsymbol{x}) \geq 0\}$,
there are two cases:

1. $f_m(\boldsymbol{p}) > 0$. Let us define $F := (f_1, \cdots f_{m-1})$, so by an argument similar to that of M, we can find a C^r diffeomorphism G_1 between an open set

W of $\mathbb{R}^{m+k}$ containing $\boldsymbol{p}$ such that $F \circ G_1 = \pi$. Here π is the projection mapping to the first $(m-1)$ coordinates

$$
\begin{aligned}
U \cap N &= U \cap \{\boldsymbol{x} : f_1(\boldsymbol{x}) = \cdots f_{m-1}(\boldsymbol{x}) = 0\} \cap \{\boldsymbol{x} : f_m(\boldsymbol{x}) \geq 0\} \\
&= G_1(W) \cap \{\boldsymbol{x} : f_1(\boldsymbol{x}) = \cdots f_{m-1})(\boldsymbol{x}) = 0\} \cap \{\boldsymbol{x} : f_m(\boldsymbol{x}) \geq 0\} \\
&= G_1(W) \cap (F \circ G_1 \circ G_1^{-1})^{-1}(\{0\}_{m-1}) \cap \{\boldsymbol{x} : f_m(\boldsymbol{x}) \geq 0\} \\
&= G_1(W) \cap (\pi_1 \circ G_1^{-1})^{-1}(\{0\}_{m-1}) \cap \{\boldsymbol{x} : f_m(\boldsymbol{x}) \geq 0\} \\
&= G_1(W) \cap G_1 \circ \pi_1^{-1}(\{0\}_{m-1}) \cap \{\boldsymbol{x} : f_m(\boldsymbol{x}) \geq 0\} \\
&= G_1(W) \cap G_1(\{0\}_{m-1} \times \mathbb{R}^{k+1}) \cap \{\boldsymbol{x} : f_m(\boldsymbol{x}) \geq 0\} \\
&= G_1\big(W \cap (\{0\}_{m-1} \times \mathbb{R}^{k+1})\big) \cap \{\boldsymbol{x} : f_m(\boldsymbol{x}) \geq 0\}
\end{aligned}
$$

When U is sufficiently small, by the continuity of f_m and the fact that $f_m(\boldsymbol{p}) > 0$, we can assume $f_m(\boldsymbol{x}) > 0$, $\forall \boldsymbol{x} \in U$. so

$$
\begin{aligned}
U \cap N &= U \cap \{\boldsymbol{x} : f_1(\boldsymbol{x}) = \cdots f_{m-1}(\boldsymbol{x}) = 0, f_m(x) > 0\} \\
&= G_1\big(W \cap (\{0\}_{m-1} \times \mathbb{R}^{k+1})\big) \cap \{\boldsymbol{x} : f_m(\boldsymbol{x}) > 0\} \\
&= G_1\big(W \cap (\{0\}_{m-1} \times \mathbb{R}^{k+1})\big) \cap G_1 \circ G_1^{-1}\{\boldsymbol{x} : f_m(\boldsymbol{x}) > 0\} \\
&= G_1\big(W \cap (\{0\}_{m-1} \times \mathbb{R}^{k+1}) \cap G_1^{-1}\{\boldsymbol{x} : f_m(\boldsymbol{x}) > 0\}\big)
\end{aligned}
$$

This shows that $\beta(x_1, \cdots, x_{k+1}) := G_1(0_{m-1}, x_1, \cdots, x_{k+1})$ is a $(k+1)$–dimensional manifold patch on N about $\boldsymbol{p}$.

2. $f_m(\boldsymbol{p}) = 0$. Here we note that $\boldsymbol{p}$ is necessarily in M. So $Df(\boldsymbol{p})$ is of rank m and there is a C^r diffeomorphism G between an open set W of $\mathbb{R}^{m+k}$ and an open set U of $\mathbb{R}^{m+k}$ containing $\boldsymbol{p}$, such that $f \circ G = \pi$ on W. So

$$
\begin{aligned}
U \cap N &= \{\boldsymbol{x} \in U : f_1(\boldsymbol{x}) = \cdots f_{m-1}(\boldsymbol{x}) = 0, f_m(\boldsymbol{x}) \geq 0\} \\
&= G(W) \cap (f \circ G \circ G^{-1})^{-1}(\{0_{m-1}\} \times [0, \infty)) \\
&= G(W) \cap (\pi \circ G^{-1})^{-1}(\{0_{m-1}\} \times [0, \infty)) \\
&= G(W) \cap G(\pi^{-1})(\{0_{m-1}\} \times [0, \infty)) \\
&= G(W \cap \pi^{-1}(\{0_{m-1}\} \times [0, \infty)) \\
&= G(W \cap 0_{m-1} \times [0, \infty) \times \mathbb{R}^k).
\end{aligned}
$$

This shows that $\gamma(x_1, \cdots, x_{k+1}) := G(0_{m-1}, x_{k+1} \geq 0, x_1, \cdots, x_k))$ is a $(k+1)$–dimensional coordinate patch on N about $\boldsymbol{p}$.

In summary, we have shown that N is a $(k+1)$–manifold. Lemma 24.2 shows that $\partial N = M$.

Comments Interestingly functions of the type $f(\boldsymbol{x}) = 0$, for $\boldsymbol{x} \in \mathbb{R}^m$ define hyper–surfaces in the m–dimensional space of rank at most, $m-1$. For example a function $f(x_1, x_0) = 0$ defines a curve in the two–dimensional space, a function $f(x_1, x_2, x_3)$ define a surface in the three–dimensional space. Then to define a hypervolume we change the symbol "=" by the symbol "≥" yielding $f(\boldsymbol{x}) \geq 0$. We learned this from linear algebra or linear optimization by selecting lines, planes and hyperplanes.

We also know that the intersection of surfaces yields curves. In this way saying $F : \mathbb{R}^m \to \mathbb{R}^k$ is like saying that we have a set of hyper–surfaces $F = (f_1, \cdots, f_k)$ of rank at most $m-1$ each, but as the number of surfaces start intersecting the rank of the intersection starts reducing at most by 1. So the rank of the resulting surface is at most $m-k$.

This is a powerful theorem and the rest of exercises on this section are applications of this theorem.

Problem 4.

Show that the upper hemisphere of $S^{n-1}(a)$, defined by the equation

$$E_+^{n-1}(a) = S^{n-1}(a) \cap H^n$$

is an $n-1$ manifold. What is its boundary?

sln. Let us write $n = 2 + (n-2) = m + k$, with $m = 2$ and $k = n-2$ in the Theorem shown in exercise 2.

Let us define

$$\begin{aligned} f : \mathbb{R}^{m+k} &\to \mathbb{R}^2 \\ \boldsymbol{x} &\mapsto (f_1, f_2) = (\|x\| - a^2, x_n), \end{aligned}$$

We have the following associations

$$\begin{aligned} M &= \{\boldsymbol{x} : f_1(\boldsymbol{x}) = 0 \quad \wedge \quad f_2(x) = x_n = 0\} \\ E_+^{n-1}(a) = N &= \{\boldsymbol{x} : f_1(\boldsymbol{x}) = 0 \quad \wedge \quad f_2(\boldsymbol{x}) = x_n \geq 0\} \end{aligned}$$

Now

$$Df = \begin{pmatrix} 2x_1 & \cdots & 2x_{n-1} & 2x_n \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

The matrix Df has rank 2 at M, where $x_n = 0$, but at least one of the other components x_i, $1 < i < n$ should be different from zero (since the sum of all squares is a^2) so (using the Theorem in exercise 2) M is a $k = n-2$–manifold without boundary in R^n.

Now,

$$\frac{\partial f_1}{\partial \boldsymbol{x}} = (2x_1, \cdots 2x_n)$$

has rank 1, since not all components could be zero at the same time, so again using the Theorem on exercise 2 we see that N is an $n-1$ manifold such that $\partial N = M = S^{n-2}(a)$.

Section 25: Integrating a Scalar Function Over a Manifold

Before going to the Problem I want to go over the Definition of "measure zero" set. I find it confusing as well as the proof of equivalence right after it.

Definition. *Let M be a compact k–manifold in $\mathbb{R}^n$, of class C^r. A subset D of M is said to have* **measure zero in** *M if it can be covered by countable many coordinate patches $\alpha_i : U_i \to V_i$ such that the set*

$$D_i = \alpha^{-1}(D \cap V_i) \tag{5.26}$$

has measure zero in $\mathbb{R}^k$ for each i.

First, there is a typo in equation 5.26. The mapping α is missing its subindex i. Let us rewrite it correctly:

$$D_i = \alpha_i^{-1}(D \cap V_i) \tag{5.27}$$

Second, let us work the equivalence after the definition. Munkres says: " An equivalent definition is to require that for *any* coordinate patch $\alpha : U \to V$, on M, the set $\alpha^{-1}(D \cap V)$ has measure zero for each i. And this follows from the fact that the set $\alpha_i^{-1}(D \cap V \cap V_i)$ has measure zero because it is a subset of D_i, and that $\alpha^{-1} \circ \alpha_i$ is of class C^r. "

I find Munkres proof confusing, so I want to rephrase it. Being an equivalence, let us prove it in two steps. The obvious implication is that if for any coordinate patch $\alpha : U \to M$, $\alpha^{-1}(D \cap V)$ has measure zero in $\mathbb{R}^k$, then in particular for each set of $\{\alpha_i\}$ countable coordinate patches which form a covering of M the measure of $D_i = \alpha^{-1}(D \cap V_i)$ is zero. The, not obvious implication is this. Assume that for each countable covering patches of M as defined above, we have that the measure of D_i is zero. Then given any arbitrary patch $\alpha : U \to V$, we want to prove that the measure of $\alpha^{-1}(D \cap V)$ is zero. We can create a new set with all countable patches α_i and the patch α. Let us index this new patch as α_J. Adding one patch to a countable set, will leave it countable. So we can apply the hypothesis of the new countable set to say that $\alpha^{-1}(D \cap V_J) = \alpha^{-1}(D \cap V)$ is of measure zero.

Problem 2.

Let $\alpha(t), \beta(t), f(t)$ be real–valued functions of class C^1 on $[0, 1]$, with $f(t) > 0$. Suppose M is a 2–manifold in $\mathbb{R}^3$ whose intersection with the plane $z = t$ is the circle

$$(x - \alpha(t))^2 + (y - \beta(t))^2 = (f(t))^2; \quad z = t \tag{5.28}$$

if $0 \leq t \leq 1$, and is empty otherwise.

(a) Set up an integral for the area of M. [*Hint:* Proceed as in Example 2.]

sln. We define the following patch from $A = (0, 2\pi) \times [0, 1]$ into our manifold M. Note that from equation 5.28 we find that

$$\begin{aligned} f(t)\cos\theta &= x - \alpha(t) \\ f(t)\sin\theta &= y - \beta(t) \end{aligned}$$

from which we can map

$$\begin{aligned} x &= \alpha(t) + f(t)\cos\theta \\ y &= \beta(t) + f(t)\sin\theta \end{aligned}$$

and define the patch

$$\gamma(\theta, t) = (\alpha(t) + f(t)\cos\theta, \beta(t) + f(t)\sin\theta, t)$$

By definition

$$v(M) = \int_A V(D\gamma)$$

and

$$D\gamma = \begin{bmatrix} -f(t)\sin\theta & \alpha'(t) + f'(t)\cos\theta \\ f(t)\cos\theta & \beta'(t) + f'(t)\sin\theta \\ 0 & 1 \end{bmatrix}.$$

To simplify notation let us define

$$c = \cos\theta, \qquad s = \sin\theta$$

and

$$\begin{aligned} g^2(t) &= (D\gamma)_{22} \\ &= (\alpha'(t) + f'(t)c)^2 + (\beta'(t) + f'(t)s^2 + 1 \\ &= 1 + f'^2(t) + \alpha'^2(t) + \beta'^(t) + 2\alpha'(t)f'(t)c + 2\beta'(t)f'(t)s, \end{aligned}$$

then

$$\begin{aligned} V(D\gamma) &= \left(\det\begin{bmatrix} f^2(t) & f(t)(\beta'(t)c.\alpha'(t)s) \\ f(t)(\beta'(t)c - \alpha'(t)s) & g^2(t) \end{bmatrix}\right)^{1/2} \\ &= (f^2(t)g^2(t) - f^2(t)(\beta'^2(t)c^2 + \alpha'^2(t)s^2))^{1/2} \\ &= f(t)\sqrt{(g^2(t) - \beta'^2(t)c^2 - \alpha'^2(t)s^2)} \\ &= f(t)\sqrt{(1 + f'^2(t) + \alpha'^2(t)c^2 + \beta'^2(t)s^2 + 2\alpha'(t)f'(t)c + 2\beta'(t)f'(t)s} \\ &= f(t)\sqrt{1 - f'^2(t) + (\alpha'^(t)c + f'(t))^2 + (\beta'(t)s + f'(t))^2} \end{aligned}$$

The integral to compute the volume is then given by

$$v(M) = \int_0^1 dt \int_0^{2\pi} d\theta f(t)\sqrt{1 - f'^2(t) + (\alpha'^(t)c + f'(t))^2 + (\beta'(t)s + f'(t))^2} \tag{5.29}$$

Before proceeding to the next part of the exercise, let us check our answer for the simplest case

$$\begin{aligned}\alpha(t) &= \text{const}\\ \beta(t) &= \text{const}\\ f^2(t) &= a^2 - t^2\end{aligned}$$

So,

$$2f(t)f'(t) = -2t \Rightarrow f'(t) = -\frac{t}{f(t)}$$

The evaluation of 5.29 turns out into

$$\begin{aligned}v(M) &= \int_0^1 dt \int_0^{2\pi} d\theta f(t)\sqrt{1+f'^2(t)} \qquad (5.30)\\ &= \int_0^1 dt \int_0^{2\pi} d\theta f(t)\sqrt{1+t^2/f^2(t)}\\ &= \int_0^1 dt \int_0^{2\pi} d\theta \sqrt{t^2+f^2(t)}\\ &= \int_0^1 dt \int_0^{2\pi} d\theta\ a \quad \text{this is good news}\\ &= 2\pi a\end{aligned}$$

We do not get $4\pi a^2$ because $t \in [0, 1]$. If $|t| < a$ we would get $4\pi a^2$ as in example 2.

(b) Evaluate when α and β are constant and $f(t) = 1 + t^2$.

sln. From $f(t) = 1 + t^2$ we find $f'(t) = 2t$, and from 5.30

$$\begin{aligned}v(M) &= \int_0^1 dt \int_0^{2\pi} d\theta f(t)\sqrt{1+f'^2(t)}\\ &= \int_0^1 dt \int_0^{2\pi} d\theta (1+t^2)\sqrt{1+4t^2}\\ &= 2\pi \int_0^1 dt (1+t^2)\sqrt{1+4t^2}\\ &= \approx 13.1022.\end{aligned}$$

(c) What form does the integral take when f is constant and $\alpha(t) = 0$ and $\beta(t) = at$? (This integral cannot be evaluated in terms of the elementary functions.)

sln. If f is constant, $\alpha(t) = 0$ and $\beta(t) = at$, then from 5.29

$$\begin{aligned} v(M) &= \int_0^1 dt \int_0^{2\pi} d\theta f \sqrt{1 + (a\sin\theta + f)^2} \\ &= f \int_0^{2\pi} \sqrt{1 + (a\sin\theta + f)^2}\, d\theta \end{aligned}$$

Problem 3.

Consider the torus T of Exercise 7 of §17.

(a) Find the area of this torus. [*Hint:* The cylindrical coordinate transformation carries a cylinder onto T. Parameterize the cylinder using the fact that its cross–section are circles.

Let us rewrite parameterization 5.24 for the torus T

$$\alpha(\theta, \varphi) = (b + a\cos\theta)\cos\varphi, (b + a\cos\theta)\sin\varphi, a\sin\theta) \qquad (5.31)$$

Then

$$D\alpha = \begin{bmatrix} -a\sin\theta\cos\varphi & -\sin\varphi(b + a\cos\theta) \\ -a\sin\theta\sin\varphi & \cos\varphi(b + a\cos\theta) \\ a\cos\theta & 0 \end{bmatrix}$$

so

$$V(D\alpha) = (\det[(D\alpha)^T D\alpha] = \det \begin{bmatrix} a^2 & 0 \\ 0 & (b + a\cos\theta)^2 \end{bmatrix})^{1/2} = a(b + a\cos\theta).$$

and so

$$v(M) = \int_A a(b + a\cos\theta) = 2\pi a \int_0^{2\pi} (b + a\cos\theta) d\theta = 2\pi a(2\pi b) = 4\pi^2 ab.$$

(b) Find the area of that portion of T satisfying the condition $x^2 + y^2 \geq b^2$.

It seems obvious that this corresponds to half of the surface. That is $2\pi^2 ab$, however this is tricky. The inside of the donut has less surface than the outside.

The outside comes from integrating only θ in the interval $[-\pi/2, \pi/2]$. Note that in parameterization 5.31 this corresponds to values of x and y larger than b.

We have

$$v(M) = 2\pi a \int_{-\pi/2}^{\pi/2} (b + a\cos\theta) d\theta = 2\pi a(2a + \pi b) = 2\pi^2 ab + 4\pi a^2.$$

Similarly the inner part of the donut is between the angles $\theta \in [\pi/2, 3\pi/2]$. Note that in parameterization 5.31 this corresponds to values of x and y smaller than b. So,

$$v(M) = 2\pi a \int_{\pi/2}^{3\pi/2} (b + a\cos\theta) d\theta = 2\pi a(-2a + \pi b) = 2\pi^2 ab - 4\pi a^2.$$

So it is interesting that the surface of a sphere with radius $r = a$ is what is the excess of area in the outside, or the defect area in the inside.

Problem 4.

Let M be a compact k–manifold in $\mathbb{R}^n$. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry; let $N = h(M)$. Let $f : N \to \mathbb{R}$ be a continuous function. Show that N is a k–manifold in $\mathbb{R}^k$, and

$$\int_N f dV = \int_M (f \circ h) dV.$$

Conclude that M and N have the same volume.

sln. Solution, from Yang Zeng's document.

We start by showing that N is a k–manifold in $\mathbb{R}^n$.

Let $\{\alpha_j\}$ be a family of coordinate patches that covers M. We show that $\{h \circ \alpha_j\}$ is a family of coordinate patches that covers N.

Each isometry is a diffeomorphism (the opposite is not always true). That is if if $|h(\boldsymbol{x}) - h(\boldsymbol{y})| = |\boldsymbol{x} - \boldsymbol{y}|$ then whatever is done in one domain does not changes distances in the other domain. Theorem 20.6 indicates that each isometry is an orthogonal transformation followed by a translation. That is

$$h(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{p}.$$

So the differential of h is $Dh = A$ and since A is orthogonal

Now, let the family of coordinate patches $\{\alpha_j\}$ cover the manifold M. That is

$$M \subset \cup_i \alpha_i(U_i) = \cup V_i,$$

so

$$N = h(M) \subset \cup_i (h \circ \alpha_i)(U_i)$$

so the family $\{h \circ \alpha_j\}$ covers N and since h is a diffeomorphism we can assure that this family is an atlas covering N. Hence, N is a manifold.

To evaluate the volume we follow Zeng. Supprose $\phi_1, \cdots \phi_l$ is a partition of unity on M that is dominated by $\{\alpha_j\}$, then $\phi_1 \circ h^{-1}, \cdots \phi_l \circ h^{-1}$ is a partition of unity on N that is dominated by $h \circ \alpha_j$. I prove this.

We want to show the three attributes under the Lemma 25.2. That is:

(a) $\phi_i \circ h^{-1} \geq 0.$

Since ϕ_i is a partition of unity, then for all $\boldsymbol{x} \in \mathbb{R}^n$ $\phi(\boldsymbol{x}) \geq 0$. In particular $\phi \circ h^{-1}(\boldsymbol{x}) \geq 0$. for each $\boldsymbol{x} \in \mathrm{R}^n$.

(b) Since, for each i, the support of ϕ_i is compact and there is a coordinate patch $\alpha_i : U_i \to V_i$ belonging to the given covering such that

$$((\text{Support } \phi_i) \cap M) \subset V_i,$$

then there is a coordinate patch $h \circ \alpha_i$ such that

$$((\text{Support } h \circ \phi_i) \cap N) \subset W_i,$$

(c)

$$\sum \phi_i \circ h^{-1}(\boldsymbol{x}) = \sum \phi_i(\boldsymbol{y})$$

with $\boldsymbol{y} = h^{-1}(\boldsymbol{x}) \in \mathbb{R}^n$. Hence by the property (3) of Lemma 25.2,

$$\sum \phi_i \circ h^{-1}(\boldsymbol{x}) = 1.$$

We then showed that the family $\{h \circ \phi_i^{-1}\}$ is a partition of unity on N dominated b y $\{h \circ \alpha_j\}$.

Let us now evaluate the integral

$$\begin{aligned}
\int_N f\,dV &= \sum_{i=1}^{l} \int_N (\phi_i \circ h^{-1}) f\,dV \\
&= \sum_{i=1}^{l} \int_{\mathrm{Int}U_i} (\phi_i \circ h^{-1} \circ h \circ \alpha_i)(f \circ h \circ \alpha_i) V(D(h \circ \alpha_i)) \\
&= \sum_{i=1}^{l} \int_{\mathrm{Int}U_i} (\phi_i \circ \alpha_i)(f \circ h \circ \alpha_i) V(D\alpha_i) \\
&= \sum_{i=1}^{l} \int_M \phi_i (f \circ h)\,dV \\
&= \int_M (f \circ h)\,dV.
\end{aligned}$$

In particular, by setting $f \equiv 1$, we get $v(N) = v(M)$.

Problem 5.

(a) Express the volume of $S^n(a)$ in terms of the volume $B^{n-1}(a)$. [*Hint:* Follow the pattern of Example 2.]

sln, method 1. We can write

$$\begin{aligned}
S^n(a) &= \{(x_1, \cdots, x_{n+1})\ ,\ x_1^2 + \cdots + x_{n+1}^2 = a^2\} \\
&= \{(x_1, \cdots, x_{n-1})\ ,\ x_1^2 + \cdots + x_{n-1}^2 = a^2 \cos^2\theta\} \\
&\quad \times \{(x_n, x_{n+1})\ ,\ x_n^2 + x_{n+1}^2 = a^2 \sin^2\theta\}
\end{aligned}$$

(this equality is easy to show and I will omit its proof)

Then we parameterized the sphere based on the single parameter θ which we integrate between 0 and $\pi/2$. That is

$$v(S^n(a)) = \int_0^{\pi/2} v(S^{n-2}(a\cos\theta))\, v(S^1(a\sin\theta)) J d\theta$$

The product of the two volumes is taken because we are integrating over cross product of independent spaces (for each fixed θ). The Jacobian $J = a$ comes from the transformation from rectangular coordinates to polar (a, θ) coordinates. To get this Jacobian requires a good amount of work. It is obvious for 2D when we say that if $x_1^2 + x_2^2 = a^2$, $x_1 = a\cos\theta$ and $x_2 = a\sin\theta$, then the Jacobian

$$J = \det\left(\frac{\partial(x_1, x_2)}{\partial(a, \theta)}\right) = a.$$

For higher dimensions the work is more complicated.

We then have,

$$\begin{aligned} v(S^n(a)) &= a\int_0^{\pi/2} v(S^{n-2}(a\cos\theta))\ 2\pi(a\sin\theta))d\theta \\ &= 2\pi a\int_0^a v(S^{n-2}(\rho))d\rho \\ &= 2a\pi v(B^{n-1}(a)). \end{aligned}$$

with the substitution $\rho = a\cos\theta$, $d\rho = -a\sin\theta d\theta$, and recognizing that

$$v(B^{n-1}(a)) = \int_0^a v(S^{n-2}(\rho))d\rho. \tag{5.32}$$

This integral is easy to see as thinking that an onion is the union of all its concentric shells.

To verify the result let us consider a few cases.

- For $n = 2$

$$\begin{aligned} v(S^2(a)) &= 4\pi a^2, \\ 2\pi a B^1(a) &= (2\pi a)(2\pi\ a) = 4\pi a^2 \end{aligned}$$

- For $n = 3$

$$\begin{aligned} v(S^3(a)) &= 2\pi^2 a^3, \\ 2\pi a B^2(a) &= (2\pi a)(\pi^2\ a^2) = 2\pi^2 a^3 \end{aligned}$$

- and for $n = 4$

$$v(S^4(a)) = \frac{8}{3}\pi^2 a^4,$$
$$2\pi a B^3(a) = (2\pi a)(\frac{4}{3}\pi a^3) = \frac{8}{3}\pi^2 a^3$$

So,

$$\frac{v(S^n(a))}{v(B^{n-1}(a))} = 2\pi a. \tag{5.33}$$

sln, method 2. Let us define the coordinate patch

$$\begin{aligned} \alpha : \mathbb{R}^n &\rightarrow \mathbb{R}^{n+1} \\ (x_1, \cdots x_n) &= (x_1, \cdots x_n, f(x_1, \cdots, x_n)) \end{aligned} \tag{5.34}$$

with

$$f(x_1, \cdots x_n) = \pm \sqrt{a^2 - \sum_{i=1}^{n} x_i^2}$$

Then,

$$D\alpha = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 \\ f_{,1} & \cdots & \cdots & f_{,n} \end{pmatrix}$$

where

$$f_{,j} = \frac{\partial f}{\partial x_j} = \mp \frac{x_j}{\sqrt{a^2 - \sum_{i=1}^{n} x_i^2}} \tag{5.35}$$

So

$$A = (D\alpha)^T (D\alpha) = \begin{pmatrix} 1 + f_{,1}^2 & f_{,1} f_{,2} & \cdots & f_{,1} f_{,n} \\ f_{,1} f_{,2} & 1 + f_{,2}^2 & \cdots & f_{,2} f_{,n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{,1} f_{,n} & \cdots & \cdots & 1 + f_{,n}^2 \end{pmatrix}$$

Before we evaluate the determinant, we see that for $n = 1, 2, 3$ the computation is trivial and yields

$$\det A = 1 + \sum_{i=1}^{n} f_{,i}^2. \tag{5.36}$$

We show that this is the case in general.

To evaluate the determinant of this matrix we observe that any entry has the form of

$$a_{ij} = \delta_{ij} + f_{,i} f_{,j},$$

and using the index notation for the determinant, we see that

$$\begin{aligned}
\det A &= \epsilon_{i_1 \cdots i_n} a_{1i_1} \cdots a_{ni_n} \\
&= \epsilon_{i_1 \cdots i_n} (\delta_{1i_1} + f_{,1} f_{,i_1}) \cdots (\delta_{ni_n} + f_{,n} f_{,i_n}) \\
&= \epsilon_{PQ} \sum_{P,Q} \prod_{P,Q} \delta_{IP} f_{,I} f_{,Q}
\end{aligned}$$

Where P and Q are all possible two ordered partitions of the set $1, \cdots, n$, and $I = (1, \cdots, n)$.

Here

$$\begin{aligned}
\delta_{IP} = \delta_{1i_1} \cdots \delta_{ki_k}, \qquad & P = (i_1 \cdots i_k) \\
f_{,Q} = f_{,j_1} \cdots f_{,j_{n-k}}, \qquad & Q = (j_1, \cdots j_{,n-k}).
\end{aligned}$$

We will add these partitions according to the cardinal number $\#P$. If $\#P = n$ then P is a permutation of I. Any permutation of P of I that is not the identity will produce some $\delta_{ij} = 0$, so the only contribution to the sum of permutations of $\#P = n$ comes from the identity and this contribution is 1, since the $f_{,Q}$ does not even enter into the picture (Q is the empty set ϕ).

Next let us assume $\#P = n - 1$. Then $\#Q = 1$, so each term in this collection will have $n - 1$ factors of δ's and one factor of the type $f_{,i} f_{,j}$. If $i \neq j$ then one of the deltas is of the form $\delta_{mn} = 0$ with $m \neq n$, since if all of them roll along the diagonal, the reminding i has to be that in the set Q. The total contributions for this case ($\#P = n-1$) is given by

$$\sum_{i=1}^{n} f_{,i}^2$$

We used equation 5.35

If $\#P = n-2$, so $\#Q = 2$, then. For the δ not to be zero, all the P members should be images of the identity, but 2 Q members are free to be permuted. Half of the permutations of those two members is even and half is odd. From the definition of the antisymmetric tensor, this brings the sum to 0.

If, in general, $\#P < n-1$, $\#Q > 2$, we see that half of the permutations of those elements of Q are even and half are odd. This yields a total sum of 0.

We found then that

$$\det A = 1 + \sum_{i=1}^{n} f_{,i}^2 = 1 + \sum_{i=1}^{n} \frac{x_i^2}{a^2 - \sum_{j=1}^{n} x_j^2} = \frac{a^2}{a^2 - \sum_{i=1}^{n} x_i^2} \tag{5.37}$$

as indicated in equation 5.36.

From the theory of manifolds, the volume surface volume can be computed as

$$v(S^n(a)) = 2 \int_{B^n(a)} \frac{a}{\sqrt{a^2 - \sum_{i=1}^{n} x_i^2}}$$

The 2 comes about because the two patches for $\pm$ in the function f. That is the upper and lower hemisphere, which have the same volume.

To evaluate this integral we assume that we can apply Fubini's rule. When the denominator goes to zero, the function is not bounded and hence the Fubini's rule will be invalid. Instead we assume that the denominator is bounded away from zero (that is, we are not yet in the sphere but inside) and apply Fubini's rule. Then we can take the limit as we approach the sphere after Fubini's rule.

That is,

$$v(S^n(a)) = 2 \lim_{\epsilon \to 0} \int_{B^{n-1}(a)} dx_1 \cdots dx_{n-1} \int_{-b+\epsilon}^{b-\epsilon} dx_n \frac{a}{\sqrt{b^2 - x_n^2}}$$

with

$$b^2 = a^2 - \sum_{i=1}^{n-1} x_i^2.$$

To evaluate the last integral we perform the substitution

$$x_n = b\sin\theta, \qquad dx_n = b\,\cos\theta d\theta,$$

which yields

$$\int_{\arcsin(-1+\epsilon)}^{\arcsin(1-\epsilon)} d\theta\, \cancel{b\cos\theta} \frac{a}{\cancel{b\cos\theta}},$$

and since $\arcsin(-1+\epsilon/b) \to -\pi/2$ and $\arcsin(1-\epsilon/b) \to \pi/2$ as $\epsilon \to 0$, then

$$v(S^n(a)) = 2\pi a \int_{B^{n-1}(a)} dx_1 \cdots dx_{n-1} = 2\pi a B^{n-1}(a).$$

(b) Show that for $t > 0$,

$$v(S^n(t)) = Dv(B^{n+1}(t)).$$

[*Hint:* Use the result of Exercise 6 of 19.] I use equation, 5.32, for $n+1$, instead of $n-1$, and t instead of a. That is,

$$v(B^{n+1}(t)) = \int_0^t v(S^n(\rho))d\rho.$$

I do not derive this equation. It should be obvious from calculus after integrating along the radial direction concentric isotropic shells.

So from, the fundamental theorem of calculus, it is obvious that

$$v(S^n(t)) = Dv(B^{n+1}(t)). \tag{5.38}$$

Problem 6.

The centroid of a compact manifold M in $\mathbb{R}^n$ is defined by a formula like that given in Exercise 3 of §22. Show that if M is symmetric with respect to the subspace $x_i = 0$ of $\mathbb{R}^n$, then $c_i(M) = 0$.

sln. It seems natural that the centroid of anything should lie in its center of symmetry. This is an easy exercise of calculus. However the formality of this using manifold rhetoric needs to be trained. I copied this proof from Yang Zeng's who has a good grasp on the language.

Let $H_i^+ = \{\boldsymbol{x} \in \mathbb{R}^n : x_i > 0\}$. Then $M \cap H_i^+$ is a manifold, for if $\alpha : U \to V$ is a coordinate patch on M, $\alpha : U \cap \alpha^{-1}(H_i^+) \to V \cap H_i^+$ is a coordinate patch on $M \cap H_i^+$. Similarly, if we let $H_i^- = \{\boldsymbol{x} \in \mathbb{R}^n : x_i < 0\}$, then $M \cap H_i^-$ is manifold.

Theorem 25.4 implies

$$c_i(M) = \frac{1}{v(M)} \int_M \pi dV = \frac{1}{v(M)} \left[\int_{M \cap H_i^+} \pi dV + \int_{M \cap H_i^-} \pi dV \right].$$

Suppose (α_j) is a family of coordinate patches on $M \cap H_i^+$ and $\{\phi_i\}_1^l$ a partition of unity on $M \cap H_i^+$ that is dominated by (α_j), then

$$\int_{M \cap H_i^+} \pi_i dV = \sum_{j=1}^{l} \int_M (\phi_j \pi_i) dV = \sum_{j=1}^{l} \int_{\operatorname{Int} U_j} (\phi_j \circ \alpha_j)(\pi_i \circ \alpha_j) V(D\alpha_j).$$

Define $f : \mathbb{R}^n \to \mathbb{R}^n$ by $f(\boldsymbol{x}) = (x_1, \cdots, -x_i, \cdots, x_n)$. It is easy to see $(f \circ \alpha_j)$ is a family of coordinate patches on $M \cap H_i^-$ and $\phi_1 \circ f, \cdots, \phi_l \circ f$ is a partition of unity on $M \cap H_i^-$ that is dominated by $(f \circ \alpha_j)$. Therefore

$$\begin{aligned} \int_{M \cap H_i^-} \pi_i dV &= \sum_{j=1}^{l} \int_{\operatorname{Int} U_j} (\phi_j \circ f \circ f \circ \alpha_j)(\pi_i \circ f \circ \alpha_j) U(D(f \circ \alpha_j)) \\ &= \sum_{j=1}^{l} \int_{\operatorname{Int} U_j} (\phi_j \circ \alpha_j)(\pi_i \circ f \circ \alpha_j) U(D(f \circ \alpha_j)). \end{aligned}$$

In order to show that $c_i(M) = 0$, it suffices to show

$$(\pi_i \circ \alpha_j) V(D\alpha_j) = -(\pi_i \circ f \circ \alpha_j) V(D(f \circ \alpha_j) V(D(f \circ \alpha_j)).$$

Indeed,

$$\begin{aligned} V^2(D(f \circ \alpha_j))(\boldsymbol{x}) &= V^2(Df(\alpha_j(\boldsymbol{x}))D\alpha_j(\boldsymbol{x})) \\ &= \det\big((D\alpha_j(\boldsymbol{x}))^T Df(\alpha_j(\boldsymbol{x}))^T Df(\alpha_j(\boldsymbol{x}))D\alpha_j(\boldsymbol{x})\big) \\ &= \det\big((D\alpha_j(\boldsymbol{x}))^T D\alpha_j(\boldsymbol{x})\big) \\ &= V^2(D\alpha(\boldsymbol{x})) \end{aligned}$$

and $\pi_i \circ f = -\pi_i$. Combined, we conclude that

$$\int_{M\cap H_i^-} \pi_i dV = -\int_{M\cap H_i^+} \pi_i dV,$$

and so

$$c_i(M) = 0.$$

Problem *7.

Let $E_+^n(a)$ denote the intersection of $S^n(a)$ with the upper half–space H^{n+1}. Let $\lambda_n = v(B^n(1))$.

(a) Find the centroid of $E_+^n(a)$ in terms of λ_n and λ_{n-1}.

sln. From exercise 4 of the section 25, we know that $E_+^n(a)$ is a manifold whose boundary is $S^{n-1}(a)$.

From the definition 5.22 we have

$$C_i(Y_\alpha) = \frac{1}{v(Y_\alpha)} \int_A \pi_i V, \tag{5.39}$$

Here $Y_\alpha = E_+^n$.

To find the volume $v(Y_\alpha)$ we observe that, from equation 5.33,

$$\begin{aligned} v(S^n(a)) &= 2\pi a v(B^{n-1}(a)) \\ &= 2\pi a^n B^{n-1}(1) \\ &= 2\pi a^n \lambda_{n-1}. \end{aligned}$$

However Y_α is only half of the sphere so,

$$v(Y_\alpha) = \pi a^n \lambda_{n-1}. \tag{5.40}$$

From the previous problem (Exercise 6) we know that the centroid c_k with respect to any direction $k = 1, \cdots, n$ is 0, since the manifold is symmetric around those directions and each x_k takes both positive and negative values there. That is

$$c_k = 0 \qquad 1 \le k \le n.$$

We should only find c_{n+1} in the upper H^{n+1} upper space.

We use the coordinate patch defined in equation 5.34. That is,

$$\begin{aligned} \alpha : \mathbb{R}^n &\rightarrow \mathbb{R}^{n+1} \\ (x_1, \cdots, x_n) &= (x_1, \cdots, x_n, f(x_1, \cdots, x_n)) \end{aligned}$$

with

$$f(x_1, \cdots, x_n) = \sqrt{a^2 - \sum_{i=1}^{n} x_i^2}$$

We found (see equation 5.37) that

$$\det(D\alpha)^T(D\alpha) = \frac{a^2}{a^2 - \sum_{i=1}^n x_i^2}$$

and so

$$V(D\alpha) = \frac{a}{\sqrt{a^2 - \sum_{i=1}^n x_i^2}} = \frac{a}{f} = \frac{a}{x_{n+1}} \tag{5.41}$$

where

$$f = x_{n+1},$$

So, from equations 5.39, 5.40 and 5.41

$$c_{n+1} = \frac{1}{\pi a^n \lambda_{n-1}} \int_A x_{n+1} \frac{a}{x_{n+1}}$$

Since A is the set such that

$$x_1^2 + \cdots + x_n^2 \leq a^2,$$

$$v(A) = v(B^n(a)) = a^n \lambda_n, \tag{5.42}$$

(I show the second equation in my notes on PDE) then

$$\begin{aligned} c_{n+1} &= \frac{1}{\pi a^n \lambda_{n-1}} a^{n+1} \lambda_n \\ &= \frac{a\, \lambda_n}{\pi \lambda_{n-1}}, \end{aligned}$$

and

$$c_i \quad = \quad \delta_{i\; n+1} \frac{a\,\lambda_n}{\pi\,\lambda_{n-1}}, \tag{5.43}$$

Let us check this formula against known cases in small dimensions. Some first few values of λ_i are

$$\begin{aligned} \lambda_0 &= 1 \\ \lambda_1 &= 2 \\ \lambda_2 &= \pi \\ \lambda_3 &= \frac{4}{3} \end{aligned}$$

So, the last (only non–trivial) coordinate c_{n+1} on each case is

$$\begin{aligned} c_1 \quad &= \quad \frac{2a}{\pi} \quad \text{semicircle} \\ c_2 = \frac{\pi a^2}{\pi(2a)} \quad &= \quad \frac{a}{2} \quad \text{see 5.23} \\ c_3 = \frac{4/3\ \pi a^3}{\pi^2 a^2} \quad &= \quad \frac{4a}{3\pi} \end{aligned}$$

(b) Find the centroid of $E^n_+(a)$ in terms of the centroid of $B^{n-1}_+(a)$. (See the exercises of §19.)

From equations 5.43, and 5.42 we find

$$c_i = \delta_{i\; n+1} \frac{a v(B^n(a))/a^n}{\pi v(B^{n-1}(a)/a^{n-1}} = \delta_{i\; n+1} \frac{1}{\pi} \frac{v(B^n(a))}{v(B^{n-1}(a))}$$

It is interesting to observe that as $n \to \infty$, $v(B^n(a)) \to 0$, $v(S^n a) \to 0$ and $c_i \to 0$, even for $i = n+1$.

Problem 8.

Let M and N be compact manifolds without boundary in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively,

(a) Let $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ be continuous. Show that

$$\int_{M\times N} f\cdot g dV = \left[\int_M f\ dV\right]\left[\int_N g\ dV\right].$$

[*Hint:* Consider the case where the supports of f and g are contained in coordinate patches.]

sln. From Yan Zeng's document. Let $\{\alpha_i\}$ be a family of coordinate patches on M and $\phi_1, \cdots, \phi_l$ a partition of unity on M dominated by $\{\alpha_i\}$. Let $\{\beta_j\}$ be a family of coordinate patches on N and $\psi_1, \cdots, \psi_k$ a partition of unity on N dominated by $\{\beta_j\}$. Then it is easy to show that $\{(\alpha_i, \beta_j)\}_{i,j}$ is a family of coordinate patches on $M \times N$ and $\{\phi_m \psi_n\}_{1 \leq m \leq l, 1 \leq n \leq k}$ is a partition of unity on $M \times N$ dominated by $\{(\alpha_i, \beta_j)\}_{i,j}$.

$$\begin{aligned}
\int_{M\times N} f \cdot g dV &= \sum_{\substack{1\leq m\leq l\\ 1\leq n\leq k}} \int_{M\times N} (\phi_m f)(\phi_n g) dV \\
&= \sum_{\substack{1\leq m\leq l\\ 1\leq n\leq k}} \int_{\text{Int}U_m \times \text{Int}V_n} (\phi_m \circ \alpha_m \cdot f \circ \alpha_m) V(D\alpha_m) \\
&\qquad (\psi_n \circ \beta_m \cdot g \circ \beta_n) V(D\beta_n) \\
&= \sum_{\substack{1\leq m\leq l\\ 1\leq n\leq k}} \int_{\text{Int}U_m} (\phi_m \circ \alpha_m \cdot f \circ \alpha_m) V(D\alpha_m) \\
&\qquad \int_{\text{Int}V_n} (\psi_n \circ \beta_n \cdot g \circ \beta_n) V(D\beta_n) \\
&= \left[\sum_{1\leq m\leq l} \int_{\text{Int}U_m} (\phi_m \circ \alpha_m \cdot f \circ \alpha_m) V(D\alpha_m)\right] \\
&\qquad \left[\sum_{1\leq n\leq k} \int_{\text{Int}V_n} (\psi_n \circ \beta_n \cdot g \circ \beta_n) V(D\beta_n)\right] \\
&= \left[\int_M f\, dV\right]\left[\int_N g\, dV\right].
\end{aligned}$$

(b) Show that $v(M \times N) = v(M) \cdot v(N)$.

sln. Set $f = g = 1$ in (a).

(c) Find the area of the 2–manifold $S^1 \times S^1$ in $\mathbb{R}^4$. In (b), we see that

$$v(S^1 \times S^1) = v(S^1) \cdot v(S^1) = (2\pi)(2\pi) = 4\pi^2.$$

Chapter 5: Differential Forms

Section 26: Multilinear Algebra

Problem 4.

Determine which of the following are tensors in $\mathbb{R}^4$, and express those that are in terms of the elementary tensors in $\mathbb{R}^4$.

(a)

$$f(\boldsymbol{x}, \boldsymbol{y}) = 3x_1y_2 + 5x_2x_3.$$

sln. Here we have $V^1 = V^2 = \mathbb{R}^4$. For, f to be a tensor, f should be linear in both $\boldsymbol{x}$ and $\boldsymbol{y}$. However the crossing term x_2x_3 spoils linearity since

$$f(c\,\boldsymbol{x}, \boldsymbol{y}) = 3\,c\,x_1y_2 + 5\,c^2\,x_2x_3 \neq c\,(f(\boldsymbol{x}, \boldsymbol{y}).$$

(b)

$$g(\boldsymbol{x}, \boldsymbol{y}) = x_1y_2 + x_2y_4 + 1,$$

sln. The constant 1 spoils linearity. A linear operator should map 0 to 0. We see that

$$g(c\,\boldsymbol{x}, \boldsymbol{y}) = c\,x_1y_2 + c\,x_2y_4 + 1 \neq c\,g(\boldsymbol{x}, \boldsymbol{y}).$$

sln. (since 1 does not get scaled).

(c)

$$h(\boldsymbol{x}, \boldsymbol{y}) = x_1y_1 - 7x_2y_3.$$

sln. h is a tensor. It is linear in both $\boldsymbol{x}$ and $\boldsymbol{y}$, now

$$h(\boldsymbol{x},\boldsymbol{y}) = x_1y_1 - 7x_2y_3 = \phi_1\phi_1 - 7\phi_2\phi_3 = \phi_{1,1} - 7\,\phi_{2,3}.$$

Problem 5.

Repeat Exercise 4 for the functions

(a)

$$f(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = 3x_1x_2z_3 - x_3y_1z_4.$$

sln. The crossing term $x_1x_2z_3$ breaks linearity. So f is not a tensor.

(b)

$$g(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{u},\boldsymbol{v}) = 5x_3\,y_2\,z_3\,u_4\,v_4.$$

sln. Here g is a tensor and we can write it as

$$g = 5\,\phi_{3,2,3,4,4}.$$

Problem 6.

Let f and g be the following tensors in $\mathbb{R}^4$.

$$\begin{aligned} f(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) &= 2x_1\,y_2\,z_2 - x_2\,y_3\,z_1, \\ g &= \phi_{2,1} - 5\phi_{3,1} \end{aligned}$$

(a) Express $f \otimes g$ as a linear combination of elementary 5–tensors.

sln.

$$f = 2\,\phi_{1,2,2} - \phi_{2,3,1}, \qquad g = \phi_{2,1} - 5\,\phi_{3,1}$$

and

$$f \otimes g = 2\,\phi_{1,2,2,2,1} - 10\,\phi_{1,2,2,3,1} - \phi_{2,3,1,2,1} + 5\,\phi_{2,3,1,3,1}$$

(b) Express $(f \otimes g)(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{u},\boldsymbol{v})$ as a function

sln.

$$(f\otimes g)(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{u},\boldsymbol{v}) = 2\,x_1\,y_2\,z_2\,u_2\,v_1 - 10\,x_1\,y_2\,z_2\,u_3\,v_1 - x_2\,y_3\,z_1\,u_2\,v_1 + 5\,x_2\,y_3\,z_1\,u_3\,v_1.$$

Problem 7.

Show that the four properties stated in Theorem 26.4 characterize the tensor product uniquely, for finite–dimensional spaces V.

sln. Assume the following tensors $f_1 = \sum_I f_{I1}\phi_I$ in V , $g_1 = \sum_J g_{I1}\,\psi_J$. If $f_2 = \sum_K f_{K2}\alpha_K = f_1$, then because the tensor is uniquely defined, then $I = K$, $\phi_I = \alpha_K$, and $f_{I1} = f_{K2}$. In the same way, if $g_2 = \sum_L g_{L2}\beta_L = g_1$ then $J = L$, $\psi_J = \beta_L$, and $g_{J1} = g_{L2}$.

So

$$\begin{aligned}
f_1\otimes g_1 &= \left(\sum_I f_{I1}\phi_I\right)\left(\sum_J g_{J1}\psi_I\right)\\
&= \left(\sum_I\sum_J f_{I1}\phi_I g_{J1}\psi_I\right)\\
&= \sum_{I,J} f_{I1}g_{J1}\psi_I\phi_J\\
&= \sum_{K,L} f_{K2}g_{L2}\alpha_K\beta_L\\
&= \left(\sum_K\sum_L f_{K2}\phi_L g_{J2}\alpha_K\right)\\
&= f_2\otimes g_2,
\end{aligned}$$

so the tensor product defines a unique representation.

Problem 8.

Let f be a 1–tensor in $\mathbb{R}^n$; then $f(\boldsymbol{y}) = A\cdot\boldsymbol{y}$ for some matrix A of size 1 by n. If $T : \mathbb{R}^m \to \mathbb{R}^n$ is the linear transformation $T(\boldsymbol{x}) = B\cdot\boldsymbol{x}$, what is the matrix of the 1–tensor T^*f on $\mathbb{R}^m$?

sln. Pick $\boldsymbol{x} \in \mathbb{R}^m$. Then

$$T^* f(\boldsymbol{x}) = f(T(\boldsymbol{x})) = f(B \cdot \boldsymbol{x}) = A \cdot B \cdot \boldsymbol{x} = (AB) \cdot \boldsymbol{x}$$

So the matrix of the $T^* f$ on $\mathbb{R}^m$ is AB.

Section 27: Alternating Tensors

Problem 1.

Which of the following are alternating tensors in $\mathbb{R}^4$?

(a)

$$f(\boldsymbol{x}, \boldsymbol{y}) = x_1\, y_2 - x_2\, y_1 + x_1 y_1$$

sln. f is a tensor. Now There are only two options $f(\boldsymbol{x}, \boldsymbol{y})$ and $f(\boldsymbol{y}, \boldsymbol{x})$, where

$$f(\boldsymbol{y}, \boldsymbol{x}) = y_1 x_2 - y_2 x_1 + y_1 x_1,$$

We see that $f(\boldsymbol{y}, \boldsymbol{x}) \neq -f(\boldsymbol{x}, \boldsymbol{y})$ if $x_1 y_1 \neq 0$. So f is not alternating.

(b)

$$g(\boldsymbol{x}, \boldsymbol{y}) = x_1 y_3 - x_3 y_3.$$

g is a tensor. Now,

$$g(\boldsymbol{y}, \boldsymbol{x}) = y_1 x_3 - y_3 x_3 \neq -(x_1 y_3 - x_3 y_3)$$

in general. For example choose $x_1 = y_1 = 0$, and $x_3 y_3 \neq 0$. So g is not alternating.

(c)

$$h(\boldsymbol{x}, \boldsymbol{y}) = x_1^3\, y_2^3 - x_2^3 y_1^3.$$

h is not multilinear, so it is not even a tensor.

Problem 2.

Let $\sigma \in S_5$ be a permutation such that

$$(\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5)) = (3, 1, 4, 5, 2).$$

Use the procedure given in the proof of Lemma 27.1 to write σ as a composite of elementary permutations.

sln. I personally believe that Lemma 27.1 is quite confuse and this exercise shows my point.

Here $i = 0$ according to the definition in the Lemma. That is, no element is fixed. So, the sequence $1, \cdots, i-1 = -1$ does not have sense and this invalidate some of the statements in the Lemma. What is $\sigma(0)$? We have then to assume to get started that $i = 1$, otherwise we can not even get started.

Instead I proof the theorem stating the bubble sort algorithm from computer science.

Let us assume that we have a sequence $\sigma = (a_1, \cdots a_n)$, that we want to sort into ascending order. If the sequence is of numbers from 1 to n then the final sequence, after sorting is $\sigma_0 = (1, \cdots, n)$.

The bubble sort algorithm consists of two loops. The internal loop takes the largest of each of the remainder (non yet sorted) elements to the top. This is done by comparing each element with the next neighbor and pushing it up if it is bigger than its following neighbor. Once the largest element is on the top, then the outer loop takes control by sorting only the remaining elements, in the same fashion.

Here there is a C++ implementation of the bubble sort algorithm:

```
void bubbleSort(int arr[], int n) {
      bool swapped = true;
      int j = 0;
      int tmp;
      while (swapped) {
            swapped = false;
            j++;
            for (int i = 0; i < n - j; i++) {
                  if (arr[i] > arr[i + 1]) {
              // swap two consecutive elements
                        tmp = arr[i];
                        arr[i] = arr[i + 1];
                        arr[i + 1] = tmp;
                        swapped = true;
                  }
```

```
            }
        }
}
```

Each swap is an elementary permutation e_i. Once the array is sorted, the final result is the identity array σ_0, and so

$$\sigma_0 = \sigma_k \circ \sigma_{k-1} \cdots \circ \sigma_1 \circ \sigma,$$

where k is the total number of permutations done. Each σ_i is a swap. Then because each elementary permutation is its inverse, we can write

$$\sigma = \sigma_1 \circ \sigma_2 \cdots \circ \sigma_k.$$

We illustrate the bubble sort with this exercise. Pop up the first bubble (that is, take 5 to the top)

$$\sigma = (3, 1, 4, 5, 2) \Rightarrow e_1 \circ \sigma = (1, 3, 4, 5, 2) \Rightarrow e_4 \circ e_1 \circ \sigma = (1, 3, 4, 2, 5).$$

Pop up the second bubble (that is, take 4 to the the fourth place)

$$e_3 \circ e_4 \circ e_1 \circ \sigma = (1, 3, 2, 4, 5).$$

Pop up the third element into its position 3.

$$e_2 \circ e_3 \circ e_4 \circ e_1 \circ \sigma = (1, 2, 3, 4, 5) = \sigma_0.$$

At this time we do not have to swap more elements. Se found

$$\sigma = e_1 \circ e_4 \circ e_3 \circ e_2.$$

Let us verify this

$$\begin{aligned} e_2 &= (1, 3, 2, 4, 5) \\ e_3 \circ e_2 &= (1, 3, 4, 2, 5) \\ e_4 \circ e_3 \circ e_2 &= (1, 3, 4, 5, 2) \\ \sigma = e_1 \circ e_4 \circ e_3 \circ e_2 &= (3, 1, 4, 5, 2) \end{aligned}$$

as desired.

Problem 3.

Let ψ_I be an elementary k–tensor on V corresponding to the basis $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ for V. If $j_1, \cdots, j_k$ is an arbitrary k–tuple of integers from the set $\{1, \cdots, n\}$, what is the value of

$$\psi_I(\boldsymbol{a}_{j_1}, \cdots, \boldsymbol{a}_{j_k})?$$

sln. We know from Theorem 27.5, that if the sequence $I = i_1, \cdots, i_k$ is ascending then

$$\psi_I(\boldsymbol{a}_{j_1}, \cdots, \boldsymbol{a}_{j_k}) = \begin{cases} 0 & \text{if} \quad I \neq J \\ 1 & \text{if} \quad I = J \end{cases}$$

where $J = (j_1, \cdots, j_n)$.

It could happen that that the sequence J is not in ascending mode. To put J in ascending mode we need to perform some number m of permutations, and each permutation would reverse the sign, so,

$$\psi_I(\boldsymbol{a}_{j_1}, \cdots, \boldsymbol{a}_{j_k}) = \begin{cases} 0 & \text{if} \quad \{I\} \neq \{J\} \\ \operatorname{sgn}(J) & \text{if} \quad \{I\} = \{J\} \end{cases}$$

where, for example, the symbol $\{I\}$ means I as a set, that is, the order does not matter.

Problem 4.

Show that if $T : V \to W$ is a linear transformation and if $f \in \mathcal{A}^k(W)$, then $T^* f \in \mathcal{A}^k(V)$.

sln. Let us assume that $\{\boldsymbol{v}_1, \cdots \boldsymbol{v}_k\}$ are base vectors for V. Let σ be a permutation of $\{1, \cdots, k\}$. So,

$$\begin{aligned} (T^* f)^\sigma(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k) &= T^* f(\boldsymbol{v}_{\sigma_1}, \cdots, \boldsymbol{v}_{\sigma_k}) \\ &= f(T(\boldsymbol{v}_{\sigma_1}, \cdots \boldsymbol{v}_{\sigma_k})) \\ &= (fT)^\sigma(\boldsymbol{v}_1, \cdots \boldsymbol{v}_k)) \\ &= \operatorname{sgn}\sigma\, f(T(\boldsymbol{v}_1, \cdots \boldsymbol{v}_k)) \qquad \text{since} \quad f \in \mathcal{A}^k(W) \\ &= \operatorname{sgn}\sigma\, (T^*\, f)(\boldsymbol{v}_1, \cdots \boldsymbol{v}_k), \end{aligned}$$

hence $T^* f \in \mathcal{A}^k(W)$.

Problem 5.

Show that

$$\psi_I = \sum_\sigma (\operatorname{sgn}\sigma)\phi_{I_\sigma},$$

where if $I = (i_1, \cdots, i_k)$, we let $I_\sigma = (i_{\sigma(1)}, \cdots, i_{\sigma(k)})$. [*Hint:* Show first that $(\phi_{I_\sigma})^\sigma = \phi_I$.]

sln. I believe Munkres meant this: *Hint:* Show first that $\phi_{I_\sigma} = (\phi_I)^\sigma$.

Let us assume some k–base vectors

$$\boldsymbol{v}_{i_{\sigma(1)}}, \cdots, \boldsymbol{v}_{i_{\sigma(k)}},$$

so

$$(\phi_{I_\sigma})^{\sigma^{-1}}(\boldsymbol{v}_{i_{\sigma(1)}}, \cdots, \boldsymbol{v}_{i_{\sigma(k)}}) = (\phi_{I_\sigma})(\boldsymbol{v}_{i_1}, \cdots, \boldsymbol{v}_{i_k}) = \begin{cases} 0 & \text{if} \quad I_\sigma \neq I \\ 1 & \text{if} \quad I_\sigma = I \end{cases}$$

In other words

$$\begin{aligned} (\phi_{I_\sigma})^{\sigma^{-1}}(\boldsymbol{v}_{i_{\sigma(1)}}, \cdots, \boldsymbol{v}_{i_{\sigma(k)}}) &= \begin{cases} 0 & \text{if} \quad I \neq I_\sigma \\ 1 & \text{if} \quad I = I_\sigma \end{cases} \\ &= \phi_I(\boldsymbol{v}_{i_{\sigma(1)}}, \cdots, \boldsymbol{v}_{i_{\sigma(k)}}). \end{aligned}$$

From which

$$(\phi_{I_\sigma}) = (\phi_I)^\sigma.$$

and from formula

$$\psi_I = \sum_\sigma (\operatorname{sgn}\sigma)(\phi_I)^\sigma$$

of Theorem 27.5 the proof follows.

Before starting to solve the problems of this section, let me do a simple exercise implied in *step 4* of Theorem 28.1.

Show that the number of inversions in the permutation

$$(\pi(1), \cdots, \pi(k+\ell)) = (k+1, k+2, \cdots, k+\ell, 1, 2, \cdots, k),$$

is $k\,l$. That is, $\operatorname{sgn}\pi = (-1)^{k\,l}$.

sln. We first move 1 to the first place by swapping it with $k+\ell$, $k+\ell-1$ and so forth until swapping it with $k+1$, we need to swap it ℓ times and end up with

$$(1, k+1, k+2, \cdots, k+\ell, 2, \cdots k)$$

We do the same for 2, moving it to the second plane; and so forth until k. This is k times ℓ, or in other words $k\,\ell$.

Section 28: The Wedge Product

Problem 1.

Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathrm{R}^5$. Let [4]

$$\begin{aligned} F(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) &= 2x_2\, y_2\, z_1 + x_1\, y_5\, z_4, \\ G(\boldsymbol{x}, \boldsymbol{y}) &= x_1\, y_3 + x_3\, y_1, \\ h(\boldsymbol{w}) &= w_1 - 2w_3. \end{aligned}$$

(a) Write AF and AG in terms of elementary alternating tensors. [*Hint:* Write F and G in terms of elementary tensors and use step 9 of the preceding proof to compute $A\phi_I$.]

sln.

$$F = 2\,\phi_2 \otimes \phi_2 \otimes \phi_1 + \phi_1 \otimes \phi_5 \otimes \phi_4.$$

So, form the property on Step 9 of Theorem 28.1

$$AF = 2\,\phi_2 \wedge \phi_2 \wedge \phi_1 + \phi_1 \wedge \phi_5 \wedge \phi_4 = -\phi_1 \wedge \phi_4 \wedge \phi_5 = -\psi_{1,4,5}.$$

Similarly

$$G = \phi_1 \otimes \phi_3 + \phi_3 \otimes \phi_1 \quad \Rightarrow \quad AG = \phi_1 \wedge \phi_3 + \phi_3 \wedge \phi_1 = 0.$$

(b) Express $(AF) \wedge h$ in terms of elementary alternating tensors.

[4] note the typographical error for the third variable of F in Munkres statement

sln.

$$\begin{aligned} AF \wedge h &= -(\phi_1 \wedge \phi_4 \wedge \phi_5) \wedge (\phi_1 - 2\phi_3) \\ &= 2\,\phi_1 \wedge \phi_4 \wedge \phi_5 \wedge \phi_3 \\ &= -2\,\phi_1 \wedge \phi_4 \wedge \phi_3 \wedge \phi_5 \\ &= 2\,\phi_1 \wedge \phi_3 \wedge \phi_4 \wedge \phi_5 \\ &= 2\,\psi_{1,3,4,5}. \end{aligned}$$

(c) Express $(AF)(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ as a function.

sln. From the Example 2, (and in general from Theorem 27.7)

$$\psi_{i,j,k}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \det \begin{bmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{bmatrix}$$

In particular

$$(AF)(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = -\psi_{1,4,5} \quad = \quad -\det \begin{bmatrix} x_1 & y_1 & z_1 \\ x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \end{bmatrix}$$

That is,

$$\begin{aligned} (AF)(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) &= -z_1(x_4\,y_5 - x_5\,y_4) - x_1(y_4 z_5 - y_5 z_4) + y_1(x_4 z_5 - x_5 z_4) \\ &= -x_4 y_5 z_1 + x_5 y_4 z_1 - x_1 y_5 z_5 + x_1 y_5 z_4 + x_4 y_1 z_5 - x_5 y_1 z_4. \end{aligned}$$

Problem 2.

If G is symmetric, show that $AG = 0$. Does the converse hold?

sln. Let us assume $G = f_1 \otimes \cdots \otimes f_m \otimes \cdots \otimes f_n \otimes \cdots \otimes f_k$, where m and n are two arbitrary indices with $m \neq n$.

$$G = f_1 \otimes \cdots \otimes f_m \otimes \cdots \otimes f_n \otimes \cdots \otimes f_k = f_1 \otimes \cdots \otimes f_n \otimes \cdots \otimes f_m \otimes \cdots \otimes f_k$$

and so

$$AG = f_1 \wedge \cdots \wedge f_m \wedge \cdots \wedge f_n \wedge \cdots \wedge f_k = f_1 \wedge \cdots \wedge f_n \wedge \cdots \wedge f_m \wedge \cdots \wedge f_k$$

But from the wedge properties

$$f_1 \wedge \cdots \wedge f_n \wedge \cdots \wedge f_m \wedge \cdots \wedge f_k = -f_1 \wedge \cdots \wedge f_m \wedge \cdots \wedge f_n \wedge \cdots \wedge f_k$$

so

$$AG = 2f_1 \wedge \cdots \wedge f_m \wedge \cdots \wedge f_n \wedge \cdots \wedge f_k = 0$$

Since n and m are arbitrary then in any case

$$AG = 0.$$

I claim that any general tensor G is a finite sum of products of 1–tensors. So $AG = 0$ is still valid.

The converse should hold since by definition

$$AG = \sum_{\sigma} (\text{sgn}\sigma) F^{\sigma} = 0,$$

then since the $\text{sgn}\,\sigma$ changes for each transposition we can divide the sum in two sets, the even and the odd transpositions. We can map each index array for the tensor F (assuming it is or rank k) $i_1, i_2, \cdots i_m \cdots i_n \ldots i_k$ to its transpoed $i_1, i_2 \cdots i_n \cdots i_m \cdots i_k$ where we just switched the index i_n with the index i_m. If F is non–symmetric then the terms will not cancel under the transposition, so we would not have the whole sum equal to 0.

Problem 3.

Show that if $f_1, \cdots, f_k$ are alternating tensors of order $\ell_1, \cdots, \ell_k$, respectively, then

$$\frac{1}{\ell_1! \cdots \ell_k!} A(f_1 \otimes \cdots \otimes f_k) = f_1 \wedge \cdots \wedge f_k.$$

sln. From step 2 we see the definition

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g),$$

where f and g are tensors of order k and ℓ respectively.

We use induction. Assume that for the result is valid up to $k-1$. That is,

$$\frac{1}{\ell_1!\cdots\ell_{k-1}!}A(f_1\otimes\cdots\otimes f_{k-1}) = f_1\wedge\cdots\wedge f_{k-1}.$$

Let us define

$$g = f_1\otimes\cdots\otimes f_{k-1}$$

The order of g is given by $\ell_1!\cdots\ell_{k-1}!$. Now we apply the definition (using associativity) to the product

$$g\wedge f_k = \frac{1}{(\ell_1!\cdots\ell_{k-1}!)\ell_k!}A(g\otimes f_k).$$

That is

$$\frac{1}{\ell_1!\cdots\ell_k!}A(f_1\otimes\cdots\otimes f_k) = f_1\wedge\cdots\wedge f_k.$$

Problem 4.

Let $\boldsymbol{x}_1,\cdots,\boldsymbol{x}_k$ be vectors in $\mathbb{R}^n$; let X be the matrix $X = [\boldsymbol{x}_1\cdots\boldsymbol{x}_k]$. If $I = (i_1,\cdots,i_k)$ is an *arbitrary* k–tuple from the set $\{1,\cdots,n\}$, show that

$$\phi_{i_1}\wedge\cdots\wedge\phi_{i_k}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_k) = \det X_I.$$

sln. We know from step 9 in Theorem 28.1 that

$$A(\phi_{i_1}\otimes\cdots\otimes\phi_{i_k})(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_k) = \phi_{i_1}\wedge\cdots\wedge\phi_{i_k}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_k).$$

(also look at the previous Problem 3, where each tensor is of order 1).
Also, from $\psi_I = A\phi_I$ we find

$$A(\phi_{i_1}\otimes\cdots\otimes\phi_{i_k})(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_k) = A\phi_I(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_k) = \psi_I(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_k).$$

and from Theorem 27.7

$$\psi_I(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_k) = \det X_I,$$

so the result

$$\phi_{i_1}\wedge\cdots\wedge\phi_{i_k}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_k) = \det X_I,$$

follows.

Problem 5.

Verify that $T^*(F^\sigma) = (T^*F)^\sigma$.

sln. Let us assume that the set $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k\}$ is a basis for W, so

$$\begin{aligned} T^*(F^\sigma)(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k) &= F^\sigma(T(\boldsymbol{v}_1), \cdots, T(\boldsymbol{v}_k)) \\ &= F(T(\boldsymbol{v}_{\sigma(1)}), \cdots, T(\boldsymbol{v}_{\sigma(k)})) \\ &= T^*F(\boldsymbol{v}_{\sigma(1)}, \cdots, \boldsymbol{v}_{\sigma(k)}) \\ &= (T^*F)^\sigma(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k), \end{aligned}$$

which proofs the statement.

Problem 6.

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be the linear transformation $T(\boldsymbol{x}) = B \cdot \boldsymbol{x}$.

(a) If ψ_I is an elementary alternating k–tensor on $\mathbb{R}^n$, then $T^*\psi_I$ has the form

$$T^*\psi_I = \sum_{[J]} c_J \psi_J,$$

where the ψ_J are the elementary alternating k–tensors on $\mathbb{R}^m$. What are the coefficients c_J?

sln. Let us assume a basis $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m\}$ in $\mathbb{R}^m$. For convenience in notation let us define

$$f(\boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{ik}) = T^*\psi_I(\boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{ik})$$

We assume $I = (i_1, \cdots, i_k)$ is an ascending order for the vectors a_i Since f is an alternating tensor in $\mathbb{R}^m$, then this can be written as

$$f(\boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{i_k}) = \sum_{|J|} c_J \psi_J(\boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{i_k}) \tag{5.44}$$

where $J = (j_1, \cdots, j_k)$ an ascending sort of k members, for the basis vectors a_j.

The only contribution to the sum (see theorem 27.5) comes from $I = J$. That is

$$f(\boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{i_k}) = c_I \psi_I(\boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{i_k}) = c_I \tag{5.45}$$

On the other hand

$$\begin{aligned}(T^*\psi_I)(\boldsymbol{a}_{i_1}, \cdots \boldsymbol{a}_{i_k}) &= \psi_I(T(\boldsymbol{a}_{i_1}), \cdots, T(\boldsymbol{a}_{i_k})) \\ &= \psi_I(B \cdot \boldsymbol{a}_{i_1}, \cdots, B \cdot \boldsymbol{a}_{i_k}).\end{aligned}$$

We want to use Theorem 27.7. Assume that the vectors $\boldsymbol{a}_i$ are the canonical vectors $\boldsymbol{e}_i$ we define the matrix

$$X = [B \cdot \boldsymbol{e}_{i_1}, \cdots, B \cdot \boldsymbol{e}_{i_k}] = B_I$$

with $B_I = [\boldsymbol{b}_{i_1}, \cdots, \boldsymbol{b}_{i_k}]$, the matrix composed of the I columns of B.

Then, from Theorem 27.7,

$$\psi_I(B \cdot \boldsymbol{a}_{i_1}, \cdots, B \cdot \boldsymbol{a}_{i_k}) = \det X_I = \det B_{II}, \tag{5.46}$$

where $X_I = B_{II}$ is the matrix composed of the I columns and rows of B.

Then from equations 5.44, 5.45, and 5.46, we find

$$c_I = \begin{cases} \det B_{JJ} & \text{if} \quad I = J \\ 0 & \text{if} \quad I \neq J \end{cases}$$

where B_{II} is the matrix formed by taking the I rows and columns of B.

(b) If $f = \sum_{[I]} d_I \psi_I$ is an alternating k–tensor on R^n, express $T^* f$ in terms of the elementary alternating k–tensors on $\mathbb{R}^m$.

sln. My opinion is that this should be part (a), and what we solved as (a) should be part (b).

As in part, (a), assume $I = (i_1, \cdots, i_k)$ is an ascending order for the vectors a_i, from a basis $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m\}$. Let us name

$$g(\boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{i_k}) = T^* f(\boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{i_k}),$$

Since g is an alternating tensor in $\mathbb{R}^m$, then this can be written as

$$g(\boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{i_k}) = \sum_{|J|} c_J \psi_J(\boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{i_m}).$$

From all combinations only the $I = J$ will produce a non–zero value. That is,

$$g(\boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{i_k}) = c_I \psi_I(\boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{i_m}) = c_I,$$

so all we have to evaluate is

$$\begin{aligned} g(\boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{i_k}) &= T^* f(\boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{i_k}) \\ &= f(T(\boldsymbol{a}_{i_1}), \cdots, T(\boldsymbol{a}_{i_k})) \end{aligned}$$

and so

$$c_I = \begin{cases} f(T(\boldsymbol{a}_{j_1}), \cdots, T(\boldsymbol{a}_{j_k})) & \text{if} \quad I = J \\ 0 & \text{if} \quad I \neq J \end{cases}$$

Section 29: Tangent Vectors and Differential Forms

Problem 1.

Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be of class C^r. Show that the velocity vector of γ corresponding to the parameter t is the vector $\gamma_*(t; \boldsymbol{e}_1)$.

sln. By definition, and Theorem 5.4

$$\begin{aligned} \gamma_*(t; \boldsymbol{e}_1) &= (\gamma(t); D\gamma(t) \cdot \boldsymbol{e}_1) \\ &= (\gamma(t); \dot{\gamma}(t)) \end{aligned}$$

which is in fact the velocity vector for the curve γ. Here

$$\dot{\gamma}(t) = \begin{bmatrix} d\gamma_1(t)/dt \\ \vdots \\ d\gamma_n(t)/dt \end{bmatrix}$$

and $\boldsymbol{e_1} = [1]$.

Problem 2.

If A is open in $\mathbb{R}^k$ and $\alpha : A \to \mathbb{R}^k$ is of class C^r, show that $\alpha_*(\boldsymbol{x};\boldsymbol{v})$ is the velocity vector of the curve $\gamma(t) = \alpha(\boldsymbol{x} + t\boldsymbol{v})$ corresponding to parameter $t = 0$.

sln. By definition

$$\alpha_*(\boldsymbol{x};\boldsymbol{v}) = (\alpha(\boldsymbol{x}); D\alpha(\boldsymbol{x}) \cdot \boldsymbol{v})$$

From the definition of directional derivative

$$\alpha'(\boldsymbol{x},\boldsymbol{v}) = \lim_{t\to 0} \frac{\alpha(\boldsymbol{x} + t\boldsymbol{v}) - \alpha(\boldsymbol{x})}{t} = \lim_{t\to 0} \frac{\gamma(t) - \gamma(0)}{t} = \dot{\gamma}(0).$$

On the other hand, from Theorem 5.1

$$\alpha'(\boldsymbol{x},\boldsymbol{v}) = D\alpha(\boldsymbol{x}) \cdot \boldsymbol{v}.$$

Hence

$$\alpha_*(\boldsymbol{x};\boldsymbol{v}) = (\gamma(0); \dot{\gamma}(0)).$$

Which is the velocity vector at $t = 0$.

Problem 3.

Let M be a k–manifold of class C^r in $\mathbb{R}^n$. Let $\boldsymbol{p} \in M$. Show that the tangent space to M at $\boldsymbol{p}$ is well–defined, independent of the choice of the coordinate patch.

sln. Assume the coordinate patch $\alpha : U_\alpha \to V_\alpha$ around point $\boldsymbol{p}$. The tangent space, $\mathcal{T}_{\boldsymbol{p}}(M)$ is a linear space spanned by the vectors of the form $(\boldsymbol{p}; \partial\alpha/\partial x_j)$ as shown in the text. Let us assume that there is another coordinate patch around the point $\boldsymbol{p}$, $\beta : U_\beta \to V_\beta$. For this coordinate patch the tangent space is generated by the points $(\boldsymbol{p}; \partial\beta/\partial y_j)$. We want to show that the two tangent spaces are actually two representations of the same object.

I follow, Yan Zeng's argument. Let $W = V_\alpha \cap V_\beta$ (the set is not empty since $\boldsymbol{p} \in W$). Let us call $U'_\alpha = \alpha^{-1}(W)$ and $U'_\beta = \beta^{-1}(W)$. Then from

Theorem 24.1, $\beta^{-1}\circ\alpha : U'_\alpha \to U'_\beta$ is a C^r–diffeomorphism. From using the chain rule,

$$D\alpha(\boldsymbol{x}) = D(\beta\circ\beta^{-1}\circ\alpha)(\boldsymbol{x}) = D(\beta)(\boldsymbol{y})\,\cdot D(\beta^{-1}\circ\alpha)(\boldsymbol{x}).$$

Since $D(\beta^{-1}\circ\alpha) : U'_\alpha \to U'_\beta$ is of rank k, then as sets,

$$\{D\alpha(\boldsymbol{x})\cdot\boldsymbol{v}, \boldsymbol{v}\in\mathbb{R}^k\} = \{D\beta(\boldsymbol{y})\cdot\boldsymbol{w}, \boldsymbol{w}\in\mathbb{R}^k\}$$

are the same, and we understand that $\boldsymbol{y} = (\beta^{-1}\circ\alpha)(\boldsymbol{x})$ is any arbitrary number in $\mathbb{R}^k$ as $\boldsymbol{x}$ is. Also $\boldsymbol{w} = D(\beta^{-1}\circ\alpha)(\boldsymbol{x})\cdot\boldsymbol{v}$, where again, since $\beta^{-1}\circ\alpha$ is a diffeomorphism then $\boldsymbol{v}$ is as arbitrary as $\boldsymbol{w}$ is.

In conclusion the tangent space $\mathcal{T}_{\boldsymbol{p}}$ is independent of the coordinate patch around $\boldsymbol{p}$.

Problem 4.

Let M be a k–manifold in $\mathbb{R}^n$ of class C^r. Let $\boldsymbol{p}\in M-\partial M$.

(a) Show that if $(\boldsymbol{p};\boldsymbol{v})$ is a tangent vector to M, then there is a parameterized–curve $\gamma : (-\epsilon,\epsilon)\to\mathbb{R}^n$ whose image set lies in M, such that $(\boldsymbol{p};\boldsymbol{v})$ equals the velocity vector of γ corresponding to parameter value $t = 0$. See Figure 29.4

sln. The idea is to pick a small line segment inside the open set in the domain of α and map this segment of a curve $\gamma(\boldsymbol{x})$ where $\boldsymbol{x}$ rides along this small line segment. Also, the direction of the segment should be along the direction of the tangent vector $\boldsymbol{v}$. Since $\boldsymbol{p}\in M-\partial M$, and $\alpha^{-1} : V\to U$ is continuous, we can pick a small open set O_v around $\boldsymbol{p}$ and its inverse image under α, that will generate an open set $O_u\subset U$. So, $\boldsymbol{x} = \alpha^{-1}(\boldsymbol{p})\subset U$ is inside some open ball $B(\boldsymbol{p},\epsilon)$ in the domain U of α, of radius ϵ. We can find a small segment such that it is totally within this ball. To find the direction of the segment we know that since $\boldsymbol{v}$ is tangent to M then there exist $\boldsymbol{u}\in U$, such that $\boldsymbol{v} = D\alpha(\boldsymbol{x})\cdot\boldsymbol{u}$. The segment is given by the points $\boldsymbol{x}+t\boldsymbol{u}$, and the mapping

$$\gamma(t) := \alpha(\boldsymbol{x}+t\boldsymbol{u})$$

where $|t|<\epsilon$, maps points from U to the manifold M. Also, from the definition of directional derivative

$$\frac{d}{dt}\gamma(t)\bigg|_{t=0} = D\alpha(\boldsymbol{x}) \cdot u = \boldsymbol{v}.$$

(b) Prove the converse. [*Hint:* Recall that for any coordinate patch α, the map α^{-1} is of class C^r. See Theorem 24.1]

sln. We assume that if the vector $(\boldsymbol{p}, \boldsymbol{v})$ is such that $\boldsymbol{v}$ equals the velocity of $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$. Then we show that $(\boldsymbol{p}, \boldsymbol{v})$ is tangent to the manifold M, that is we show that

$$(\boldsymbol{p}, \boldsymbol{v}) = \alpha_*(\boldsymbol{x}, \boldsymbol{u}),$$

for some $\boldsymbol{u}$ and $\alpha(\boldsymbol{x}) = \boldsymbol{p}$. In the previous problem we used the definition $\boldsymbol{v} = D\alpha(\boldsymbol{x}) \cdot \boldsymbol{u}$. Using the hint, since α^{-1} is of class C^r then we can uniquely and in the same way define

$$\boldsymbol{u} = (D\alpha)^{-1}(\boldsymbol{x}) \cdot \boldsymbol{v}.$$

and from the definition of tangent vector to a manifold,

$$\begin{aligned} \alpha_*(\boldsymbol{x}, \boldsymbol{u}) &= (\boldsymbol{p}; D\alpha(\boldsymbol{x}) \cdot \boldsymbol{u}) \\ &= (\boldsymbol{p}; (D\alpha)(\boldsymbol{x}) \cdot (D\alpha)^{-1}(\boldsymbol{x}) \cdot \boldsymbol{v}) \\ &= (\boldsymbol{p}, \boldsymbol{v}) \end{aligned}$$

which verifies the statement.

Problem 5.

Let M be a k–manifold in $\mathbb{R}^n$ of class C^r. Let $q \in \partial M$.

(a) Show that if $(\boldsymbol{q}, \boldsymbol{v})$ is a tangent vector to M at $\boldsymbol{q}$, then there is a parameterized–curve $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$, where γ carries either $(-e, 0]$ or $[0, \epsilon)$ into M, such that $(\boldsymbol{q}, \boldsymbol{v})$ equals the velocity vector of γ corresponding to the parameter value $t = 0$.

sln. This is quite similar to Problem 4, but now we are on the boundary of the manifold. Still, the details are bit more difficult because the boundary. I will mimic Problem 4 solution here.

Since $\boldsymbol{p}$ is in ∂M then $\alpha^{-1}(\boldsymbol{p})$ is in H^k, then from the counter–reciprocal of part (b) of Lemma 24.2, we have that $\boldsymbol{x} = \alpha^{-1}(\boldsymbol{p}) \notin H^k_+$. So $x_k = 0$. Because α is C^r then for any open set O_M in M having $\boldsymbol{p}$ the inverse α^{-1} is an open set O_U in the domain of α.

We want to construct a small segment in the domain of α. The segment is the set of points $\boldsymbol{x} + t\boldsymbol{u}$, with $\boldsymbol{x}$, such that $\alpha(\boldsymbol{x}) = \boldsymbol{q}$, and $\boldsymbol{u}$ such that $\boldsymbol{v} = D\alpha(\boldsymbol{x}) \cdot \boldsymbol{u}$. The existence if $\boldsymbol{u}$ is guaranteed because $\boldsymbol{v}$ is tangent to M and the definition of $\boldsymbol{v} \in \mathcal{T}_M$.

There could be three possibilities

(a) $\boldsymbol{u}$ is along the direction with $u_k = 0$, that is pointing along the H_k hyper–plane interface. In this case we can have $t \in (-\epsilon, \epsilon)$ such that $\boldsymbol{x} + t\boldsymbol{u}$ is totally within the domain of α, U.

(b) The direction of $\boldsymbol{u}$ is toward H^k_+, and in this case $t \in [0, \epsilon)$ would be such that the segment $\boldsymbol{x} + t\boldsymbol{u}$ is completely immersed in U.

(c) The direction of $\boldsymbol{u}$ is toward H^k_-, and in this case $t \in (-\epsilon, 0]$ is such that the set $\boldsymbol{x} + t\boldsymbol{u}$ is contained in U.

In any case, the definition

$$\gamma(t) = \alpha(\boldsymbol{x} + t\boldsymbol{u}),$$

is such that $\gamma(0) = \alpha(\boldsymbol{x}) = \boldsymbol{q}$, and

$$\frac{d}{dt}\gamma(t)\bigg|_{t=0} = D\alpha(\boldsymbol{x}) \cdot u = \boldsymbol{v}.$$

(b) Prove the converse.

sln. We assume that $(\boldsymbol{p}, \boldsymbol{v})$ is such that $\boldsymbol{v}$ is the velocity of $\gamma : A \to \mathbb{R}^n$, where A is an interval that could be $(-\epsilon, \epsilon)$, $(-\epsilon, 0]$ or $[0, \epsilon)$, according to the three cases considered above. We want to prove that

$$(\boldsymbol{p}, \boldsymbol{v}) = \alpha_*(\boldsymbol{x}, \boldsymbol{u}),$$

for some $\boldsymbol{u}$ and $\alpha(\boldsymbol{x}) = \boldsymbol{p}$. Proceeding as before with $\boldsymbol{v} = D\alpha(\boldsymbol{x} \cdot \boldsymbol{u})$ and since α^{-1} is of class C^r, then we can uniquely define

$$\boldsymbol{u} = (D\alpha)^{-1}(\boldsymbol{x}) \cdot \boldsymbol{v}.$$

Now, from the definition of tangent vector to a manifold,

$$\begin{aligned}
\alpha_*(\boldsymbol{x}, \boldsymbol{u}) &= (\boldsymbol{p}; D\alpha(\boldsymbol{x}) \cdot \boldsymbol{u}) \\
&= (\boldsymbol{p}; (D\alpha)(\boldsymbol{x}) \cdot (D\alpha)^{-1}(\boldsymbol{x}) \cdot \boldsymbol{v}) \\
&= (\boldsymbol{p}, \boldsymbol{v})
\end{aligned}$$

which verifies the statement.

Bibliography

[1] J.R. Munkres. *Analysis On Manifolds*. Advanced Book Classics. Basic Books, 1997.