Partial Differential Equations

Herman Jaramillo

May 10, 2016

Contents

Introduction

In chapter 1 I show the method of characteristics by a simple example for first order, quasi–linear Partial Differential Equations (PDEs). The second order PDEs are also studied using the method of characteristics in chapter 2. Chapter 3 deals with Fourier methods, and chapter 4 with Green functions.

Chapter 1

First Order

Solve

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} = 0. \tag{1.0.1}$$

1.1 Finding the characteristics

Let us define a vector of the coefficients of the equation as:

$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Now it is clear that equation 2.1.9 is:

$$a \cdot \nabla U = 0.$$

So the vector $\boldsymbol{a}$ is perpendicular to the gradient ∇U, and therefore parallel to contour levels of U defined by $U = (x,t) = \text{constant}$. If C is a contour level of the field U along its arc length s (assuming it is rectifiable) then

$$\frac{dU}{ds} = \frac{\partial U}{\partial x}\frac{dx}{ds} + \frac{\partial U}{\partial t}\frac{dt}{ds} = 0. \tag{1.1.2}$$

So we see that the vector $\boldsymbol{b}$ defined as:

$$\boldsymbol{b} = \begin{pmatrix} \dfrac{dx}{ds} \\ \\ \dfrac{dt}{ds} \end{pmatrix}.$$

is also perpendicular to the gradient ∇U and so it is parallel to $\boldsymbol{a}$. Therefore we have:

$$\begin{aligned} \frac{dx}{ds} &= 1.F(s) \\ \\ \frac{dt}{ds} &= 1.F(s) \end{aligned} \tag{1.1.3}$$

Given that

$$\boldsymbol{t}\cdot\boldsymbol{t}=\left(\frac{dx}{ds}\right)^2+\left(\frac{dt}{ds}\right)^2$$

then dx/ds and dt/ds cannot vanish simultaneously. We use equation 1.1.3 to find the curve C. From 1.1.3

$$\frac{dx}{dt}=\frac{dx/ds}{dt/ds}=1/1=1.$$

This is the equation of the characteristics. That is (integrating), the path C satisfies

$$x=t+c$$

where c is a constant. So we can say that U is constant along all the 45 degree lines that cross the t axis.

The next step is to solve the equation along the characteristic.

1.1.1 Initial conditions

Because the PDF 2.1.9, is equivalent to an ordinary differential equation along the characteristic curve, Eq. is is not possible to prescribe the dependence of U along a characteristic curve.

Let N be any curve such that the slope of N at any point (x,t) is not equal to the slope of the characteristic curve through (x,t). Prescribing the initial values of U on N is enough to determine a unique solution of the original PDF 2.1.9 on each characteristic curve that issues from a point on N. Therefore the values of U on N uniquely determine U, and conversely.

Since U is constant along the characteristic we have that the value of U is only determined by the value c in the equation $x=t+c$, that is by the value $x-t$. That is,

$$U(x,t)=f(x-t)=f(c)$$

Let as assume that the initial condition is $f(x)=\sin 4\pi x$ at the time $t=0$. So the solution is

$$U(x,t)=f(x-t)=\sin 4\pi(x-t).$$

1.1.2 Description of the General Algorithm: Following Bleistein's [2] book

Given the firs–order quasi–linear equation

$$a(x,y,u)u_x+b(x,y,u)u_y=c(x,y,u).$$

Find the equations of the characteristics:

$$\frac{a}{dx} = \frac{b}{dy} = \frac{c}{du}$$

Note that I inverted the fractions since I want c upstairs, to consider the homogeneous case $c = 0$.

Bleistein's idea here is to parametrize the solution surface $U(x, y)$ by two parameters τ and σ such that

$$x = x(\sigma, \tau) \quad , \quad y = y(\sigma, \tau) \quad , \quad U = U(\sigma, \tau). \tag{1.1.4}$$

In such a way that one parameter is along the characteristics (say σ) and the other (τ) in a trajectory that cuts through the characteristics (nowhere kissing them).

The equations of the characteristics in the new parameterization are:

$$\frac{dx}{d\sigma} = \lambda\, a \quad , \quad \frac{dy}{d\sigma} = \lambda\, b \quad , \quad \frac{du}{d\sigma} = \lambda\, c. \tag{1.1.5}$$

We can pick

$$\lambda = \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

to make σ be the arc length along the characteristics, or

$$\lambda = \frac{1}{\sqrt{a^2 + b^2}}$$

to make σ be the arc length along the base characteristics, or simply $\lambda = 1$.

Data specification

For example if $\sigma = 0$, we can have a curve parameterized by τ

$$x = x(0, \tau) \quad , \quad y = y(0, \tau) \quad , \quad U = U(0, \tau).$$

For each fixed τ we solve equations 1.1.5 and get to the solution in the form of 1.1.4.

In order, for the parameterization 1.1.4 to work we need a non–vanishing Jacobian. That is:

$$J = \begin{vmatrix} x_\sigma & x_\tau \\ y_\sigma & y_\tau \end{vmatrix} \neq 0.$$

Bleistein show that the Jacobian vanishes along tangential (kissing) points of the initial curve with the characteristics. If the Jacobian vanishes everywhere then the initial curve is a characteristic. There could be infinite solutions.

The method of envelopes

We describe the method following Bleistein's example

$$u_x + u_y = u$$

The characteristics are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{u}$$

The integration of the characteristics in terms of "x" is:

$$y = x + c_1 \quad , \quad u = c_2\, e^x \tag{1.1.6}$$

Each choice of the constants c_1, c_2 produces a characteristic curve. That is, 1.1.6 is a two–parameter family of characteristic curves. A solution surface could be determined by prescribing a functional relationship between c_1 and c_2, that i,s by prescribing, for example,

$$c_2 = f(c_1)$$

Or equivalently

$$u\, e^{-x} = f(y - x)$$

solving for u we conclude

$$u = e^x\, f(y - x)$$

By direct substitution one can verify that this is a solution to the differential equation

Chapter 2

second order

2.1 constant coefficients

The characteristic method for solving the second order partial differential equations is based on the book by [3].

Let us consider the equation of the form

$$Au \equiv a\,\frac{\partial^2 u}{\partial t^2} + 2\,b\,\frac{\partial^2 u}{\partial t\,\partial x} + c\,\frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R} \quad t > 0 \tag{2.1.1}$$

with initial conditions

$$\begin{aligned} u(x,0) &= \varphi(x) \\ \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} &= \psi(x). \end{aligned} \tag{2.1.2}$$

We further assume that $b^2 - ac > 0$ (so that equation 2.1.1 is hyperbolic). The idea behind the method of characteristics is to change this second order equation into two simpler equation in terms of first order derivatives. We do this by factoring the equation. To simplify notation let us define

$$T = \frac{\partial}{\partial t} \qquad X = \frac{\partial}{\partial x}$$

Now using this notation into equation 2.1.1 we find

$$aT^2 + 2bTX + cX^2 = 0. \tag{2.1.3}$$

This changes equation 2.1.1 into an algebraic equation which is easy to factor. The roots of equation 2.1.3 are given by

$$\frac{-2\,bX \pm \sqrt{4b^2X^2 - 4a\,cX^2}}{2\,a} \quad = \quad \frac{-b \pm \sqrt{b^2 - a\,c}}{a}\,X = X\lambda_{1,2}$$

with

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - a\,c}}{a} X.$$

since $b^2 - 4a\,c > 0$ the two roots are real. We then write

$$(T - \lambda_1 X)(T - \lambda_2 X) = 0.$$

We write the factored differential equation as

$$A\,u \equiv \left(\frac{\partial}{\partial t} - \lambda_1 \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \lambda_1 \frac{\partial}{\partial x}\right) u = 0, \tag{2.1.4}$$

We split this equation into two

$$\begin{aligned} L_{(-\lambda_1,1)} u = \left(\frac{\partial}{\partial t} - \lambda_1 \frac{\partial}{\partial x}\right) u = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u \cdot (-\lambda_1, 1) = \nabla u \cdot (-\lambda_1, 1) &= 0 \\ L_{(-\lambda_2,1)} u = \left(\frac{\partial}{\partial t} - \lambda_2 \frac{\partial}{\partial x}\right) u = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u \cdot (-\lambda_2, 1) = \nabla u \cdot (-\lambda_2, 1) &= 0 \end{aligned} \tag{2.1.5}$$

and by using the definition of directional derivative, we conclude that the derivative along the directions $(-\lambda_1, 1)$ and $(-\lambda_2, 1)$ is zero. Any line in the (x, t) plane, with direction $(-\lambda_1, 1)$ has the slope $-1/\lambda_1$ and so can be written as

$$t - t_0 = -\frac{1}{\lambda_1}(x - x_0).$$

which we can write as

$$x + \lambda_1 t = \lambda_1 t_0 + x_0 = \xi$$

along the same lines we find that the directional derivative of u is zero along any line of the form

$$x + \lambda_2 t = \lambda_2 t_0 + x_0 = \eta$$

That is the directional derivatives of u along the lines $\xi =$ const and $\eta =$ const are zero. This lines are know as the "*characteristics*" for the differential equation 2.1.1. This provides a method to change variables in a way to simplify the second order differential equation 2.1.1. We define the coordinate transformation

$$u = u(x(\xi, \eta), t(\xi, \eta)).$$

Then

$$\begin{aligned} L_{(-\lambda_1,1)} \xi &= \frac{\partial \xi}{\partial t} - \lambda_1 \frac{\partial \xi}{\partial x} = \lambda_1 - \lambda_1 = 0 \\ L_{(-\lambda_2,1)} \eta &= \frac{\partial \eta}{\partial t} - \lambda_1 \frac{\partial \eta}{\partial x} = \lambda_2 - \lambda_2 = 0 \end{aligned}$$

$L_{(-\lambda_1,1)}$ is the operator of differentiation along the lines $\xi = \text{const}$, while $L_{(-\lambda_2,1)}$ is the operator of differentiation along the lines $\eta = \text{const}$. So,

$$L_{(-\lambda_1,1)} = c_1 \frac{\partial}{\partial \eta}\bigg|_{\xi=\text{const}} \quad ; \quad L_{(-\lambda_2,1)} = c_2 \frac{\partial}{\partial \xi}\bigg|_{\eta=\text{const}} \tag{2.1.6}$$

From equations 2.1.4 and 2.1.6 we find

$$\frac{\partial^2 u}{\partial \eta\, \partial \xi} = 0. \tag{2.1.7}$$

This is the canonical form of the second order hyperbolic PDE. To solve this equation, let denote

$$\frac{\partial u}{\partial \eta}(\xi, \eta) = v(\xi, \eta). \tag{2.1.8}$$

so equation 2.1.7 can be written as

$$\frac{\partial v}{\partial \xi} = \frac{dv}{d\xi}\bigg|_{\eta=\text{const}} = 0.$$

It then follows that $v|_{\eta=\text{const}}$ does not depend on ξ, that is,

$$v(\xi, \eta) = c(\eta),$$

and from 2.1.8

$$\frac{d}{d\eta} u\bigg|_{\xi=\text{const}} = c(\eta).$$

Integrating this ordinary differential equation, we obtain

$$u\bigg|_{\xi=\text{const}} = \int c(\eta) d\eta + c_1(\xi).$$

Thus,

$$u(x,t) = f(\xi) + g(\eta) = f(x + \lambda_1 t) + g(x + \lambda_2 t).$$

where f and g are functions of one variable. To find the form of f and g we use the initial conditions 2.1.2. From the first of these conditions we see that

$$u(x, 0) = f(x) + g(x) = \varphi(x), \tag{2.1.9}$$

and from the second condition

$$\frac{\partial u(x,t)}{\partial t}\bigg|_{t=0} = \lambda_1\, f'(x) + \lambda_2\, g'(x) = \psi(x);$$

we integrate this equation between 0 and x and find

$$\lambda_1 f(x) + \lambda_2 g(x) = \int_0^x \psi(s)\, ds \tag{2.1.10}$$

by inserting 2.1.9 into 2.1.10

$$\lambda_1 [\varphi(x) - g(x)] + \lambda_2 g(x) = \int_0^x \psi(s)\, ds \tag{2.1.11}$$

so

$$\begin{aligned} g(x) &= \frac{1}{\lambda_2 - \lambda_1} \int_0^x \psi(s)\, ds - \frac{\lambda_1}{\lambda_2 - \lambda_1} \varphi(x) \\ f(x) &= \varphi(x) - g(x) \\ &= \left(1 + \frac{\lambda_1}{\lambda_2 - \lambda_1}\right) \varphi(x) - \frac{1}{\lambda_2 - \lambda_1} \int_0^x \psi(s)\, ds \\ &= \frac{\lambda_2}{\lambda_2 - \lambda_1} \varphi(x) - \frac{1}{\lambda_2 - \lambda_1} \int_0^x \psi(s)\, ds \end{aligned}$$

so

$$\begin{aligned} u(x,t) &= f(x + \lambda_1 t) + g(x + \lambda_2 t) \\ &= \frac{\lambda_2}{\lambda_2 - \lambda_1} \varphi(x + \lambda_1 t) - \frac{\lambda_1}{\lambda_2 - \lambda_1} \varphi(x + \lambda_1 t) + \\ &\quad \frac{1}{\lambda_2 - \lambda_1} \int_0^{x+\lambda_2 t} \psi(s)\, ds - \frac{1}{\lambda_2 - \lambda_1} \int_0^{x+\lambda_1 t} \psi(s)\, ds. \end{aligned}$$

Finally we obtain the d'Alembert solution

$$u(x,t) = \frac{\lambda_2 \varphi(x + \lambda_1 t) - \lambda_1 \varphi(x + \lambda_2)}{\lambda_2 - \lambda_1} + \frac{1}{\lambda_2 - \lambda_1} \int_{x+\lambda_1}^{x+\lambda_2 t} \psi(s)\, ds.$$

example

Let us consider the simple equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad t > 0.$$

with 2.1.2 initial conditions. The corresponding characteristic polynomial is given by

$$\lambda^2 - a^2 = (\lambda + a)(\lambda - a).$$

with roots $\lambda_{1,2} = \mp a$ The d'Alembert solution is then given by

$$u(x,t) = \frac{\varphi(x + at) + \varphi(x - at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(s)\, ds. \tag{2.1.12}$$

2.1.1 varying coefficients: general hyperbolic equation

We assume the equation

$$\begin{aligned} Au \ &\equiv\ a(x,t)\,\frac{\partial^2 u}{\partial t^2} + 2\,b(x,t)\,\frac{\partial^2 u}{\partial t\,\partial x} + c(x,t)\,\frac{\partial^2 u}{\partial x^2} + d(x,y)\frac{\partial u}{\partial t} \\ &+ e(x,t)\frac{\partial u}{\partial x} + f(x,y)u = g, \quad x \in \mathbb{R} \qquad t > 0 \end{aligned} \tag{2.1.13}$$

where now the coefficients are varying. For the moment we will ignore the initial conditions and focus in the general solution. We want to find a solution around a point (x_0, y_0) that

$$\Delta = b^2(x,t) - a(x,t)\,c(x,t)\Big|_{x=x_0,y=y_0} > 0.$$

We call Δ the discriminant and the point (x_0, y_0) a hyperbolic point.

At this moment I consider convenient to change notation.

New notation

From now on, partial derivatives are recognized by a subindex. For example

$$\begin{aligned} \frac{\partial u}{\partial t} &= u_t \\ \frac{\partial^2 u}{\partial t^2} &= u_{tt} \\ \frac{\partial u}{\partial x} &= u_x \\ \frac{\partial \xi}{\partial x} &= \xi_x \\ \cdots \qquad \text{and so on} \end{aligned}$$

observe that each time we use this notation we change 5 characters by 2 characters on a first derivative and from 7 to 3 in a second derivative, saving lots in typing without loosing the clarity in exposition.

In the new notation equation 2.1.14 becomes

$$\begin{aligned} &a(x,t)\,u_{tt} + 2\,b(x,t)\,u_{tx} + c(x,t)\,u_{xx} + \ d(x,t)\,u_t + e(x,t)u_x + f(x,t)u = g, \\ &\quad x \in \mathbb{R} \quad t > 0 \end{aligned} \tag{2.1.14}$$

Change of coordinates

A way to solve equations is by a change of variables that simplify the equation. In the case of equation 2.1.14 we want to defined new coordinates ξ, η such that the equation in the new coordinate system is simpler (or canonical). That is we would like to have equation 2.1.14 converted to the form

$$b^*(\xi,\eta)u_{\xi\eta}(\xi,\eta) + d^*(\xi,\eta)u_\xi + e^*(\xi,\eta)u_\eta + f^*(\xi,\eta)u = g^*(\xi,\eta).$$

with some new coefficients $b^*(\xi,\eta), d^*(\xi,\eta), e^*(\xi,\eta), f^*(\xi,\eta)$ and, $g^*(\xi,\eta)$. Note that the idea is to annihilate the coefficients $a^*(\xi,\eta)$ and $c^*(\xi,\eta)$. In what follows we write a^* instead of $a^*(\xi,\eta)$ to simplify notation.

Let us then define the coordinate transformation

$$\begin{aligned}\xi &= \xi(x,t)\\ \eta &= \eta(x,t)\end{aligned} \tag{2.1.15}$$

This coordinate transformation is well defined if

$$J = \begin{vmatrix} \xi_t & \xi_x \\ \eta_t & \eta_x \end{vmatrix} = \xi_t\eta_x - \eta_t\xi_x \neq 0. \tag{2.1.16}$$

We first find the first partial derivatives of u with respect to t and x and then the second partial derivatives. By using the chain rule

$$\begin{aligned}u_t &= u_\xi\,\xi_t + u_\eta\,\eta_t\\ u_x &= u_\xi\,\xi_x + u_\eta\,\eta_x\end{aligned}$$

Applying the chain rule again

$$\begin{aligned}u_{tx} &= (u_t)_x = (u_\xi\,\xi_t + u_\eta\,\eta_t)_x\\ &= (u_\xi\,\xi_t)_x + (u_\eta\,\eta_t)_x\\ &= (u_\xi)_x\,\xi_t + u_\xi\xi_{tx} + (u_\eta)_x\,\eta_t + u_\eta\,\eta_{t\,x}\end{aligned}$$

we use the second of equations 2.1.17 and so

$$u_{tx} = (u_{\xi\xi}\xi_x + u_{\eta\xi}\eta_x)\,\xi_t + u_\xi\xi_{t\,x} + (u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x)\eta_t + u_\eta\eta_{tx}$$

and rearranging,

$$u_{tx} = u_{\xi\xi}\xi_t\xi_x + u_{\xi\eta}(\xi_t\eta_x + \xi_x\eta_t) + u_{\eta\eta}\eta_t\eta_x + u_\xi\xi_{tx} + u_\eta\eta_{tx}. \tag{2.1.17}$$

if the derivative is with respect to tt, change x by t in the previous equation, and if the derivative is with respect to xx, change t by x in the previous equation, so

$$u_{tt} = u_{\xi\xi}\xi_t^2 + 2u_{\xi\eta}\xi_t\eta_t + u_{\eta\eta}\eta_t^2 + u_\xi\xi_{tt} + u_\eta\eta_{tt}. \tag{2.1.18}$$

and

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx}. \tag{2.1.19}$$

we plug equations 2.1.17, 2.1.18, and 2.1.19 into the original equation 2.1.14 and find

$$a^* u_{\xi\xi} + 2\, b^* u_{\xi\eta} + c^* u_{\eta\eta} + d^* u_\xi + \, e^* u_\eta + f^* u = g^*, \tag{2.1.20}$$

where the new coefficients a^*, b^* and c^* are the following functions of ξ and η

$$\begin{aligned}
a^* &= a\,\xi_t^2 + 2b\,\xi_t\,\xi_x + c\,\xi_x^2 \\
b^* &= a\,\xi_t\,\eta_t + b\,(\xi_t\,\eta_x + \xi_x\,\eta_t) + c\,\xi_x\,\eta_x \\
c^* &= a\,\eta_t^2 + 2b\,\eta_t\,\eta_x + c\eta_x^2 \\
d^* &= a\,\xi_{tt} + b\,\xi_{tx} + c\xi_{tt} + d\xi_t + e\xi_x \\
e^* &= a\,\eta_{tt} + b\,\eta_{tx} + c\eta_{xx} + d\eta_t + e\eta_x \\
f^* &= f \\
g^* &= g.
\end{aligned} \tag{2.1.21}$$

We now show that the discriminant Δ^* in the new coordinate system is still greater than zero. That is, the point (x_0, y_0) in the new coordinate system (ξ, η) is still hyperbolic. Let us see, after sorting in the order $b^2, ac, a^2, c^2, ab, bc$

$$\begin{aligned}
\Delta^* &= (b^*)^2 - (a^*)(c^*) \\
&= b^2[(\xi_t\eta_x + \xi_x\eta_t)^2 - 4\,\xi_t\xi_x\eta_t\eta_x] - ac[\xi_t^2\eta_x^2 + \eta_t^2\xi_x^2 - 2\xi_t\eta_t\xi_x\eta_x] \\
&\quad + a^2[\xi_t^2\eta_t^2 - \xi_t^2\eta_t^2] + \, c^2[\xi_x^2\eta_x^2 - \xi_x^2\eta_x^2] \\
&\quad + ab[2\,\xi_t\eta_t(\xi_t\eta_x + \xi_x\eta_t) - 2\,\xi_t^2\eta_t\eta_x - 2\,\eta_t^2\xi_t\xi_x] \\
&\quad + bc[2\,\xi_x\eta_x(\xi_t\eta_x + \xi_x\eta_x) - 2\,\eta_x^2\xi_t\xi_x - 2\,\xi_x^2\eta_t\eta_x \\
&= b^2(\xi_t\eta_x - \xi_x\eta_t)^2 - ac(\xi_t\eta_x - \xi_x\eta_t)^2 + a^2\,0 + a\,b\,0 + b\,c\,0 \\
&= (b^2 - ac)J^2
\end{aligned}$$

where J is given in equation 2.1.16, and since $J \neq 0$ then $\Delta^* > 0$ and the change of coordinates preserve the hyperbolic point.

The idea is to make $a^* = c^* = 0$. Since $J \neq 0$, then $b \neq 0$. That is, we want

$$a\,\zeta_t^2 + 2b\,\zeta_t\,\zeta_x + c\,\zeta_x^2 = 0. \tag{2.1.22}$$

where ζ is either ξ or η. We divide by ζ_x^2 and find

$$a\left(\frac{\zeta_t}{\zeta_x}\right)^2 + 2b\left(\frac{\zeta_t}{\zeta x}\right) + c \;=\; 0 \tag{2.1.23}$$

If we pick $\zeta(x,t) = \text{const}$ then

$$d\zeta = \zeta_t dt + \zeta_x dx = 0,$$

which is

$$\frac{\zeta_t}{\zeta_x} = -\frac{dx}{dt},$$

and equation 2.1.23 becomes

$$a\left(\frac{dx}{dt}\right)^2 - 2b\left(\frac{dx}{dt}\right) + c \quad = \quad 0. \tag{2.1.24}$$

We arrived to a quadratic equation of ordinary derivatives. A mnemotechnic trick to get to here is to make the substitution

$$\frac{\partial}{\partial t} \mapsto dx \qquad \frac{\partial}{\partial x} \mapsto -dt \tag{2.1.25}$$

on the original equation 2.1.13 and then divide by dt^2. Equation 2.1.25 turns into the two ordinary differential equations

$$\frac{dx}{dt} = \lambda_{1,2} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

The solutions, which are the characteristics, are integrated and the two constants of integration named ξ and η. We proceed to illustrate the methods by solving the problems in Problem 4.6. section of Komech's [3]'s book.

2.2 Problems

1. . Find the general solution for the following equations:

   (a)

   $$u_{xx} + 2u_{xy} - 3u_{yy} + 2u_x + 6u_y = 0.$$

   solution: To find the coordinates that transform this form into a canonical form, by using the mapping 2.1.25, on the quadratic terms, then the characteristics are given by the solutions of

   $$\left(\frac{dy}{dx}\right)^2 - 2\left(\frac{dy}{dx}\right) - 3 = 0.$$

   Let us identify the following coefficients

   $$a = 1, \quad b = 1, \quad c = -3, \quad d = 2, \quad e = 6, \quad f = 0, \quad g = 0.$$

This equation can be factored out as

$$\left(\frac{dy}{dx} - 3\right)\left(\frac{dy}{dx} + 1\right) = 0,$$

from which we find two solutions

$$\begin{aligned} \int dy &= 3\int dx \\ y - 3x &= \xi \end{aligned}$$

and

$$\begin{aligned} \int dy &= -\int dx \\ y + x &= \eta \end{aligned} \tag{2.2.26}$$

From equations 2.2.26 we find

$$\begin{aligned} \xi_x &= -3, \quad \xi_y = 1, \quad \xi_{xx} = 0, \quad \xi_{xy} = 0, \quad \xi_{yy} = 0 \\ \eta_x &= 1, \quad \eta_y = 1, \quad \eta_{xx} = 0, \quad \eta_{xy} = 0, \quad \eta_{yy} = 0 \end{aligned} \tag{2.2.27}$$

With these values we build the coefficients of the new differential equation. As checking we first verify that $a^* = c^* = 0$.

$$\begin{aligned} a^* &= a\,\xi_x^2 + 2b\,\xi_x\,\xi_y + c\,\xi_y^2 = 9 - 6 - 3 = 0 \\ c^* &= a\,\eta_x^2 + 2b\,\eta_x\,\eta_y + c\eta_y^2 = 1 + 2 - 3 = 0 \end{aligned}$$

so indeed $a^* = b^* = 0$. Let us now compute the rest of coefficients.

$$\begin{aligned} b^* &= a\,\xi_x\,\eta_x + b\,(\xi_x\,\eta_y + \xi_y\,\eta_x) + c\,\xi_y\,\eta_y = -3 + (-3 + 1) - 3 = -8 \\ d^* &= a\,\xi_{xx} + b\,\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y = 0 + 0 + 0 - 6 + 6 = 0 \\ e^* &= a\,\eta_{xx} + b\,\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y = 0 + 0 + 0 + 2 + 6 = 8 \\ f^* &= f = 0 \\ g^* &= g = 0, \end{aligned}$$

with this, equation 2.1.20 turns out to be

$$-16u_{\xi\eta} + 8u_\eta = 0.$$

That is

$$2\,u_{\xi\eta} - u_\eta = 0. \tag{2.2.28}$$

This equation is in canonical form. Let $v = u_\eta$. So we have the equation

$$2v_\xi - v = 0.$$

This is an ordinary differential equation in ξ that we can integrate. That is

$$2\frac{dv}{d\xi} = v \Rightarrow 2\int \frac{dv}{v} + C(\eta) = \int d\xi,$$

where $C(\eta)$ does not depend on ξ. Then

$$2\ln v + C(\eta) = \xi$$

and this can be written as

$$v(\xi, \eta) = \varphi(\eta)\, e^{\xi/2}.$$

and because $v = u_\eta$ then

$$\frac{du}{d\eta} = \phi(\eta)\, e^{\xi/2} \Rightarrow du = \phi(\eta)\, e^{\xi/2} d\eta$$

we integrate this equation with respect to η and find

$$u = \int \varphi(\eta) e^{\xi/2} d\eta + \psi(\xi) = e^{\xi/2} \int \varphi(\eta) d\eta + \psi(\xi).$$

we can rename

$$F(\eta) = \int \varphi(\eta) d\eta$$
$$G(\xi) = \psi(\xi)$$

and rewrite

$$u = e^{\xi/2} F(\eta) + G(\xi).$$

in terms of x and y this is

$$u = e^{-(3x-y)/2} F(x+y) + G(y-3x).$$

(b)

$$x\, u_{xx} - y u_{yy} + \frac{1}{2}(u_x - u_y) = 0, \quad x > 0 \quad y > 0.$$

solution: Let us first, normalize the equation. Since $x > 0$ we can divide by x so we can write the equation as

$$u_{xx} - \frac{y}{x} u_{yy} + \frac{1}{2\,x}(u_x - u_y) = 0, \quad x > 0 \quad y > 0.$$

Here the coefficients are:

$$a = 1, \quad b = 0, \quad c = -\frac{y}{x}, \quad d = \frac{1}{2x}, \quad e = -\frac{1}{2x}, \quad f = g = 0. \tag{2.2.29}$$

The characteristics are given by the solution of

$$\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0,$$

which can be factored out as

$$\left(\frac{dy}{dx} - \sqrt{\frac{y}{x}}\right)\left(\frac{dy}{dx} + \sqrt{\frac{y}{x}}\right) = 0.$$

The solutions are given by

$$\begin{aligned} \int \frac{dy}{\sqrt{y}} &= \int \frac{dx}{\sqrt{x}} + \xi \\ \int \frac{dy}{\sqrt{y}} &= \int -\frac{dx}{\sqrt{x}} + \eta \end{aligned}$$

That is

$$\begin{aligned} \xi &= 2(\sqrt{y} - \sqrt{x}) \\ \eta &= 2(\sqrt{y} + \sqrt{x}), \end{aligned} \tag{2.2.30}$$

and from here

$$\begin{aligned} \xi_x &= -\frac{1}{\sqrt{x}}, \quad \xi_y = \frac{1}{\sqrt{y}}, \quad \xi_{xx} = \frac{1}{2\,x\sqrt{x}}, \quad \xi_{xy} = 0 \quad \xi_{yy} = -\frac{1}{2\,y\sqrt{y}} \\ \eta_x &= \frac{1}{\sqrt{x}}, \quad \eta_y = \frac{1}{\sqrt{y}}, \quad \eta_{xx} = -\frac{1}{2\,x\sqrt{x}}, \quad \eta_{xy} = 0 \quad \eta_{yy} = -\frac{1}{2\,y\sqrt{y}}. \end{aligned}$$

We now verify that $a^* = c^* = 0$.

$$\begin{aligned} a^* &= a\,\xi_x^2 + 2b\,\xi_x\,\xi_y + c\,\xi_y^2 = \frac{1}{x} - \frac{y}{x}\frac{1}{y} = 0 \\ c^* &= a\,\eta_x^2 + 2b\,\eta_x\,\eta_y + c\eta_y^2 = \frac{1}{x} - \frac{y}{x}\frac{1}{y} = 0. \end{aligned}$$

The other coefficients are:

$$\begin{aligned}
b^* &= a\,\xi_x\,\eta_x + b\,(\xi_x\,\eta_y + \xi_y\,\eta_x) + c\,\xi_y\,\eta_y \\
&= -\frac{1}{\sqrt{x\,y}} - \frac{y}{x}\frac{1}{\sqrt{x\,y}} \\
&= -\frac{1}{\sqrt{xy}}\left(1 + \frac{y}{x}\right) \\
d^* &= a\,\xi_{xx} + b\,\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y \\
&= \frac{1}{2x\sqrt{x}} + \frac{y}{x}\frac{1}{2y\sqrt{y}} - \frac{1}{2x}\frac{1}{\sqrt{x}} - \frac{1}{2x}\frac{1}{\sqrt{y}} \\
&= 0 \\
e^* &= a\,\eta_{xx} + b\,\eta_{xy} + c\,\eta_{yy} + d\,\eta_x + e\,\eta_y \\
&= -\frac{1}{2x\sqrt{x}} + \frac{y}{x}\frac{1}{2y\sqrt{y}} - \frac{1}{2x\sqrt{x}} + \frac{1}{2x\sqrt{y}} \\
&= 0 \\
f^* &= f = 0 \\
g^* &= g = 0
\end{aligned}$$

We now build the differential equation 2.1.20 in the new coordinates (ξ, η),

$$-\frac{2}{\sqrt{xy}}\left(1 + \frac{y}{x}\right) u_{\xi\eta} = 0. \tag{2.2.31}$$

which simplifies to

$$u_{\xi\eta} = 0.$$

The solutions of this equation are given by

$$u = F(\xi) + G(\eta) = F[2(\sqrt{y} - \sqrt{x})] + G[2(\sqrt{y} + \sqrt{x})].$$

(c)

$$x^2 u_{xx} - y^2 u_{yy} - 2y u_y = 0.$$

We further assume that $x \neq 0$ and $y \neq 0$ otherwise the equation would not be hyperbolic. We can divide by x^2 and find

$$u_{xx} - \left(\frac{y}{x}\right)^2 u_{yy} - 2\left(\frac{y}{x}\right) u_y = 0. \tag{2.2.32}$$

The coefficients are

$$a = 1, \quad b = 0, \quad c = -\left(\frac{y}{x}\right)^2, \quad d = 0, \quad e = -2\left(\frac{y}{x^2}\right), \quad f = 0, \quad g = 0.$$

We map the partial derivatives into total derivatives by using mapping 2.1.25 and find

$$\left(\frac{dy}{dx}\right)^2 - \left(\frac{y}{x}\right)^2 = 0. \tag{2.2.33}$$

The factorization of this quadratic equation This yields

$$\left(\frac{dy}{dx} - \frac{y}{x}\right)\left(\frac{dy}{dx} + \frac{y}{x}\right) = 0, \tag{2.2.34}$$

from which we find

$$\begin{aligned} \int \frac{dy}{y} - \int \frac{dx}{x} &= \xi \\ \int \frac{dy}{y} + \int \frac{dx}{x} &= \eta \end{aligned} \tag{2.2.35}$$

so the canonical transformation is given by

$$\begin{aligned} \xi &= \ln y - \ln x \\ \eta &= \ln y + \ln x \end{aligned}$$

and from here

$$\begin{aligned} \xi_x &= -\frac{1}{x}, \quad \xi_y = \frac{1}{y}, \quad \xi_{xx} = \frac{1}{x^2}, \quad \xi_{xy} = 0 \quad \xi_{yy} = -\frac{1}{y^2} \\ \eta_x &= \frac{1}{x}, \quad \eta_y = \frac{1}{y}, \quad \eta_{xx} = -\frac{1}{x^2}, \quad \eta_{xy} = 0 \quad \eta_{yy} = -\frac{1}{y^2}. \end{aligned}$$

and also

$$x = e^{(\eta-\xi)/2}, \quad y = e^{(\eta+\xi)/2} \tag{2.2.36}$$

Let us verify that $a^* = b^* = 0$

$$\begin{aligned} a^* &= a\,\xi_x^2 + 2b\,\xi_x\,\xi_y + c\,\xi_y^2 = \frac{1}{x^2} - \frac{y^2}{x^2}\frac{1}{y^2} = 0 \\ c^* &= a\,\eta_x^2 + 2b\,\eta_x\,\eta_y + c\eta_y^2 = \frac{1}{x^2} - \frac{y^2}{x^2}\frac{1}{y^2} = 0 \end{aligned}$$

The other coefficients are:

$$\begin{aligned}
b^* &= a\,\xi_x\,\eta_x + b\,(\xi_x\,\eta_y + \xi_y\,\eta_x) + c\,\xi_y\,\eta_y \\
&= \frac{1}{x\,y} - \frac{y^2}{x^2}\frac{1}{y^2} \\
&= \frac{1}{x}\left(\frac{1}{y} - \frac{1}{x}\right) \\
d^* &= a\,\xi_{xx} + b\,\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y \\
&= \frac{1}{x^2} + \frac{y^2}{x^2}\frac{1}{y^2} - \frac{2y}{x^2}\frac{1}{y} \\
&= 0 \\
e^* &= a\,\eta_{xx} + b\,\eta_{xy} + c\,\eta_{yy} + d\,\eta_x + e\,\eta_y \\
&= -\frac{1}{x^2} + \frac{y^2}{x^2}\frac{1}{y^2} - \frac{2y}{x^2}\frac{1}{y} + \\
&= -\frac{2}{x^2} \\
f^* &= f = 0 \\
g^* &= g = 0
\end{aligned}$$

Equation 2.1.20 turns out to be

$$\frac{2}{x^2 y}(x-y)u_{\xi\eta} - \frac{2}{x^2}u_\eta = 0,$$

after we multiply by $x^2y/2$ we find

$$(x-y)u_{\xi\eta} - yu_\eta = 0,$$

and from 2.2.36

$$(e^{(\eta-\xi)/2} - e^{(\eta+\xi)/2})u_{\xi\eta} - e^{(\eta+\xi)/2}u_\eta = 0.$$

We further multiply by $e^{-\eta/2}e^{\xi/2}$ and find

$$(1-e^\xi)\,u_{\xi\eta} - e^\xi\;u_\eta = 0.$$

Let us call $v = u_\eta$ so

$$(1-e^\xi)\,v_\xi - e^\xi v = 0. \tag{2.2.37}$$

we consider this as an ordinary differential equation in ξ. To solve this equation we have two options

- $\xi = 0$ (this happens along the line $y = x$) This implies

$$v = 0,$$

so

$$u_\eta = 0, \Rightarrow u = F(\xi) = F(\ln y - \ln x) = F(0),$$

That is the solution is a constant,

$$u(x, y) = \text{constant}.$$

- $\xi \neq 0$ We divide 2.2.37 by $1 - e^\xi$

$$v_\xi - \frac{2e^\xi}{1 - e^\xi} v = 0. \tag{2.2.38}$$

Let us use the method integrating factor. Consider a function $M(\xi)$. We multiply both sides of 2.2.39 by $M(\xi)$ and get

$$M(\xi)\, v_\xi - M(\xi)\, \frac{e^\xi}{1 - e^\xi} v = 0. \tag{2.2.39}$$

We want the left hand side to be in the form of the derivative of a product, such that this equation can be written as

$$(M(\xi)v)' = 0, \tag{2.2.40}$$

which after integration becomes

$$vM(\xi) = \varphi(\eta) \Rightarrow v = \frac{\varphi(\eta)}{M(\xi)}. \tag{2.2.41}$$

We apply the product rule to 2.2.40 and equating to the left hand side of 2.2.39

$$M'(\xi)v + M(\xi)v_\xi = M(\xi)\, v_\xi - M(\xi)\, \frac{e^\xi}{1 - e^\xi} v$$

(note that $v' = v_\xi$) so

$$\frac{M'(\xi)}{M(\xi)} = -\frac{e^\xi}{1 - e^\xi}$$

and from here

$$M(\xi) = e^{-\int_o^\xi e^s/(1-e^s)} d\, s \tag{2.2.42}$$

where o is any constant. Now

$$-\int_o^{\xi} \frac{e^s}{1-e^s} d\, s = \ln(1-e^{\xi}) + \text{constant}$$

so equation 2.2.42 becomes

$$M(\xi) = C(1-e^{\xi})^2 \tag{2.2.43}$$

for a constant C. So, from 2.2.41

$$v = \frac{\varphi(\eta)}{(1-e^{\xi})}.$$

where now the constant C was absorbed into $\varphi(\eta)$. Since $v = u_\eta$ we have

$$u_\eta = \frac{\varphi(\eta)}{1-e^{\xi}}.$$

and so, by integration

$$u = \frac{1}{1-e^{\xi}} \int \varphi(\eta) + G(\xi).$$

and so

$$u(\xi, \eta) = \frac{1}{1-e^{\xi}}\ F(\eta) + G(\xi)$$

Finally, using 2.2.36 we see that

$$u(x,y) = \frac{1}{1-x\, y}\ F(\ln y + \ln x) + G(\ln y + \ln x)$$

(d)

$$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) = x^2\, \frac{\partial^2 u}{\partial y^2}.$$

We start by using the product rule of differentiation

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2x\frac{\partial u}{\partial x} - x^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

Dividing by x^2 and re–arranging we find

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \frac{2}{x}\,\frac{\partial u}{\partial x} = 0.$$

This defines the coefficients

$$a = 1, \quad b = 0, \quad c = -1, \quad d = \frac{2}{x}, \quad e = f = g = 0. \tag{2.2.44}$$

The characteristics are given by solutions of

$$\left(\frac{dy}{dx}\right)^2 - 1 = 0.$$

That is, the solutions of

$$\left(\frac{dy}{dx} - 1\right)\left(\frac{dy}{dx} + 1\right) = 0.$$

Integrating,

$$\int dy = \int dx + \xi \qquad \int dy = -\int dx + \eta$$

$$\xi = y - x \tag{2.2.45}$$

$$\eta = y + x \tag{2.2.46}$$

So,

$$\begin{aligned} \xi_x &= -1, \quad \xi_y = 1, \quad \xi_{xx} = 0, \quad \xi_{xy} = 0 \quad \xi_{yy} = 0 \\ \eta_x &= 1, \quad \eta_y = 1, \quad \eta_{xx} = 0, \quad \eta_{xy} = 0 \quad \eta_{yy} = 0. \end{aligned}$$

Let us verify that $a^* = b^* = 0$

$$\begin{aligned} a^* &= a\,\xi_x^2 + 2b\,\xi_x\,\xi_y + c\,\xi_y^2 = 1 - 1 = 0 \\ c^* &= a\,\eta_x^2 + 2b\,\eta_x\,\eta_y + c\eta_y^2 = 1 - 1 = 0 \end{aligned}$$

The other coefficients are:

$$\begin{aligned} b^* &= a\,\xi_x\,\eta_x + b\,(\xi_x\,\eta_y + \xi_y\,\eta_x) + c\,\xi_y\,\eta_y = -1 - 1 = -2 \\ d^* &= a\,\xi_{xx} + b\,\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y = -\frac{2}{x} \\ e^* &= a\,\eta_{xx} + b\,\eta_{xy} + c\,\eta_{yy} + d\,\eta_x + e\,\eta_y = \frac{2}{x} \\ f^* &= f = 0 \\ g^* &= g = 0 \end{aligned}$$

Equation 2.1.20 turns out to be

$$u_{\xi\eta} - \frac{1}{2x}u_\xi + \frac{1}{2x}u_\eta = 0. \tag{2.2.47}$$

From equation 2.2.45 we find

$$y = \frac{\eta + \xi}{2} \quad x = \frac{\eta - \xi}{2} \tag{2.2.48}$$

and equation 2.2.47 becomes

$$u_{\xi\eta} - \frac{1}{\eta - \xi} u_\xi + \frac{1}{\eta - \xi} u_\eta = 0. \cdots \tag{2.2.49}$$

Problems with initial conditions

1. (a)

$$u_{xx} + 4u_{xy} - 5u_{yy} + u_x - u_y = 0, \quad u\big|_{y=0} = f(x), \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = F(x).$$

We identify the coefficients

$$a = 1, \quad b = 2, \quad c = -5, \quad d = 1, \quad e = -1, \quad f = 0, \quad g = 0.$$

The characteristics are given by the solution of the quadratic equation

$$\left(\frac{dy}{dx}\right)^2 - 4\left(\frac{dy}{dx}\right) - 5 = 0.$$

We factor this quadratic form into

$$\left(\frac{dy}{dx} - 5\right)\left(\frac{dy}{dx} + 1\right)$$

which produces the following change coordinate transformations

$$\begin{aligned} y - 5x &= \xi \\ y + x &= \eta \end{aligned}$$

so

$$\begin{aligned} \xi_x &= -5, \quad \xi_y = 1, \quad \xi_{xx} = 0, \quad \xi_{xy} = 0, \quad \xi_{yy} = 0 \\ \eta_x &= 1, \quad \eta_y = 1, \quad \eta_{xx} = 0, \quad \eta_{xy} = 0, \quad \eta_{yy} = 0 \end{aligned}$$

first verify that $a^* = c^* = 0$.

$$\begin{aligned} a^* &= a\,\xi_x^2 + 2b\,\xi_x\,\xi_y + c\,\xi_y^2 = 25 - 20 - 5 = 0 \\ c^* &= a\,\eta_x^2 + 2b\,\eta_x\,\eta_y + c\eta_y^2 = 1 + 4 - 5 = 0. \end{aligned}$$

the other coefficients are

$$\begin{aligned}
b^* &= a\,\xi_x\,\eta_x + b\,(\xi_x\,\eta_y + \xi_y\,\eta_x) + c\,\xi_y\,\eta_y = -5 - 8 - 5 = -18 \\
d^* &= a\,\xi_{xx} + b\,\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y = -5 - 1 = -6 \\
e^* &= a\,\eta_{xx} + b\,\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y = 1 - 1 = 0 \\
f^* &= f = 0 \\
g^* &= g = 0,
\end{aligned}$$

Equation 2.1.20 turns out to be

$$-36u_{\xi\eta} - 6u_\xi = 0.$$

which is

$$6u_{\xi\eta} + u_\xi = 0.$$

We factor this equation as

$$\frac{\partial}{\partial \xi}\left(6\frac{\partial u}{\partial \eta} + u\right) = 0.$$

and so

$$\left(6\frac{\partial u}{\partial \eta} + u\right) = h(\eta),$$

for some h function only on η. We can look this as an ordinary differential equation

$$\frac{du}{d\eta} + \frac{u}{6} + h(\eta) = 0.$$

To solve this we multiply it by the integrating factor $e^{\eta/6}$

$$e^{\eta/6}\frac{du}{d\eta} + e^{\eta/6}\,\frac{u}{6} + e^{\eta/6}h(\eta) = \frac{d(e^{\eta/6}u)}{d\eta} + e^{\eta/6}h(\eta) = 0.$$

which can be integrated as

$$u(\xi, \eta) = e^{-\eta/6}g(\eta) + p(\xi). \tag{2.2.50}$$

where

$$g(\eta) = -\int e^{\eta/6}h(\eta)d\eta.$$

To find the exact form of f and g we use the initial conditions. Let us transform 2.2.50 into the original variables x, y, before applying the initial conditions. That is

$$u(x,y) = e^{-(x+y)/6} g(x+y) + p(y-5x).$$

From the first initial condition $u(x,0) = f(x)$ we find

$$u(x,0) = e^{-x/6} g(x) + p(-5x) = f(x), \tag{2.2.51}$$

and from the second condition

$$\left.\frac{du(x,0)}{dy}\right|_{y=0} = F(x),$$

we see

$$F(x) = -\frac{1}{6} e^{-x/6} g(x) + e^{-x/6} g'(x) + p'(-5x). \tag{2.2.52}$$

We should find $g(x)$ and $p(x)$ as functions of $f(x)$ and $F(x)$. From 2.2.51 taking the derivative with respect to x

$$f'(x) = e^{-x/6} g'(x) - \frac{1}{6} e^{-x/6} g(x) - 5p'(-5x) \tag{2.2.53}$$

From 2.2.51 and 2.2.53 into 2.2.52

$$\begin{aligned} F(x) &= -\frac{1}{6}[f(x) - p(-5x)] + f'(x) + \frac{1}{6}[f(x) - p(-5x)] \\ &\quad +5p'(-5x) + p'(-5x) = f'(x) + 6p'(-5x). \end{aligned}$$

so

$$f(s) - \frac{6}{5} p(-5s) = \int_0^s F(t)dt$$

or

$$p(-5s) = \frac{6}{5}\left(f(s) - \int_0^s F(t)dt\right)$$

and from equation 2.2.51

$$\begin{aligned} g(s) = e^{s/6}[f(s) - p(-5s)] &= e^{s/6}\left(f(s) - \frac{5}{6} f(s) + \frac{5}{6}\int_0^s F(t)dt\right) \\ &= \frac{e^{s/6}}{6}\left(f(s) - 5\int_0^s F(t)dt\right) \end{aligned}$$

and the solution for u is

$$\begin{aligned} u(x,y) &= \frac{1}{6}\left(f(x+y) - 5\int_0^{-5(x+y)} F(s)ds\right) + \\ &\quad \frac{6}{5}\left(f\left(\frac{5-y}{5}\right) - \int_0^{((5-y)/5} F(s)ds\right) \end{aligned}$$

2.3 Semi–infinite string

2.3.1 Mixed problem for the d'Alembert equation

The semi–infinite string can be modeled with the d'Alembert equation

$$u_{tt} = a^2 u_{xx}, \quad x > 0 \quad t > 0,$$

where one end is located at $x = 0$ and the other is located at infinite (which for practical purposes is $\gg at$) and a represents the wavespeed. We add to this initial and boundary conditions. The initial conditions are given by

$$u(x,0) = \varphi(x), \qquad \frac{\partial u}{\partial t}(x,0) = \psi(x), \quad x > 0.$$

The boundary condition, for the string fixed at the left ($x = 0$) extreme should be a zero condition (since the string is fixed), that is

$$u(t,0) = 0, \qquad t > 0. \tag{2.3.54}$$

The solution for the infinite string is given by equation 2.1.12, that is

$$u(x,t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)\,ds. \tag{2.3.55}$$

This equation is valid as long as $x > at$. Since $t, a, x > 0$ then $x + at > 0$, however $x - at$ could be negative and we have that φ is not defined in the negative real line. Therefore we have to modify this solution. The general solution of the equation 2.3.54 is given by

$$u(x,t) = f(x-at) + g(x+at). \tag{2.3.56}$$

We know that at $x = 0$,

$$u(0,t) = f(-at) + g(at) = 0, \quad t > 0. \tag{2.3.57}$$

so we can have the function f in its negative real line, defined as a function of g which is defined in the positive real line. So, we need to find g and that is what we do now. From the initial conditions ($t = 0$)

$$u(x,0) = f(x) + g(x) = \varphi(x) \tag{2.3.58}$$

$$\frac{\partial}{\partial t}u(x,0) = -a\,f'(x) + a\,g'(x) = \varphi(x) \tag{2.3.59}$$

from the second of these equations, after integration

$$-af(x) + ag(x) = \int_0^x \psi(s)ds + c.$$

which when added with the first of equations 2.3.58 (scaled by a) yields

$$2\,a\,g(x) = a\,\varphi(x) + \int_0^x \psi(s)ds + c.$$

so

$$g(x) = \frac{\varphi(x)}{2} + \frac{1}{2\,a}\int_0^x \psi(s)ds + \frac{c}{2a}. \tag{2.3.60}$$

From 2.3.57, we see that for negative argument, $-s < 0$

$$f(-s) = -g(s),$$

so for $x - at < 0$,

$$\begin{aligned} f(x-at) &= -g(at-x) = -\frac{\varphi(at-x)}{2} - \frac{1}{2\,a}\int_0^{a\,t-x}\psi(s)ds - \frac{c}{2a} \\ &= -\frac{\varphi(x-a\,t)}{2} + \frac{1}{2\,a}\int_{a\,t-x}^0 \psi(s)ds - \frac{c}{2a}. \end{aligned} \tag{2.3.61}$$

and then for $0 < x < a\,t$, the solution is given by

$$\begin{aligned} u(x,t) &= f(x-a\,t) + g(x+a\,t) \\ &= \frac{\varphi(x+at) - \varphi(a\,t-x)}{2} + \frac{1}{2\,a}\int_{a\,t-x}^{x+a\,t}\psi(s)ds \end{aligned}$$

2.3.2 Other boundary conditions

Instead of specifying the displacement data at $x = 0$ (Dirichlet), we can specify the derivative (Neumann) boundary conditions. Particularly

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad t > 0. \tag{2.3.62}$$

Below the principal characteristic curve $x > at$, equation 2.3.55 is still valid. Let us consider the case above the principal characteristic curves, that is, for $x < at$. We start with the general equation 2.3.56, which after substituting the boundary condition 2.3.62 is

$$f'(-at) + g'(at) = 0, \quad t > 0.$$

We use the substitution $z = -at$ so

$$f(z) + g'(-z) = 0, \quad z < 0.$$

Integrating, we get

$$f(z) - g(-z) = c_1 = \text{const}, \quad z < 0,$$

so from 2.3.60 , and for $x < at$

$$f(x-at) = g(at-x) + c_1 = \frac{1}{2}\varphi(at-x) + \frac{1}{2a}\int_0^{at-x}\psi(s)ds + \frac{c}{2a} + c_1 \tag{2.3.63}$$

and using 2.3.60 again we find

$$u(x,t) = \frac{\varphi(at-x)+\varphi(x+at)}{2} + \frac{1}{2a}\int_0^{at-x}\psi(s)ds + c_2.$$

The constant c_2 could be found continuity. That is, we assume that $u(x,t)$ is continuous at the characteristic curve $x = at$. Continuity is necessary for a string or a rod, but it is not for gases where shock waves could be present.

2.3.3 Discontinuities of a solution along a principal characteristic curve. Continuity conditions

In the previous section we solved the initial value and boundary at one side, one–dimensional (string) wave problem. We found that the solution is expressed as two branches. One branch for $x > at$ and another for $x < at$. We wonder about the continuity along the characteristic $x = at$. From the general solution

$$\begin{aligned} u(x,t) &= f(x-a\,t) + g(x+a\,t) \\ &= \frac{\varphi(x+at)-\varphi(a\,t-x)}{2} + \frac{1}{2\,a}\int_{a\,t-x}^{x+a\,t}\psi(s)ds \end{aligned}$$

The wave $g(x+at)$ is continuous when passing through the principal characteristic curve, since its level curves $x+at = \text{const}$ intersect the line $x = at$. The wave $f(x-at)$ below the principal characteristic curve $x - at = 0$ has a limit equal to $f(0-)$. Thus,

$$u\Big|_{x-at=0-} - u\Big|_{x-at=0+} = f(0-) - f(0+).$$

where

$$f(a\pm) := \lim_{x\to a\pm 0} f(x).$$

A solution $u(x,t)$ is continuous on the principal curve characteristic if

$$f(0-) = f(0+). \tag{2.3.64}$$

We now find, under which conditions the semi–infinite string problem has continuous solutions through the principal characteristic curve $x - at = 0$. Since g is continuous, we focus in the f component. Let us recall the initial conditions

$$\begin{aligned} u(x,0) &= \varphi(x) \\ \frac{\partial u(x,t)}{\partial t}\Big|_{t=0} &= \psi(x). \end{aligned}$$

which imply

$$
\begin{aligned}
f(x)+g(x) &= \varphi(x) \qquad (2.3.65)\\
-f'(x)a+g'(x)a &= \psi(x) \qquad (2.3.66)
\end{aligned}
$$

integrating the second of equations 2.3.65

$$-f(x)+g(x) \;=\; \frac{1}{a}\int_0^x \varphi(s)dx+\frac{c}{a}.$$

Subtracting this, from the first of equations 2.3.65, we find, after scaling by 1/2,

$$f(x)=\frac{1}{2}\varphi(x)-\frac{1}{2a}\int_0^x \psi(s)ds-\frac{c}{2a},$$

so

$$f(0+)=\frac{\varphi(0)}{2}-\frac{c}{2a}, \qquad (2.3.67)$$

and from 2.3.61

$$f(0-)=-\frac{\varphi(0)}{2}-\frac{c}{2a},$$

Therefore, continuity is achieved if

$$-\frac{\varphi(0)}{2}=\frac{\varphi(0)}{2}, \quad \text{hence} \quad \varphi(0)=0. \qquad (2.3.68)$$

Due to the boundary condition $u(0,t)=0, t>0$ we have that making $u(0,0)=0$ is imposing continuity at the point $(0,0)$.

Let us now study the Neumann boundary condition 2.3.62

$$\frac{\partial u}{\partial x}(0,t)=0, \quad t>0.$$

and how continuity is related to this condition. We use the relation 2.3.63, and find

$$f(0-)=\frac{\varphi(0)}{2}+\frac{c}{2\,a}+c_1.$$

From, here, equation 2.3.67 and the continuity condition 2.3.64 we see that

$$\frac{\varphi(0)}{2}+\frac{c}{2\,a}+c_1=\frac{\varphi(0)}{2}-\frac{c}{2\,a}, \quad \text{hence} \quad c_1+\frac{c}{a}=c_2=0.$$

In a string or a road, discontinuity $c_2\neq 0$ does not make sense since this implies the braking of the string or the road. In gas dynamics discontinuity is achieved along the principal characteristics and this is known in the fluid dynamics literature as shock waves. This type of discontinuity needs additional conditions not considered on this document.

Example: Find a continuous solution to the problem

$$\begin{cases} u_{tt} = 9u_{xx} & x > 0, & t > 0; \\ u(x,0) = e^{-x}, & u_t(x,0) = \cos 5x; & u_x(0,t) = u(0,t) + t. \end{cases}$$

We identify

$$\begin{aligned} \varphi(x) &= e^{-x} \\ \psi(x) &= \cos 5x. \end{aligned}$$

and so from the general solution for $x \geq 3t$ 2.1.12 we have

$$\begin{aligned} u(x,t) &= \frac{e^{-(x-3t)} + e^{-(x+3t)}}{2} + \frac{1}{6}\int_{x-3t}^{x+3t} \cos 5s \, ds \\ &= \frac{e^{-(x-3t)} + e^{-(x+3t)}}{2} + \frac{1}{6}\frac{\sin[5(x+3t)] - \sin[5(x-3t)]}{5}. \end{aligned} \tag{2.3.69}$$

Let us now assume $x < 3t$. The general solution for u is given by

$$u(x,t) = f(x-3t) + g(x+3t)$$

Let us first find $g(s)$. From the initial conditions

$$\begin{aligned} u(x,0) &= e^{-x} = f(x) + g(x) \quad x > 0 \qquad (2.3.70) \\ u_t(x,0) &= \cos 5x = -3f'(x) + 3g'(x) \quad x > 0 \end{aligned}$$

From the second equation

$$f(x) - g(x) = -\frac{\sin 5x}{15}$$

and subtracting this from the first of equations 2.3.70

$$g(x) = \frac{e^{-x}}{2} + \frac{\sin 5x}{30}.$$

We can not obtain the function $f(x)$ as we did for $g(x)$ because, the argument of f in $u(x,t)$ is $x - 3t$ which is negative for $x < 3t$, and we have that f is not defined (see for example equation 2.3.70) for negative argument. Hence we have

$$\begin{aligned} u(x,t) &= f(x-3\,t) + g(x+3\,t) \\ &= f(x-3\,t) + \frac{e^{-(x+3t)}}{2} + \frac{\sin 5(x+3t)}{30}. \end{aligned}$$

We use the boundary condition in u, that is

$$f'(-3t) - \frac{e^{-3t}}{2} + \frac{\cos 15t}{6} = f(-3t) + \frac{e^{-3t}}{2} + \frac{\sin 15t}{30} + t, \quad t > 0.$$

Before solving this ordinary differential equation we make the change of variables $y = -3t$.

$$f'(y) - f(y) - e^y + \frac{\cos 5y}{6} + \frac{\sin 5y}{30} + \frac{y}{3} = 0, \quad y < 0,$$

multiplying by the integrating factor e^{-y}

$$e^{-y} f'(y) - e^{-y} f(y) - 1 + \frac{e^{-y}\cos 5y}{6} + \frac{e^{-y}\, \sin 5y}{30} + \frac{e^{-y}\, y}{3}, \quad y < 0,$$

so

$$[e^{-y} f(y)]' - 1 + \frac{e^{-y}\cos 5y}{6} - \frac{e^{-y}\, \sin 5y}{30} + \frac{e^{-y}\, y}{3} = 0, \quad y < 0,$$

which we integrate

$$e^{-y} f(y) = \int dy \left(1 - \frac{e^{-y}\cos 5y}{6} + \frac{e^{-y}\, \sin 5y}{30} - \frac{e^{-y}\, y}{3}\right) + c, \quad y < 0,$$

for some constant c. That is

$$\begin{aligned} e^{-y} f(y) \;\; &= \;\; y + \frac{1}{156} e^{-y}(\cos 5y - 5\sin 5y) - \frac{e^{-y}}{780}(\sin 5y - 5\cos 5y) \\ &\quad + \frac{e^{-y}(y+1)}{3} + c, \end{aligned}$$

so

$$f(y) = e^y y + \frac{1}{156}(\cos 5y - 5\sin 5y) - \frac{1}{780}(\sin 5y - 5\cos 5y) + \frac{y+1}{3} + ce^y$$

and after simplifying

$$f(y) = e^y y + c\, e^y + \frac{y+1}{3} + \frac{1}{78}\cos 5y - \frac{1}{30}\sin 5y \tag{2.3.71}$$

We now use the continuity condition

$$f(0-) = f(0+)$$

For $f(0+)$ we use, from 2.3.69, the partition with the $x - 3t$ argument, that is

$$f(y) = \frac{e^{-y}}{2} - \frac{\sin 5y}{30}.$$

and for $f(0-)$ we use 2.3.71.

$$c + \frac{1}{3} + \frac{1}{78} = \frac{1}{2},$$

so

$$c = \frac{1}{2} - \frac{1}{3} - \frac{1}{78} = \frac{2}{13},$$

and so

$$\begin{aligned} u(x,t) &= e^{x-3t}(x-3t) + \frac{2}{13}\, e^{x-3t} + \frac{x-3t+1}{3} + \frac{1}{78}\cos 5(x-3t) \\ &\quad -\frac{1}{30}\sin 5(x-3t) + \frac{e^{-(x+3t)}}{2} + \frac{\sin 5(x+3t)}{30}. \end{aligned}$$

Chapter 3

Green Functions

3.1 Problems

3.1.1 The sifting property of a delta distribution

Show that

$$\langle \delta(x - x_0), \phi(x) \rangle = \phi(x_0).$$

We show this formally using distribution and then present a heuristic view through delta sequences.

sln.

Proof using distributions

$$\langle \delta(x - x_0), \phi(x) \rangle = \int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)dx = \int_{-\infty}^{\infty} \delta(y)\phi(y + x_0) = \phi(y + x_0)\Big|_{y=0} = \phi(x_0).$$

Note that we use the change of variable (substitution) on integration

$$y = x - x_0$$

Proof using delta sequences

Let us assume that f_n is a delta sequence around x_0. That is for each $n \in \mathbb{N}$:

1. $$f_n(x) \geq 0, \quad x \in \mathbb{R},$$

2. $$\int_{-\infty}^{\infty} f_n(x)dx = 1, \tag{3.1.1}$$

3.

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = 1, \quad \text{for any} \quad a < 0 < b. \tag{3.1.2}$$

We take the inner product with a test function $\phi(x)$. That is

$$\begin{aligned}
\langle f_n(x-x_0), \phi(x)\rangle &= \int_{-\infty}^{\infty} f_n(x-x_0)\phi(x)dx \\
&= \int_{-\infty}^{\infty} f_n(y)\phi(y+x_0)dy \\
&= \int_{-\infty}^{-1/n} f_n(y)\phi(y+x_0)dy + \int_{-1/n}^{1/n} f_n(y)\phi(y+x_0)dy \\
&\quad + \int_{1/n}^{\infty} f_n(y)\phi(y+x_0)dy
\end{aligned}$$

From equations 3.1.1 and 3.1.2 the first and third integrals above go to zero as $n \to \infty$. The second integral, after using the mean theorem for integrals is

$$\int_{-1/n}^{1/n} f_n(y)\phi(y+x_0)dy = \phi(y_0+x_0)\int_{-1/n}^{1/n} f_n(y) = \phi(y_0+x_0)$$

as $n \to \infty$, since $-1/n < y_0 < 1/n$, we see that $y_0 \to 0$, and from equation 3.1.2,

$$\int_{-1/n}^{1/n} f_n(y)dy = 1$$

and so

$$\lim_{n\to\infty}\langle f_n(x-x_0), \phi(x)\rangle = \phi(x_0).$$

3.1.2 Problem 23.7 c

Solve

$$\begin{aligned}
x^2u''(x) + 2xu'(x) &= f(x), \quad 1 < x < 2 \\
u'(1) &= 0, \\
u(2) + 5u'(2) &= 0,
\end{aligned}$$

sln.

Normalization

The normalized problem is

$$\begin{aligned} u''(x) + \frac{2}{x}u'(x) &= \frac{f(x)}{x^2}, \quad 1 < x < 2 && (3.1.3) \\ u'(1) &= 0, && (3.1.4) \\ u(2) + 5u'(2) &= 0, && (3.1.5) \\ & && (3.1.6) \end{aligned}$$

Note that it is okay to divide by x^2, since $x \neq 0$ in the $(1, 2)$ interval.

Finding the solution to the homogeneous equation

The homogeneous equation is

$$x^2u''(x) + 2xu'(x) = 0$$

which can be normalized as

$$u''(x) + \frac{2}{x}u'(x) = 0 \tag{3.1.7}$$

The integrating factor is

$$e^{\int 2dx/x} = e^2 \ln x = x^2,$$

that is, we multiply 3.1.7 by the integrating factor

$$x^2u''(x) + 2xu'(x) = 0$$

and write this equation as

$$\frac{d[x^2u'(x)]}{dx} = 0$$

so

$$x^2u'(x) = c$$

for a constant c, then

$$u'(x) = \frac{c}{x^2}$$

and integrating we see that

$$u(x) = -\frac{c}{x} + d$$

for another constant d. We now set up the Green function as two branches of this solution.

Setting up the Green's function with unknowns A,B,C,D

The Green function is given by

$$G(x,y) = \begin{cases} -A\frac{1}{x} + B & x < y \\ \\ -C\frac{1}{x} + D & x > y \end{cases} \tag{3.1.8}$$

Solving the system of 4 equations (2 boundary conditions and 2 jumping conditions) for the four unknowns A,B,C,D

The first boundary condition $u'(1) = 0$ implies

$$\left.\frac{A}{x^2}\right|_{x=1} = 0 \Rightarrow A = 0.$$

The second boundary condition $u(2) + 5u'(2) = 0$ implies

$$-\left.\frac{C}{x}\right|_{x=2} + D + 5\left.\frac{C}{x^2}\right|_{x=2} = 0 \Rightarrow D - \frac{C}{2} + \frac{5C}{4} = 0,$$

that is

$$\frac{3C}{4} + D = 0. \tag{3.1.9}$$

The jumping conditions are continuity of the Greens function at $x = y$ and jump discontinuity of 1 (due to the 1 in front of the normalized differential equation) of the first derivative. That is

$$-\frac{A}{y} + B = -\frac{C}{y} + D \tag{3.1.10}$$

and

$$\frac{C}{y^2} = \frac{A}{y^2} + 1 \tag{3.1.11}$$

from 3.1.10 and $A = 0$ we find

$$B = -\frac{C}{y} + D \tag{3.1.12}$$

and from 3.1.13 and $A = 0$ we find

$$C = y^2. \tag{3.1.13}$$

From 3.1.9 and 3.1.13

$$D = -3\frac{y^2}{4},$$

and from 3.1.12

$$B = -y - \frac{3y^2}{4} = -y\left(1 + \frac{3y}{4}\right).$$

The Green's function is then

$$G(x,y) = \begin{cases} -y\left(1+\frac{3y}{4}\right). & x < y \\ -y^2\left(\frac{1}{x}+\frac{3}{4}\right) & x > y \end{cases} \tag{3.1.14}$$

Setting up the solution

The solution is based on the normalized equation 3.1.3. That is

$$u''(x) + \frac{2}{x}u'(x) = \frac{f(x)}{x^2}, \quad 1 < x < 2$$

That is, the source term is $f(x)/x^2$ and not just $f(x)$. The formula for the solution is

$$\begin{aligned} u(x) &= \int_1^x G(x,y)\frac{f(y)}{y^2} + \int_x^2 G(x,y)\frac{f(y)}{y^2}dy \\ &= -\int_1^x y\left(1+\frac{3y}{4}\right)\frac{f(y)}{y^2}dy - \int_x^2 y^2\left(\frac{1}{x}+\frac{3}{4}\right)\frac{f(y)}{y^2}dy \\ &= -\int_1^x \left(1+\frac{3y}{4}\right)\frac{f(y)}{y}dy - \int_x^2 \left(\frac{1}{x}+\frac{3}{4}\right)f(y)dy \end{aligned}$$

3.1.3 Problem 23.7 d

Solve

$$\begin{aligned} (3+x^2)u''(x) + 2xu'(x) &= f(x), \quad 1 < x < 2 \\ u'(0) &= u(0), \\ u(1) &= 0, \end{aligned}$$

sln.

The normalized problem

The normalized problem is

$$\begin{aligned} u''(x) + \frac{2x}{3+x^2}u'(x) &= \frac{f(x)}{3+x^2}, \quad 0 < x < 1 \\ u'(0) &= u(0), \\ u(1) &= 0, \end{aligned}$$

as before, $3 + x^2 \neq 0$ in the $(0, 1)$ interval, so it is okay to divide by $3 + x^2$.

Finding the solution to the homogeneous equation

It is easy to see that the homogeneous equation

$$(3 + x^2)u''(x) + 2xu'(x) = 0,$$

can be written as

$$\frac{d}{dx}[(3 + x^2)u'(x)] = 0,$$

This can be found by the integration factor shown in the previous problem, but an easy observation is that the second coefficient (of $u''(x)$) is exactly the derivative of the first coefficient (of $u'(x)$). We have then

$$(3 + x^2)u'(x) = c$$

for a constant c, and

$$u'(x) = \frac{c}{3 + x^2}$$

which is easily integrated as

$$u(x) = \frac{c}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + d$$

for a constant d.

Setting up the Green's function with unknowns A,B,C,D

$$G(x, y) = \begin{cases} \frac{A}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + B & x < y \\ \\ \frac{C}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + D & x > y \end{cases} \tag{3.1.15}$$

Solving the system of 4 equations (2 boundary conditions and 2 jumping conditions) for the four unknowns A,B,C,D

From the first boundary condition

$$u'(0) = u(0) \Rightarrow \frac{A}{3} = B, \tag{3.1.16}$$

from the second boundary condition

$$u(1) = 0 \Rightarrow \frac{C}{\sqrt{3}}\frac{\pi}{6} + D = 0. \tag{3.1.17}$$

From the first of the jumping conditions

$$\frac{A}{\sqrt{3}} \arctan\left(\frac{y}{\sqrt{3}}\right) + B = \frac{C}{\sqrt{3}} \arctan\left(\frac{y}{\sqrt{3}}\right) + D \tag{3.1.18}$$

and from the jump of 1 in the derivative at $x = y$,

$$\frac{C}{3+y^2} = \frac{A}{3+y^2} + 1. \tag{3.1.19}$$

From equations 3.1.16 and 3.1.17

$$A = 3B, \qquad C = -\frac{6\sqrt{3}}{\pi} D. \tag{3.1.20}$$

and inserting this into the equations 3.1.18 and 3.1.19 we find

$$\begin{aligned} \frac{3B}{\sqrt{3}} \arctan\left(\frac{y}{\sqrt{3}}\right) + B &= -\frac{6}{\pi} D \arctan\left(\frac{y}{\sqrt{3}}\right) + D \\ -\frac{6\sqrt{3}}{\pi(3+y^2)} D &= \frac{3B}{3+y^2} + 1. \end{aligned}$$

From the second of these equations

$$\begin{aligned} D &= -\frac{\pi}{2\sqrt{3}} B - \frac{\pi(3+y^2)}{6\sqrt{3}} \\ &= -\frac{\pi}{6\sqrt{3}} \left(3B + 3 + y^2\right) \end{aligned} \tag{3.1.21}$$

and plugging this into the first of the two equations

$$\frac{3B}{\sqrt{3}} \arctan\left(\frac{y}{\sqrt{3}}\right) + B = \left(\frac{3}{\sqrt{3}} B + \frac{3+y^2}{\sqrt{3}}\right) \arctan\left(\frac{y}{\sqrt{3}}\right) + -\frac{\pi}{2\sqrt{3}} B - \frac{\pi(3+y^2)}{6\sqrt{3}}$$

which simplifies to

$$B = \left(\frac{3+y^2}{\sqrt{3}}\right) \arctan\left(\frac{y}{\sqrt{3}}\right) - \frac{\pi}{2\sqrt{3}} B - \frac{\pi(3+y^2)}{6\sqrt{3}}$$

from which

$$B = -\frac{3+y^2}{\sqrt{3}} \frac{\left[\arctan\left(\frac{y}{\sqrt{3}}\right) - \frac{\pi}{6}\right]}{1 + \frac{\pi}{2\sqrt{3}}} = \frac{(6+2y^2)\left[\arctan\left(\frac{y}{\sqrt{3}}\right) - \frac{\pi}{6}\right]}{2\sqrt{3} + \pi}$$

and from 3.1.21

$$D = -\frac{\pi}{6\sqrt{3}} \left(\frac{(18+6y^2)\left[\arctan\left(\frac{y}{\sqrt{3}}\right) - \frac{\pi}{6}\right]}{2\sqrt{3} + \pi} + 3 + y^2\right)$$

from $A = 3B$

$$A = \frac{(18+6y^2)\left[\arctan\left(\frac{y}{\sqrt{3}}\right) - \frac{\pi}{6}\right]}{2\sqrt{3}+\pi}$$

and finally from the second of equations 3.1.20

$$C = \frac{(18+6y^2)\left[\arctan\left(\frac{y}{\sqrt{3}}\right) - \frac{\pi}{6}\right]}{2\sqrt{3}+\pi} + 3 + y^2$$

So the Green function is

$$G(x,y) = \begin{cases} \frac{1}{\sqrt{3}}\frac{(18+6y^2)\left[\arctan\left(\frac{y}{\sqrt{3}}\right)-\frac{\pi}{6}\right]}{2\sqrt{3}+\pi} \arctan\left(\frac{x}{\sqrt{3}}\right) + \frac{(6+2y^2)\left[\arctan\left(\frac{y}{\sqrt{3}}\right)-\frac{\pi}{6}\right]}{2\sqrt{3}+\pi} & x < y \\ \left(\frac{1}{\sqrt{3}}\frac{(18+6y^2)\left[\arctan\left(\frac{y}{\sqrt{3}}\right)-\frac{\pi}{6}\right]}{2\sqrt{3}+\pi} + 3 + y^2\right)\arctan\left(\frac{x}{\sqrt{3}}\right) - \frac{\pi}{6\sqrt{3}}\left(\frac{(18+6y^2)\left[\arctan\left(\frac{y}{\sqrt{3}}\right)-\frac{\pi}{6}\right]}{2\sqrt{3}+\pi} + 3 + y^2\right) & \end{cases}$$

Setting up the solution

The solution is given by

$$\begin{aligned} u(x) &= \int_0^x G(x,y)\frac{f(y)}{3+y^2} + \int_x^2 G(x,y)\frac{f(y)}{3+y^2}dy \\ &= \int_0^x \left[\frac{1}{\sqrt{3}}\frac{(18+6y^2)\left[\arctan\left(\frac{y}{\sqrt{3}}\right)-\frac{\pi}{6}\right]}{2\sqrt{3}+\pi} \arctan\left(\frac{x}{\sqrt{3}}\right) + \right. \\ &\left. \frac{(6+2y^2)\left[\arctan\left(\frac{y}{\sqrt{3}}\right)-\frac{\pi}{6}\right]}{2\sqrt{3}+\pi}\right]\frac{f(y)}{3+y^2} \\ &+ \int_x^1 \left[\left(\frac{1}{\sqrt{3}}\frac{(18+6y^2)\left[\arctan\left(\frac{y}{\sqrt{3}}\right)-\frac{\pi}{6}\right]}{2\sqrt{3}+\pi} + 3 + y^2\right)\arctan\left(\frac{x}{\sqrt{3}}\right) \right. \\ &\left. -\frac{\pi}{6\sqrt{3}}\left(\frac{(18+6y^2)\left[\arctan\left(\frac{y}{\sqrt{3}}\right)-\frac{\pi}{6}\right]}{2\sqrt{3}+\pi} + 3 + y^2\right)\right]\frac{f(y)}{3+y^2} \end{aligned}$$

Chapter 4

Higher dimensions

4.1 Problems

4.1.1 Five different derivations of the volume and surface area of a hyper–sphere

Using a Gaussian Function

By far the fastest (and quite elegant) method to get there.

Notation: We call $S^n(a)$ the sphere

$$S^n(a) = \{(x_1, \cdots x_{n+1}) : \sum_{i=1}^{n+1} x_i^2 = 1\}$$

and the ball $B^n(a)$

$$B^n(a) = \{(x_1, \cdots x_n) : \sum_{i=1}^{n} x_i^2 \leq 1\}$$

It is customary to use the following notation

$$\begin{aligned} \omega_n &= v(S^n(1)) \\ \Omega_n &= v(B^n(1)) \end{aligned}$$

where $v(A)$ stands for the volume (in the sense of measure in the manifold context) of set A

The idea is to use the standard result from Gaussian functions:

$$I_n = \int_{\mathbb{R}^n} \mathrm{e}^{-\pi \|\boldsymbol{x}\|^2} d\boldsymbol{x} = 1.$$

We prove this for completeness.

In the one–dimensional case

$$I_1 = \int_{\mathbb{R}} \mathrm{e}^{-\pi t^2} dt,$$

since

$$\begin{aligned}
(I_1)^2 &= \int_{\mathbb{R}} \mathrm{e}^{-\pi x^2} dx \int_{\mathbb{R}} \mathrm{e}^{-\pi y^2} dy \\
&= \int_{\mathbb{R}^2} dx\, dy \mathrm{e}^{-\pi\,(x^2+y^2)} \quad \text{Fubini's Rule} \\
&= \int_{r>0\ 0\le\theta\le 2\pi} dr\, d\theta\, r\, \mathrm{e}^{-\pi\, r^2} \quad \text{converting to polar coordinates} \\
&= \int_{\mathrm{R}^+} dr\, r \mathrm{e}^{-\pi, r^2} \int_0^{2\pi} d\theta \quad \text{Fubini's Rule Again} \\
&= \cancel{2\pi}\, \frac{\mathrm{e}^{-\pi\, r^2}}{-\cancel{2\pi}} \Bigg|_0^\infty \\
&= 1.
\end{aligned}$$

We use this result to generalize the statement. Since $\|x\| = x_1^2 + \cdots + x_n^2$,

$$\begin{aligned}
I_n = \int_{\mathbb{R}^n} \mathrm{e}^{-\pi \|\boldsymbol{x}\|^2} d\boldsymbol{x} &= \int_{\mathbb{R}} \mathrm{e}^{-\pi x_1^2} dx_1 \cdots \int_{\mathbb{R}} \mathrm{e}^{-\pi x_n^2} dx_n \quad \text{Fubini's Rule} \\
&= 1.
\end{aligned}$$

The trick to find the surface area of an sphere in n dimensions is to realize that the whole space R^n can be seen as a ball of radius r, where $r \to \infty$. We also assume that the area of the sphere $S^{n-1}(a)$ is proportional to r^{n-1} (which can be guessed from knowing the solution in lower dimensions. The proof of this is in appendix A where I show it in the generalized polar–spherical coordinates). The Jacobian of the transformation is then r^{n-1} times the angular contribution, which is all coded in $d\omega_{n-1}$.

So, from the results above,

$$\begin{aligned}
1 &= \int_{\mathbb{R}^n} \mathrm{e}^{-\pi \|\boldsymbol{x}\|^2} d\boldsymbol{x} \\
&= \int_{S^{n-1}} d\omega_{n-1} \int_0^\infty dr\, r^{n-1}\, \mathrm{e}^{-\pi r^2} \quad \text{polar coordinates} \\
&= \omega_{n-1} \int_0^\infty dr\, r^{n-1}\, \mathrm{e}^{-\pi r^2},
\end{aligned}$$

with $\omega_{n-1} = \int_{S^{n-1}} d\omega_{n-1}$, not that ω_{n-1} is the solid angle in n dimensions. We perform the following change of variables

$$u = \pi r^2, \quad du = 2\pi r dr,$$

so

$$\omega_{n-1} = \frac{1}{\displaystyle\int_0^\infty (\frac{u}{\pi})^{(n-1)/2} \frac{1}{2\pi(\frac{u}{\pi})^{1/2}} \mathrm{e}^{-u} du} = \frac{2\pi^{n/2}}{\Gamma(n/2)} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

where

$$\Gamma(z) = \int_0^\infty u^{z-1} \mathrm{e}^{-u} du,$$

is the Euler Gamma function.

We then can write

$$\omega_n = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$$

To find the volume of the n–th unit ball $B^n(1)$, we see integrating concentric surfaces from 0 to 1 get us there. Each concentric surface has the area of $r^{n-1}\omega_n$. So

$$\Omega_n = \int_0^1 dr\, r^{n-1} \omega_{n-1} = \frac{\omega_{n-1}}{n}$$

from which

$$\Omega_n = \frac{2\pi^{n/2}}{n\Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{n/2}}{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

For a ball and sphere of radius r, we have

$$v(S^n(r)) = r^n \omega_n$$
$$v(B^n(r)) = r^n \Omega_n.$$

Using a recursion equation

The volume Let us denote $V_n(r)$ and $S_n(r)$ the volume and surface area of a hypersphere in the n–dimensional space with radius r.

We claim that the volume of $V_n(r) = K_n r^n$ for some constant K_n. We know from high school math that

$$k_1 = 2, \quad K_2 = \pi, \quad K_3 = 4\pi/3. \tag{4.1.1}$$

This is a naturally inductive argument. We show this statement is true in general by induction and at the same time find the general coefficient K_n.

We know it is true for dimensions 0 through 3. Let as assume that it is true for the dimension $n-1$. To find the volume $V_n(r)$ we divide the hypersphere in horizontal slabs along the n–th dimension, which I call x. I claim that each slab is a hypersphere of radius $\sqrt{r^2-x^2}$, in the $n-1$–th dimensional space. This is obvious in the three–dimensional space. In higher dimensions think that the hypersphere $V_n(r)$ is defined as

$$x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x^2 = r^2$$

so by fixing the coordinate x between 0 and r, we find that each slice is given by the equation

$$x_1^2 + x_2^2 + \cdots + x_{n-1}^2 = r^2 - x^2$$

which is the hypersphere $V_{n-1}(\sqrt{r^2-x^2})$, and by the inductive hypothesis

$$V_{n-1}(\sqrt{r^2-x^2}) = K_{n-1}(\sqrt{r^2-x^2})^{n-1} = K_{n-1}(r^2-x^2)^{\frac{n-1}{2}}$$

so the volume of $V_n(r)$ by integration is given by

$$V_n(r) = 2\int_0^r K_{n-1}(r^2-x^2)^{\frac{n-1}{2}}\,dx$$

where instead of integrating between $-r$ and r we integrated between 0 and r and multiplied by two, since the volume on each side (up or down) is the same. We do the following change of variables to simplify the integral,

$$x = r\cos\theta, \quad dx = -r\sin\theta d\theta$$

and θ goes between $\pi/2$ and 0 as x goes between 0 and r, so

$$\begin{aligned} V_n(r) &= 2K_{n-1}\int_0^{\pi/2} r(r^2 - r^2\cos^2\theta)^{\frac{n-1}{2}}\sin\theta\,d\theta \\ &= 2K_{n-1}\int_0^{\pi/2} r^n(1-\cos^2\theta)^{\frac{n-1}{2}}\sin\theta\,d\theta \\ &= 2K_{n-1}r^n\int_0^{\pi/2}\sin^n\theta\,d\theta. \end{aligned}$$

We then showed that

$$V_n(r) = K_n r^n \tag{4.1.2}$$

with

$$K_n = 2K_{n-1}\int_0^{\pi/2}\sin^n\theta\,d\theta, \tag{4.1.3}$$

which proofs our induction hypothesis. Next we search to evaluate K_n. We need to solve the integral

$$a_n = \int_0^{\pi/2} \sin^n \theta \, d\theta.$$

This integral can be found in tables, could be computed using *Maple* or *Mathematica* or using the Beta functions. However I will compute it from scratch assuming no knowledge of the Beta functions and without the help of a computer. The idea is to set up a recursion using integration by parts and then solve the recursive equation. By using integration by parts with

$$\begin{aligned} u &= \sin^{n-1} x, & dv &= \sin x dx \\ du &= (n-1)\sin^{n-2} x \cos x dx, & v &= -\cos x, \end{aligned}$$

we see that, for $n > 1$,

$$\begin{aligned} \int_0^{\pi/2} \sin^n \theta \, d\theta &= -\sin^{n-1} x \cos x \Big|_{x=0}^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^{n-2} \cos^2 x dx \\ &= (n-1) \int_0^{\pi/2} \sin^{n-2} x (1 - \sin^2 x) dx \\ &= (n-1) \left(\int_0^{\pi/2} \sin^{n-2} x dx - \int_0^{\pi 2} \sin^n x) dx \right) \end{aligned}$$

and rearranging

$$[1 + (n-1)] \int_0^{\pi/2} \sin^n x dx = (n-1) \int_0^{\pi/2} \sin^{n-2} x dx,$$

hence

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx.$$

We now solve the recursive equation

$$a_n = \frac{n-1}{n} a_{n-2},$$

Trivially

$$\begin{aligned} a_0 &= \frac{\pi}{2} \\ a_1 &= \int_0^{\pi/2} \sin \theta d\theta = 1, \end{aligned}$$

so

$$\begin{aligned} a_2 &= \frac{1}{2}a_0 = \frac{\pi}{4} \\ a_3 &= \frac{2}{3}a_1 = \frac{2}{3}. \end{aligned}$$

In general, if $n = 2m$ (even) then

$$\begin{aligned} a_n = a_{2m} &= \left(\frac{2m-1}{2m}\right)\left(\frac{2m-3}{2m-2}\right)\left(\frac{2m-5}{2m-4}\right)\cdots\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) \\ &= \frac{2^m}{2^m}\left(\frac{m-\frac{1}{2}}{m}\right)\left(\frac{m-\frac{3}{2}}{m-1}\right)\left(\frac{m-\frac{5}{2}}{m-2}\right)\cdots\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) \\ &= \left(\frac{m-\frac{1}{2}}{m}\right)\left(\frac{m-\frac{3}{2}}{m-1}\right)\left(\frac{m-\frac{5}{2}}{m-2}\right)\cdots\left(\frac{1}{2}\right)\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{\pi}{2}\right) \\ &= \frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(m+1\right)}\frac{\pi/2}{\Gamma\left(\frac{1}{2}\right)} \\ &= \frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(m+1\right)}\frac{\Gamma\left(\frac{1}{2}\right)^2}{2\Gamma\left(\frac{1}{2}\right)} \\ &= \frac{\Gamma\left(m+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(m+1\right)} \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)\,\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{n}{2}+1\right)}, \end{aligned}$$

where $\Gamma(x)$ is the Gamma function and $\Gamma(1/2) = \sqrt{\pi}$.

If $n = 2m + 1$ (odd) then

$$\begin{aligned} a_n = a_{2m+1} &= \left(\frac{2m}{2m+1}\right)\left(\frac{2m-2}{2m-1}\right)\left(\frac{2m-4}{2m-3}\right)\cdots\left(\frac{2}{1}\right) \\ &= \frac{2^{m-1}}{2^{m-1}}\left(\frac{m}{m+\frac{1}{2}}\right)\left(\frac{m-1}{m-\frac{1}{2}}\right)\left(\frac{m-2}{m-\frac{1}{3}}\right)\cdots\left(\frac{1}{2}\right) \\ &= \frac{\Gamma\left(m+1\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(m+\frac{3}{2}\right)} \end{aligned}$$

and from $n = 2m + 1$,

$$a_n = \frac{\Gamma\left(\frac{n+1}{2}\right)\,\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{n}{2}+1\right)}. \tag{4.1.4}$$

Interestingly in any case for n even or odd the previous formula is the same. We return back to our problem of computing the volume of a hypersphere. From equation 4.1.3

$$K_n = 2K_{n-1}a_n = K_{n-1}\,\frac{\Gamma\left(\frac{n+1}{2}\right)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}.$$

recalling from 4.1.1 that $K_1 = 2, K_2 = \pi, K_3 = 4\pi/3$ we check

$$\begin{aligned} K_2 &= K_1 \frac{2\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \pi \\ K_3 &= K_2 \frac{\Gamma\left(\frac{4}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} = \frac{\pi\sqrt{\pi}}{\frac{3}{2}\frac{1}{2}\sqrt{\pi}} = \frac{4\pi}{3}. \end{aligned}$$

In general,

$$\begin{aligned} K_n &= 2K_{n-1}a_n \\ &= 2^2 K_{n-2}a_n a_{n-1} \\ &= 2^3 K_{n-3}a_n a_{n-1}a_{n-2} \\ &\vdots \\ &= 2^{n-1}K_1 a_n a_{n-1} a_{n-2}\cdots a_2 \\ &= 2^n \frac{\cancel{\Gamma\left(\frac{n+1}{2}\right)}\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)} . \frac{\cancel{\Gamma\left(\frac{n}{2}\right)}\Gamma\left(\frac{1}{2}\right)}{2\cancel{\Gamma\left(\frac{n+1}{2}\right)}} . \frac{\cancel{\Gamma\left(\frac{n-1}{2}\right)}\Gamma\left(\frac{1}{2}\right)}{2\cancel{\Gamma\left(\frac{n}{2}\right)}} \cdots \frac{\Gamma\left(\frac{4}{2}\right)\ \Gamma\left(\frac{1}{2}\right)}{2\cancel{\Gamma\left(\frac{5}{2}\right)}} a_2 \\ &= \frac{2^n}{2^{n-2}} \frac{\Gamma\left(\frac{1}{2}\right)^{n-2}}{\Gamma\left(\frac{n+2}{2}\right)} \frac{\pi}{4} \\ &= \frac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)} \end{aligned}$$

So finally the volume of of the n–dimensional hyperball is given by

$$V_n(r) = K_n r^n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)} r^n \tag{4.1.5}$$

If $n = 2m$ is even

$$V_n(r) = V_{2m}(r) = K_{2m}r^{2m} = \frac{\pi^m}{\Gamma(m+1)} r^{2m} = \frac{\pi^m}{m!} r^{2m} \tag{4.1.6}$$

If $n = 2m + 1$ is odd

$$\begin{aligned} V_n(r) &= V_{2m+1}(r) = K_{2m+1}r^{2m+1} = \frac{\pi^{(2m+1)/2}}{\Gamma\left(\frac{2m+3}{2}\right)} r^{2m+1} \\ &= \frac{\pi^m\sqrt{\pi}}{\left(\frac{2m+1}{2}\right)\left(\frac{2m-1}{2}\right)\left(\frac{2m-3}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} r^{2m+1} \\ &= \frac{2^{m+1}\,\pi^m}{(2m+1)(2m-1)(2m-3)\cdots 1} r^{2m+1} \\ &= \frac{\pi^m(2m)(2m-2)(2m-4)\cdots 2}{(2m+1)!} r^{2m+1} && (4.1.7) \\ &= \frac{\pi^m\, m!}{(2m+1)!}(2r)^{2m+1}. && (4.1.8) \end{aligned}$$

Let us write a few formulas

$$\begin{aligned} V_1(r) &= \frac{2r}{1} = 2r \\ V_2(r) &= \frac{\pi r^2}{1} = \pi r^2 \\ V_3(r) &= \frac{\pi(2r)^3}{3!} = \frac{8\pi r^2}{6} = \frac{4\pi r^3}{3} \\ V_4(r) &= \frac{\pi^2 r^4}{2!} = \frac{\pi^2 r^4}{2} \\ V_5(r) &= \frac{2\pi^2 2^5 r^5}{5!} = \frac{64\pi^2 r^5}{120} = \frac{8\pi^2 r^5}{15} \\ V_6(r) &= \frac{\pi^3 r^6}{3!} = \frac{\pi^3 r^6}{6} \end{aligned} \tag{4.1.9}$$

The surface area There is a closed relationship between the volume and the surface area of a hypersphere. To compute the volume of a hypersphere we can integrate the ball along the radius adding up all the shells from inside out. That is

$$V_n(r) = \int_0^r S_n(x)dx,$$

so

$$V_n'(r) = S_n(r) = nK_n r^{n-1} = \frac{n\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)} r^{n-1}. \tag{4.1.10}$$

Clearly

$$\frac{V_n(r)}{S_n(r)} = \frac{r}{n} = \frac{V_n(1)}{S_n(1)} \tag{4.1.11}$$

If $n = 2m$ is even, then from differentiating 4.1.6 we find

$$V_n'(r) = \frac{2m\pi^m}{m!} r^{2m-1} = \frac{2\pi^m}{(m-1)!} r^{2m-1},$$

That is

$$S_n(r) = \frac{2\pi^m}{(m-1)!} r^{2m-1},$$

on the other hand, if $n = 2m + 1$ is odd, from 4.1.8

$$S_n(r) = \frac{\pi^m 2(2m+1)(2r)^{2m}}{(2m+1)!} = \frac{2\pi^m (2r)^{2m}}{(2m)!}$$

so taking derivatives in formulas 4.1.9 we find

$$\begin{aligned} S_1(r) &= 2 \\ S_2(r) &= 2\pi r \\ S_3(r) &= 4\pi r^2 \\ S_4(r) &= 2\pi^2 r^3 \\ S_5(r) &= \frac{8\pi^2 r^4}{3} \\ S_6(r) &= \pi^3 r^5. \end{aligned}$$

Using a mapping in Cartesian coordinates

Let us define the coordinate patch

$$\begin{aligned} \alpha : \mathbb{R}^n &\to \mathbb{R}^{n+1} \\ (x_1, \cdots x_n) &= (x_1, \cdots x_n, f(x_1, \cdots, x_n)) \end{aligned}$$

with

$$f(x_1, \cdots x_n) = \pm\sqrt{a^2 - \sum_{i=1}^{n} x_i^2}$$

Then,

$$D\alpha = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 \\ f_{,1} & \cdots & \cdots & f_{,n} \end{pmatrix}$$

where

$$f_{,j} = \frac{\partial f}{\partial x_j} = -\frac{x_j}{\sqrt{a^2 - \sum_{i=1}^{n} x_i^2}} \tag{4.1.12}$$

So

$$A = (D\alpha)^T (D\alpha) = \begin{pmatrix} 1 + f_{,1}^2 & f_{,1} f_{,2} & \cdots & f_{,1} f_{,n} \\ f_{,1} f_{,2} & 1 + f_{,2}^2 & \cdots & f_{,2} f_{,n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{,1} f_{,n} & \cdots & \cdots & 1 + f_{,n}^2 \end{pmatrix}$$

Before we evaluate the determinant, we see that for $n = 1, 2, 3$ the computation is trivial and yields

$$\det A = 1 + \sum_{i=1}^{n} f_{,i}^2. \tag{4.1.13}$$

We show that this is the case in general.

To evaluate the determinant of this matrix we observe that any entry has the form of

$$a_{ij} = \delta_{ij} + f_{,i} f_{,j},$$

and using the index notation for the determinant, we see that

$$\begin{aligned} \det A &= \epsilon_{i_1 \cdots i_n} a_{1i_1} \cdots a_{ni_n} \\ &= \epsilon_{i_1 \cdots i_n} (\delta_{1i_1} + f_{,1} f_{,i_1}) \cdots (\delta_{ni_n} + f_{,n} f_{,i_n}) \\ &= \epsilon_{PQ} \sum_{P,Q} \prod_{P,Q} \delta_{IP} f_{,I} f_{,Q} \end{aligned}$$

Where P and Q are all possible two ordered partitions of the set $1, \cdots, n$, and $I = (1, \cdots, n)$. Here

$$\begin{aligned} \delta_{IP} = \delta_{1i_1} \cdots \delta_{ki_k}, \qquad P = (i_1 \cdots i_k) \\ f_{,Q} = f_{,j_1} \cdots f_{,j_{n-k}}, \qquad Q = (j_1, \cdots j_{,n-k}). \end{aligned}$$

We will add these partitions according to the cardinal number $\#P$. If $\#P = n$ then P is a permutaion of I. Any permutation of P of I that is not the identity will produce some $\delta_{ij} = 0$, so the only contribution to the sum of permutations of $\#P = n$ comes from the identity and this contribution is 1, since the $f_{,Q}$ does not even enter into the picture (Q is the empty set ϕ).

Next let us assume $\#P = n - 1$. Then $\#Q = 1$, so each term in this collection will have $n - 1$ factors of δ's and one factor of the type $f_{,i} f_{,j}$. If $i \neq j$ then one of the deltas is of the form $\delta_{mn} = 0$ with $m \neq n$, since if all of them roll along the diagonal, the reminding i has to be that in the set Q. The total contributions for this case ($\#P = n - 1$) is given by

$$\sum_{i=1}^{n} f_{,i}^2$$

We used equation 4.1.12

If $\#P = n - 2$, so $\#Q = 2$, then. For the δ not to be zero, all the P members should be images of the identity, but 2 Q members are free to be permuted. Half of the permutations of those two members is even and half is odd. From the definition of the antisymmetric tensor, this brings the sum to 0.

If, in general, $\#P < n - 1$, $\#Q > 2$, we see that half of the permutations of those elements of Q are even and half are odd. This yields a total sum of 0.

We found then that

$$\det A = 1 + \sum_{i=1}^{n} f_{,i}^2 = 1 + \sum_{i=1}^{n} \frac{x_i^2}{a^2 - \sum_{j=1}^{n} x_j^2} = \frac{a^2}{a^2 - \sum_{i=1}^{n} x_i^2}$$

as indicated in equation 4.1.13.

From the theory of manifolds, the volume surface volume can be computed as

$$v(S^n(a)) = 2\int_{B^n(a)} \frac{a}{\sqrt{a^2 - \sum_{i=1}^n x_i^2}}$$

The 2 comes about because the two patches for $\pm$ in the function f. That is the upper and lower hemisphere, which have the same volume.

To evaluate this integral we assume that we can apply Fubini's rule. When the dominator goes to zero, the function is not bounded and hence the Fubini's rule will be invalid. Instead we assume that the denominator is bounded away from zero (that is, we are not yet in the sphere but inside) and apply Fubini's rule. Then we can take the limit as we approach the sphere after Fubini's rule.

That is,

$$v(S^n(a)) = 2\lim_{\epsilon\to 0}\int_{B^{n-1}(a)} dx_1\cdots dx_{n-1}\int_{-b+\epsilon}^{b-\epsilon} dx_n \frac{a}{\sqrt{b^2 - x_n^2}}$$

with

$$b^2 = a^2 - \sum_{i=1}^{n-1} x_i^2.$$

To evaluate the last integral we perform the substitution

$$x_n = b\sin\theta, \qquad dx_n = b\,\cos\theta d\theta,$$

which yields

$$\int_{\arcsin(-1+\epsilon/b)}^{\arcsin(1-\epsilon/b)} d\theta\, \cancel{b\cos\theta}\frac{a}{\cancel{b\cos\theta}},$$

and since $\arcsin(-1+\epsilon/b) \to -\pi/2$ and $\arcsin(1-\epsilon/b) \to \pi/2$ as $\epsilon \to 0$, then

$$v(S^n(a)) = 2\pi a\int_{B^{n-1}(a)} dx_1\cdots dx_{n-1} = 2\pi a v(B^{n-1}(a)). \tag{4.1.14}$$

We now assume (this is shown in appendix A) that the area of a sphere $S^n(a)$ is given by

$$v(S^n(a)) = a^n v(S^n(1))$$

Also,

$$v(B^{n-1}(a)) = \int_0^a S^{n-2}(1) r^{n-2} dr$$

since we can see the ball $B^{n-1}(a)$ as a continuum of concentric shells $S^{n-2}(r)$ where $0 \leq r \leq a$. So,

$$v(B^{n-1}(a)) = v(S^{n-2}(1)) \frac{a^{n-1}}{n-1} = v(S^{n-2}(a)) \frac{a}{n-1} \tag{4.1.15}$$

So, combining 4.1.14 and 4.1.15 we find the two order recursion

$$v(S^n(a)) = \frac{2\pi a^2}{n-1} v(S^{n-2}(a)).$$

With this recursion formula we find the general formula for the surface area.

$$\begin{aligned} v(S^0(a)) &= 2 \\ v(S^1(a)) &= 2\pi a \\ v(S^2(a)) = 2(2\pi a^2) &= 4\pi a^2 \\ v(S^3(a)) = \frac{2\pi a^2}{2} v(S^1(a)) &= 2\pi^2 a^3, \end{aligned}$$

If $n = 2k+1$ is odd,

$$\begin{aligned} v(S^{2k+1}(a)) &= \frac{2\pi a^2}{2k} \frac{2\pi a^2}{2k-2} \cdots \frac{2\pi a^2}{2} v(S^1(a)) \\ &= a^{2\,k} \frac{\pi^k}{k} \frac{\pi^k}{k-2} \cdots \frac{\pi}{1} (2\pi a) \\ &= \frac{2a^{2k+1} \pi^{k+1}}{\Gamma(k+1)} \end{aligned}$$

and since $k = (n-1)/2$, then

$$v(S^n(a)) = \frac{2a^n \pi^{(n+1)/2}}{\Gamma((n+1)/2)}.$$

It is interesting to observe that this formula is also valid if n is even. Then

$$v(S^n(a)) = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)} a^n,$$

and using recursion 4.1.14 we find

$$\begin{aligned} v(B^n(a)) &= \frac{v(S^{n+1}(a))}{2\pi a} \\ &= \frac{2a^{n+1}\pi^{(n+2)/2}}{2\pi a\ \Gamma((n+2)/2)} \\ &= \frac{a^n \pi^{n/2}}{\Gamma((n/2+1)}. \end{aligned}$$

Using generalized polar–spherical coordinates

The idea is to change coordinates into a polar coordinate system that takes advantage of the decoupling of the radial and angular variables. This type of coordinate system is ideal for operators (and in particular for integration, and the Laplacian which will be considered ahead) in isotropic (angular independent) conditions. Appendix A shows a derivation of the generalized polar spherical coordinates, and the derivation of volume and area measures using this approach.

Using a coordinate system sitting in a sphere

sln. Before defining the mapping let us find a "natural" coordinate system for this problem.

Call $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{u}$ a unit vector aligned with $\boldsymbol{x}$. The unit vector has coordinates

$$\boldsymbol{u} = (u_1, \cdots, u_n),$$

where

$$u_i = \frac{\boldsymbol{x} \cdot \boldsymbol{e}_i}{\|x\|} = \frac{x_i}{a} = \cos\theta_i$$

with θ_i is the director angle between the vector $\boldsymbol{x}$ and the coordinate base vector $\boldsymbol{e}_i$.

Since

$$u_1^2 + \cdots + u_n^2 = 1,$$

all u_i coordinates are not independent and we can write the last coordinate as

$$u_n = \pm\sqrt{1 - (u_1^2 + \cdots + u_{n-1}^2)}.$$

We define the mapping

$$\begin{aligned} \beta : B &\rightarrow \mathbb{R}^n \\ (u_1, \cdots, u_{n-1}, r) &\mapsto (ru_1, \cdots, ru_n), \end{aligned} \tag{4.1.16}$$

It is clear that this mapping turns our coordinates into a ball of radius r, which for $0 \leq r \leq a$ is $B^n(a)$.

The Jacobian of this transformation is given by:

$$D\beta = \begin{bmatrix} r & 0 & \cdots & 0 & u_1 \\ 0 & r & 0\cdots & 0 & u_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r & u_{n-1} \\ -\frac{ru_1}{u_n} & -\frac{ru_2}{u_n} & \cdots & -\frac{ru_{n-1}}{u_n} & u_n \end{bmatrix}$$

Since $u_n = \pm\sqrt{1 - \sum_{i=1}^{n-1} u_i^2}$, that is, u_n is multivalued and we need to consider two patches. Each patch (due to symmetry) has the same area and volume. We then only evaluate one patch and duplicate our result.

Let us evaluate $\det D\beta(\boldsymbol{u}, r)$. For this we perform Gaussian elimination to put zeroes in the last raw, except for the last entry of that raw. We find

$$\begin{aligned}
\det D\beta(\boldsymbol{u}, r) &= \frac{u_n}{u_1} \det \begin{bmatrix} \frac{ru_1}{u_n} & 0 & \cdots & 0 & \frac{u_1^2}{u_n} \\ 0 & r & 0 \cdots & 0 & u_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r & u_{n-1} \\ -\frac{ru_1}{u_n} & -\frac{ru_2}{u_n} & \cdots & -\frac{ru_{n-1}}{u_n} & u_n \end{bmatrix} \\
&= \frac{u_n}{u_1} \det \begin{bmatrix} \frac{ru_1}{u_n} & 0 & \cdots & 0 & \frac{u_1^2}{u_n} \\ 0 & r & 0 \cdots & 0 & u_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r & u_{n-1} \\ 0 & -\frac{ru_2}{u_n} & \cdots & -\frac{ru_{n-1}}{u_n} & \frac{u_1^2+u_n^2}{u_n} \end{bmatrix} \\
&= \frac{u_n}{u_1} \frac{u_n}{u_2} \det \begin{bmatrix} \frac{ru_1}{u_n} & 0 & \cdots & 0 & \frac{u_1^2}{u_n} \\ 0 & \frac{ru_2}{u_n} & 0 \cdots & 0 & \frac{u_2^2}{u_n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r & u_{n-1} \\ 0 & 0 & \cdots & -\frac{ru_{n-1}}{u_n} & \frac{u_1^2+u_2^2+u_n^2}{u_n} \end{bmatrix}
\end{aligned}$$

That is

$$\det D\beta(\boldsymbol{u}, r) = \frac{u_n}{u_1}\frac{u_n}{u_2}\cdots\frac{u_n}{u_{n-1}} \det\begin{bmatrix} \frac{ru_1}{u_n} & 0 & \cdots & 0 & \frac{u_1^2}{u_n} \\ 0 & \frac{ru_2}{u_n} & 0\cdots & 0 & \frac{u_2^2}{u_n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{ru_{n-1}}{u_n} & \frac{u_{n-1}^2}{u_n} \\ 0 & 0 & \cdots & 0 & \frac{\sum_{i=1}^{n-1} u_i^2 + u_n^2}{u_n} = \frac{1}{u_n} \end{bmatrix}$$

from which

$$\det D\beta(\boldsymbol{u}, r) = \frac{r^{n-1}}{u_n} \tag{4.1.17}$$

Hence, the volume of the $B^{n-1}(a)$ ball is given by (recall that there are two patches with equal volume)

$$\begin{aligned} v(B^n(a)) &= 2\int_{B_+^n(a)} \det D\beta(\boldsymbol{u}, r) \\ &= 2\int_0^a r^{n-1}dr \int_{B_+^n(1)} \frac{1}{u_n} du_1 \cdots du_{n-1} \\ &= \frac{2a^n}{n}\int_{B_+^n(1)} \frac{1}{u_n} du_1 \cdots du_{n-1} \\ &= a^n \lambda_n \end{aligned} \tag{4.1.18}$$

with

$$\lambda_n = \frac{2}{n}\int_{B_+^n(1)} \frac{1}{u_n} du_1 \cdots du_{n-1} \tag{4.1.19}$$

We found the formula

$$\begin{aligned} v(B^n(a)) &= 2\int_{B_+^n(a)} \det D\beta(\boldsymbol{u}, r) \\ &= 2\int_0^a r^{n-1}dr \int_{B_+^{n-1}(1)} \frac{1}{\sqrt{1 - \sum_{i=1}^{n-1} u_i^2}} du_1 \cdots du_{n-1}. \\ &= \frac{2a^n}{n}\int_{B_+^{n-1}(1)} \frac{1}{\sqrt{1 - \sum_{i=1}^{n-1} u_i^2}} du_1 \cdots du_{n-1}. \end{aligned}$$

Let us test this formula with a few simple cases

- If $n = 1$ (in the one–dimensional space, then

$$v(B^0(a)) = \frac{2a}{1} = 2a,$$

- If $n = 2$ (in the 2D space)

$$v(B^1(a)) = \frac{2a^2}{2} \int_{-1}^{1} \frac{1}{\sqrt{1-u_1^2}} du_1 = \pi a^2.$$

- If $n = 3$ (in the 3D space)

$$\begin{aligned} v(B^2(a)) &= \frac{2a^3}{3} \int_{-1}^{1} \frac{1}{\sqrt{1-u_1^2-u_2^2}} du_1 du_2 \\ &= \frac{2a^3}{3} \int_{-1}^{1} du_1 \int_{-1}^{1} du_2 \frac{1}{\sqrt{b^2-u_2^2}} \end{aligned}$$

where $b^2 = 1 - u_1^2$. The substitution $u_2 = b\sin\theta$, $du_2 = b\cos\theta d\theta$, where $\theta \in [-\pi/2, \pi/2]$ provides

$$\begin{aligned} v(B^2(a)) &= \frac{2a^3}{3} \int_{-1}^{1} du_1 \int_{-\pi/2}^{\pi/2} \frac{b\cos\theta d\theta}{b\cos\theta} \\ &= \frac{2a^3}{3} \int_{-1}^{1} du_1 \pi \\ &= \frac{4\pi a^3}{3} \end{aligned}$$

which corresponds to the three dimensional volume of a sphere.

In the last derivation we assumed that the variables u_1 and u_3 can be integrated iteratively. That is they are independent variables along orthogonal directions. This is not always true and we were lucky in obtaining the right answer with a weak assumption. In general we could have trouble.

In general, let us assume that we are considering the n–dimensional space. We will assume that the variable of integration r is decoupled from the rest of the variables (and this is right) there is not dependence between the radius and any of the polar/azimuthal directions. So we can write

$$v(B^n(a)) = \frac{2a^n}{n} \int_{B_+^{n-1}(1)} \frac{1}{\sqrt{1-\sum_{i=1}^{n-1} u_i^2}} du_1 \cdots du_{n-1}.$$

To solve this integral we assume that the denominator is bounded away from zero, so we can apply Fubini's rule, and then after we apply the rule we can take the limit as the denominator (u_n) goes to zero. Recall that the domain of integration $B_+^{n-1}(a)$ is the manifold of $n-1$–tuples $(u_1, \cdots, u_{n-1})$ under the mapping

$$u_1^2 + \cdots + u_{n-1}^2 = 1 - u_n^2,$$

and since $0 < u_n \leq 1$ then $1 - \sum_{i=1}^{n-1} u_i^2 > 0$.

Let us take the last coordinate u_{n-1} and let it be in the interval $u_{n-1} \in [-1+\epsilon, 1-\epsilon]$, $0 < \epsilon << 1$ and write

$$v(B^n(a)) = \frac{2a^n}{n} \int_{-1}^{1} du_{n-1} \int_{B_+^{n-1}(1)} \frac{1}{\sqrt{1 - \sum_{i=1}^{n-1} u_i^2}} du_1 \cdots du_{n-2}.$$

with $B_+^{n-1} = B_+^{n-2} \times [-1, 1]$. B_+^{n-2} is the manifold defined by the $(n-2)$–tuples $(u_1, \cdots u_{n-2})$ such that

$$u_1^2 + \cdots u_{n-2}^2 \leq 1.$$

with $u_{n-1} \geq 0$. We rewrite the integral as

$$v(B^n(a)) = \lim_{\epsilon \to 0} \frac{2a^n}{n} \int_{B_+^{n-2}(1)} du_1 \cdots du_{n-2} \int_{-1+\epsilon}^{1-\epsilon} \frac{du_{n-1}}{\sqrt{b^2 - u_{n-1}^2}}.$$

with $b^2 = 1 - \sum_{i=1}^{n-2} u_i^2$ and make the change of variables $u_{n-1} = b \sin \theta$, $du = b \cos \theta$, so

$$\begin{aligned} v(B^n(a)) &= \lim_{\epsilon \to 0} \frac{2a^n}{n} \int_{B_+^{n-2}(1)} du_1 \cdots du_{n-2} \int_{\arcsin(-1+\epsilon)}^{\arcsin(1-\epsilon)} d\theta \\ &= \frac{2a^n \pi}{n} \int_{B_+^{n-2}(1)} du_1 \cdots du_{n-2} \\ &= \frac{2a^n \pi}{n} v(B^{n-2})(1) \end{aligned} \tag{4.1.20}$$

This provides us with the following recursion formula. Starting at $n = 1$.

$$\begin{aligned} v(B^0(a)) &= 2a \\ v(B^2(a)) &= \frac{2a^3 \pi}{3} v(B^0(1)) = \frac{4\pi a^3}{3} \\ v(B^4(a)) &= \frac{2\pi a^5}{5} v(B^2(1)) = \frac{8\pi^2 a^5}{15} \end{aligned}$$

on the other hand, starting at $n = 2$

$$\begin{aligned} v(B^1(a)) &= \pi a^2 \\ v(B^3(a)) &= \frac{2a^4\pi}{4} v(B^1(1)) = \frac{a^4\pi^2}{2} \\ v(B^5(a)) &= \frac{2a^6\pi}{6} v(B^3(1)) = \frac{a^6\pi^3}{6}. \end{aligned}$$

In general, by simple induction and the recursions above, it is easy to observe that the formula for the volume of the n–dimensional ball

$$v(B^n(a)) = \frac{\pi^{n/2}a^n}{\Gamma\left(\frac{n}{2}+1\right)}. \tag{4.1.21}$$

To find the area of $S^n(a)$ we realize that

$$v(B^n(a)) = \int_0^a v(S^{n-1}(\rho))d\rho.$$

This integral is easy to see as thinking that an onion is the union of all its concentric shells.

Now, from the fundamental theorem of calculus,

$$v(S^n(t)) = Dv(B^{n+1}(t)). \tag{4.1.22}$$

so

$$\begin{aligned} v(S^n(a)) &= Dv(B^{n+1}(a)) \\ &= D\left(\frac{\pi^{(n+1)/2}a^{n+1}}{\Gamma(\frac{n+1}{2}+1)}\right) \\ &= 2a^n \frac{\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})} \end{aligned}$$

4.1.2 Fundamental solutions of the Laplacian

From the theory of tensor a vector and a scalar are invariant under rotations or flips. The Laplacian is the divergence (scalar) of the gradient (vector). That is

$$\nabla^2 F = \nabla \cdot \nabla F.$$

Therefore the Laplacian ∇^2 is invariant under rotations. On the other hand, the Dirac delta distribution $\delta(r)$, is only function of r. That is it is also invariant under rotations or flips. Therefore we expect the fundamental solution of

$$\nabla^2 E(\boldsymbol{x}) = \delta(\boldsymbol{x}). \tag{4.1.23}$$

to be only a function of $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Let us compute the Laplacian in polar spherical coordinates and ignore all derivatives that are not in the radial direction. By using the chain rule:

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2)^{-1/2}(2x_i) = \frac{x_i}{r}, \quad r \neq 0,$$

and

$$\frac{\partial E}{\partial x_i} = E'(r)\frac{x_i}{r},$$

and taking one more derivative

$$\frac{\partial^2 E}{\partial x_i^2} = E''(r)\frac{x_i^2}{r^2} + E'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right),$$

summing up over all partial derivatives $\partial/\partial x_i$ we find

$$\nabla^2 E(r) = E''(r) + \frac{n-1}{r}E'(r). \tag{4.1.24}$$

The integrating factor for this equation is

$$M = e^{\int (n-1)/r} dr = e^{(n-1)\ln r} = r^{n-1}.$$

So using the the integrator factor in 4.1.24 we find

$$r^{n-1}E''(r) + (n-1)r^{n-2}E'(r) = \left(r^{n-1}E'\right)'.$$

Now the Dirac delta suffers a transformation as well when going from regular Cartesian coordinates to polar spherical coordinates. We want to integrate a surface of the n–dimensional sphere and still get 1. So we should normalized the Delta by the volume of an n–dimensional ball. This can be also seen in the distributional sense by the computation of the following integral.

$$\int_{\mathbb{R}^n} \delta(\boldsymbol{x})d\boldsymbol{x} = \int_0^\infty \delta(r)dr \int_\Omega dS_r = S_r.$$

Where we separated the solid angle (Σ) contribution from the radial contribution. Sometimes the notation $\Sigma_r = S_r$ is used.

We find then

$$\nabla^2 E(r) = \frac{1}{r^{n-1}}\frac{d}{dr}\left[r^{n-1}\frac{dE}{dr}\right] = \frac{\delta(r)}{S_r}. \tag{4.1.25}$$

This is the equation that we use for all dimensions. The obvious advantage of this method is that we converted a partial differential equation (multiple dimensions) into an ordinary differential equation (one dimension) which is much easier to solve.

Before we find the fundamental solutions we solve the homogeneous equation

$$\nabla^2 E_0(r) = \frac{1}{r^{n-1}} \frac{d}{dr}\left[r^{n-1}\frac{dE_0}{dr}\right] = 0.$$

This is

$$\frac{d}{dr}\left[r^{n-1}\frac{dE_0}{dr}\right] = 0,$$

or

$$\frac{dE_0}{dr} = \frac{c_1}{r^{n-1}}. \tag{4.1.26}$$

Finally, for $r > 0$.

$$E_0(r) = \begin{cases} c_1 \ln r + c_2 & n = 2 \\ \\ \frac{(n-2)\, c_1}{r^{n-2}} + c_2 & n > 2 \end{cases} \tag{4.1.27}$$

We start with one dimension, then jump to n dimensions from which we obtain the particular cases of $n = 2$ and $n = 4$.

The Laplacian equation in one dimension

According to equation 4.1.25 for $n = 1$, We want to find $E(x)$ such that

$$\frac{d^2E}{dx^2} = \delta(x), \tag{4.1.28}$$

for $x \in \mathbb{R}$. We integrate once in an interval $(a, r), a < 0$ [1],

$$\int_a^r \frac{d^2E}{dx^2} dx = \int_a^r \delta(x) dx. \tag{4.1.29}$$

Now

$$\int_a^r \delta(x) dx = \Theta(r) = \begin{cases} 0 & r < 0 \\ 1 & r > 0. \end{cases} \tag{4.1.30}$$

So by applying the fundamental theorem of calculus in 4.1.29 and using equation 4.1.30 into equation 4.1.28 we find

$$\frac{dE}{dx}(r) - \frac{dE}{dx}(a) = \Theta(r)$$

[1] If $a > 0$, then we are integrating in a domain outside of the support of the Dirac delta which is the source function

where we rename the constant of integration as $c_1 = dE/dx(r)$ and rewrite

$$\frac{dE}{dx}(r) = \Theta(r) + c_1$$

We proceed in exact the same way and do one more integration to get

$$E(r) = r\,\Theta(r) + c_1 r + c_2. \tag{4.1.31}$$

There is an infinite number of fundamental solutions. The only way to constrain the solution to be unique is by imposing initial condition and/or radiation conditions (conditions at ∞). Not all solutions have the form shown in equation 4.1.31. Since for each solution $E(r)$, $E(-r)$ is also a solution, then the equation

$$E(-r) = -r\Theta(-r) - c_1 r + c_2$$

is also a solution. In particular if $c_1 = c_2 = 0$ then

$$E_+(r) = r\,\Theta(r) \quad \text{and} \quad E_-(r) \quad - r\Theta(-r)$$

are solutions and so

$$\frac{d^2E_+}{dr^2} + \frac{d^2E_-}{dr^2} = 2\delta(r),$$

Now

$$\frac{E_+(r) + E_-(r)}{2} = \frac{r\Theta(r) - r\Theta(-r)}{2} = \frac{|x|}{2}$$

and so

$$E(r) = \frac{|r|}{2} = \frac{E_+(r) + E_-(r)}{2} = \frac{|x|}{2},$$

is also a fundamental solution of $u''(r) = \delta(r)$, up to any linear shift.

The Laplacian equation in $n > 1$ dimensions

We should solve

$$\nabla^2 E(r) = \frac{1}{r^{n-1}}\frac{d}{dr}\left[r^{n-1}\frac{dE}{dr}\right] = \frac{\delta(r)}{S_r}.$$

We use the solution of the homogeneous problem 4.1.27, which is valid for all $r > 0$ and find appropriate c_1 and c_2 such that in the limit as $r \to 0$ the solution is fundamental. To do this we integrate the Laplace equation 4.1.23 along a ball of radius r around 0. That is

$$\int_{\|\boldsymbol{x}\|\le r} \nabla^2 E(\boldsymbol{x}) = \int_{\|\boldsymbol{x}\|\le r} \delta(\boldsymbol{x}) = 1$$

then we apply the divergence theorem on

$$\nabla^2 E(\boldsymbol{x}) = \nabla \cdot \nabla E(\boldsymbol{x})$$

and find

$$\int_{\|\boldsymbol{x}\|=1} \frac{dE(\boldsymbol{x})}{dn} dS = 1.$$

where d/dn is the normal derivative out of the surface. That is, the derivative with respect to r. We then have

$$\int_{\|\boldsymbol{x}\|=1} \frac{dE(\boldsymbol{x})}{dr} dS = 1.$$

We know, from equation 4.1.26 (for $r > 0$) that

$$\frac{dE_0}{dr} = \frac{(n-2)\,c_1}{r^{n-1}}$$

so we set up the equation

$$\int_{r=1} \frac{c_1}{r^{n-1}} dr dS = \frac{c_1}{r^{n-1}} S_n(r) = 1,$$

and from 4.1.10

$$c_1 = \frac{1}{S_n(r)} = \frac{\Gamma\left(\frac{n+2}{2}\right)}{n\pi^{n/2}}$$

So back to problem 4.1.27, we rewrite

$$E_0(r) = \begin{cases} c_1 \ln r + c_2 & n = 2 \\ \\ \frac{c_1}{r^{n-2}} + c_2 & n > 2 \end{cases}$$

and make c_2 zero for convenience [2].

We found

$$E(r) = \begin{cases} \frac{1}{2\pi} \ln r & n = 2 \\ \\ \frac{\Gamma\left(\frac{n}{2}\right)}{2(n-2)\pi^{n/2} r^{n-2}} & n > 2 \end{cases}$$

In particular, for $n = 4$ we find

$$E(r) = \frac{1}{4\pi^2 r^2}.$$

[2] solutions should have finite energy, then $c_2 = 0$, this is a radiation condition to infinity

4.1.3 The Helmholtz equation

Find the fundamental solution of the Helmholtz equation in three dimensions. That is, find $E(x)$ such that

$$\nabla^2 E(\boldsymbol{x}) \pm k^2 E(\boldsymbol{x}) = \delta(x).$$

We transform the equation into polar spherical coordinates. A lot of the work is already done because the Laplacian is most of what we have in the Helmholtz equation and we already did the transformation for the Laplacian and for $\delta(x)$ in section 4.1.2. Now, k^2 is constant, so the transformation to any coordinate system will remain as k^2.

$$\nabla^2 E(r) = \frac{1}{r^{n-1}} \frac{d}{dr}\left[r^{n-1}\frac{dE}{dr}\right] \pm k^2 E(r) = \frac{\delta(r)}{S_r}.$$

In particular for 3 dimensions we have

$$\nabla^2 E(r) = \frac{1}{r^2} \frac{d}{dr}\left[r^2\frac{dE}{dr}\right] \pm k^2 E(r) = \frac{\delta(r)}{4\pi r}. \tag{4.1.32}$$

We proceed in two steps. First we solve the homogeneous problem, which is valid everywhere for $r > 0$, in terms of two constants c_1 and c_2. Then by integrating 4.1.32 we find the constants c_1 and c_2. [3]

The homogeneously problem

which we rewrite equation 4.1.32 as

$$\left[r^2 E'(r)\right]' \pm k^2 r^2 G(r) = 0 \qquad r > 0.$$

This is a Bessel function and one trick to solve it is to make the change of variables

$$E = \frac{f}{r}$$

So we find

$$\begin{aligned}
\left[r^2 E'(r)\right]' \pm k^2 G(r) &= \left[r^2 (f/r)'\right]' \pm \frac{k^2 f r^2}{r} \\
&= 2r\left(\frac{f}{r}\right)' + r^2\left(\frac{f}{r}\right)'' \pm k^2 f r \\
&= \frac{2r}{r} f' - \frac{2rf}{r^2} + r^2\left(\frac{f'}{r} - \frac{f}{r^2}\right)' \pm k^2 f r \\
&= 2f' - \frac{2f}{r} + r^2\left(\frac{f''}{r} - \frac{f'}{r^2} - \frac{f'}{r^2} + \frac{2f}{r^3}\right) \pm k^2 f r \\
&= 2f' - \frac{2f}{r} + r f'' - 2f' + \frac{2f}{r} \pm k^2 f r = 0
\end{aligned}$$

[3] And assuming finite energy physical principles

From which the simpler equation results (in terms of f)

$$(f'' \pm k^2)f = 0.$$

The obvious solutions for this homogeneous equations are

- Complex exponential functions ($+k^2$). The two solutions here are given by

$$E(r) = \frac{c_{\pm} e^{\pm ikr}}{4\pi r}$$

Constants of integration

The constants of integration $c_{\pm}$ are chosen from the two solutions. The solution with the "minus" sign is picked, since it represents an outgoing wave (causal). By convenience we can pick $c_{-} = 1$, and so

$$E(r) = \frac{e^{-ikr}}{4\pi r} \qquad r > 0.$$

- Real exponential functions. ($-k^2$) If instead we choose to solve $(f'' - k^2)f = 0$. The solutions are real exponentials. That is

$$E(r) = \frac{c_{\pm} e^{-\pm kr}}{4\pi r} \qquad r > 0.$$

To have finite energy we pick the negative exponent, and for convenience $c_{-} = 1$. So

$$E(r) = \frac{e^{-kr}}{4\pi r} \qquad r > 0.$$

4.1.4 The Cauchy–Riemann equation

Find the fundamental function of the equation

$$\frac{\partial}{\partial \bar{z}} u$$

Let us first give meaning to the expression $\partial/\partial\bar{z}$. We assume $z = x + \mathrm{i}y$, with x, y real numbers and $\mathrm{i} = \sqrt{-1}$. Then for $f(z) = f(x + \mathrm{i}y)$,

$$\begin{aligned} df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy &= \frac{\partial f}{\partial x}\left[\frac{1}{2}(dz + d\bar{z})\right] + \frac{\partial f}{\partial y}\left[-\frac{\mathrm{i}}{2}(dz - d\bar{z})\right] \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right)d\bar{z}. \end{aligned}$$

from which we see that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right) \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right).$$

Then

$$\begin{aligned} \frac{dz}{dz} &= \frac{1}{2}\Big(1+0-0-(-1)\Big) = 1 \\ \frac{dz}{d\bar{z}} &= \frac{1}{2}\Big(1+0+0-1\Big) = 0 \\ \frac{d\bar{z}}{dz} &= \frac{1}{2}\Big(1+0-0-1\Big) = 0 \\ \frac{d\bar{z}}{d\bar{z}} &= \frac{1}{2}\Big(1+0+0-(-1)\Big) = 1. \end{aligned}$$

We observe that

$$\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} f = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = \frac{1}{4}\nabla^2 f. \tag{4.1.33}$$

We can take advantage of knowing that in 2D the fundamental solution of the Laplacian is

$$\nabla^2 E(x,y) = \delta(x,y) \quad \text{with} \quad E(x,y) = \frac{\ln r}{2\pi}$$

with $r = z\bar{z} = \sqrt{x^2+y^2}$. Then

$$\ln r = \ln(z\bar{z})^{1/2} = \frac{1}{2}\ln z\bar{z}$$

and

$$\frac{\partial z\bar{z}}{\partial z} = \frac{\partial(x^2+y^2)}{\partial z} = \frac{1}{2}\left(\frac{\partial(x^2+y^2)}{\partial x} - \mathrm{i}\frac{\partial(x^2+y^2)}{\partial y}\right) = \bar{z}.$$

Now if $f(x,y) = E(x,y)$ then

$$\frac{\partial f}{\partial z} = \frac{1}{2\pi}\frac{1}{2}\frac{1}{z\bar{z}}\bar{z} = \frac{1}{4\pi z.}$$

From 4.1.33

$$\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} f = \frac{1}{4}\nabla^2 f = \frac{\delta(x,y)}{4}.$$

Therefore by setting

$$u(x) = 4\frac{\partial f}{\partial z} = \frac{1}{\pi z},$$

we found the fundamental solution of $\partial/\partial\bar{z}$.

Appendix A

The generalized polar–spherical coordinates

.

A.1 Derivation

The natural generalization of the polar coordinates for 2D and spherical coordinates for 3D can be built upon the concepts of linear algebra. Figure A.1 shows an sketch of the polar–spherical coordinates in 3D.

Let us then assume that we have an n–dimensional euclidean space E_n, and a basis $\{\boldsymbol{e}_i\}$, $i = 1, 2, \cdots, n$. A vector $\boldsymbol{x}$ can be written in these basis as

$$\boldsymbol{x} = \sum_{i=1}^{n} x_i \boldsymbol{e}_i. \tag{A.1.1}$$

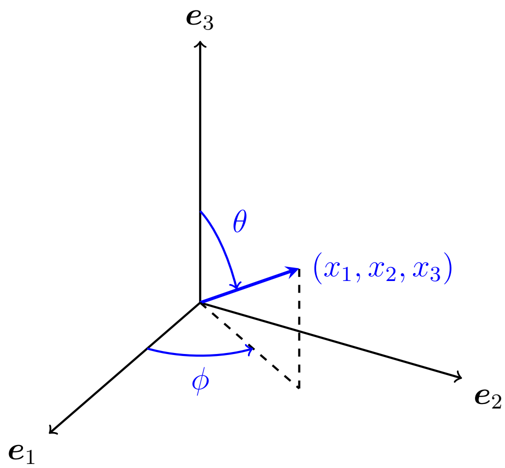
The idea is to find x_i in terms of certain polar angles ϕ_i, $i = 0, \cdots n-2$ and the vector radius $r = \|\boldsymbol{x}\|$.

There is not a unified convention for the notation of the n parameters used, except for the radius r (...and not even here; some authors use ρ for the radius depending on the dimension). I will use the following convention for the angles. The first parameter introduced is the angle θ that the vector makes with the $\boldsymbol{e}_n$ axis. Note that in the 2D case, this parameter is taken with respect to the first axis, but after that, θ is used for the polar angle, the angle that the radio vector makes with the $\boldsymbol{e}_n$ axis.

In 3D the parameters are as shown in figure A.1. That is the polar angle θ and the azimuthal angle ϕ. The starting point is in 3D. The 2D case can be seen embeded in 3D when the polar angle $\phi_0 = \theta = 0$. What about 4D and on? We introduce the azimuthal angles ϕ_i angles one–by–one, in reverse order, and to make the notation more homogeneous define $\phi_0 = \theta$.

That is, first $\phi_0 = \theta$ angle is the angle between the vector $\boldsymbol{x}$ and the $\boldsymbol{e}_n$ axis. Since

Figure A.1: Polar–Spherical Coordinates in 3D



for each x_i

$$x_i = \langle \boldsymbol{x}, \boldsymbol{e}_i \rangle.$$

then

$$x_n = \langle \boldsymbol{x}, \boldsymbol{e}_n \rangle = r \cos \phi_0 = r \cos \theta$$

and we can write A.1.1 as

$$\boldsymbol{x} = r \cos \phi_0 \; \boldsymbol{e}_n + \sum_{i=1}^{n-1} x_i \boldsymbol{e}_i, \tag{A.1.2}$$

$0 \leq \phi_0 \leq \pi$. This restriction guarantees the unique value of the x_n coordinate as a function of the angle ϕ_0, since the cosine function is one–to–one in this interval. The angle ϕ_1 will be the angle between the projection of $\boldsymbol{x}$ into the hyper–space span$\{\boldsymbol{e}_n\}\perp$ and the axis $\boldsymbol{e}_{n-1}$. This is the new "polar" angle in the $n-1$ projected dimensional space.

If we call the unit projection vector above as $\boldsymbol{u}_1$ then

$$\boldsymbol{u}_1 = \frac{\sum_{i=1}^{n-1} x_i \boldsymbol{e}_i}{\sqrt{\sum_{i=1}^{n-1} x_i^2}}$$

The angle between $\boldsymbol{e}_{n-1}$ and $\boldsymbol{u_1}$ is $\cos\phi_1$, that is

$$\begin{aligned}\cos\phi_1 &= \langle \boldsymbol{u}_1, \boldsymbol{e}_{n-1}\rangle \\ &= \frac{x_{n-1}}{\sqrt{\sum_{i=1}^{n-1} x_i^2}} \\ &= \frac{x_{n-1}}{\sqrt{r^2 - x_n^2}} \\ &= \frac{x_{n-1}}{\sqrt{r^2 - r^2\cos^2\phi_0}} \\ &= \frac{x_{n-1}}{r\sqrt{1-\cos^2\phi_0}} \\ &= \frac{x_{n-1}}{r\sin\phi_0},\end{aligned}$$

so

$$x_{n-1} = r\sin\phi_0\cos\phi_1,$$

where $0 \leq \phi_1 \leq \pi$.

We rewrite A.1.2 as

$$\boldsymbol{x} = r\cos\phi_0 + r\sin\phi_0\cos\phi_1 + \sum_{i=1}^{n-2} x_i\boldsymbol{e}_i.$$

Following the same inductive idea, we define ϕ_2 as the angle of the projection of $\boldsymbol{x}$ into the orthogonal complement of the hyper–plane spanned by the set $\{\boldsymbol{e}_n, \boldsymbol{e}_{n-1}\}$, and the vector $\boldsymbol{e}_{n-2}$. A unit vector in this hyperplane is written as

$$\boldsymbol{u}_2 = \frac{\sum_{i=1}^{n-2} x_i\boldsymbol{e}_i}{\sqrt{\sum_{i=1}^{n-2} x_i^2}},$$

and the angle between $\boldsymbol{x}$ and $\boldsymbol{u}_2$ is found by

$$\begin{aligned}\cos\phi_2 &= \langle \boldsymbol{u}_2, \boldsymbol{e}_{n-2}\rangle \\ &= \frac{x_{n-2}}{\sqrt{\sum_{i=1}^{n-2} x_i^2}} \\ &= \frac{x_{n-2}}{\sqrt{r^2 - x_n^2 - x_{n-1}^2}} \\ &= \frac{x_{n-2}}{\sqrt{r^2 - r^2\cos^2\phi_0 - r^2\sin^2\phi_0\cos^2\phi_1}} \\ &= \frac{x_{n-2}}{r\sqrt{\sin^2\phi_0 - \sin^2\phi_0\cos^2\phi_1}} \\ &= \frac{x_{n-2}}{r|\sin\phi_0|\sin\phi_1},\end{aligned}$$

Since, $\sin\phi_0 \geq 0$, for $0 \leq \phi_0 \leq \pi$, we have then

$$x_{n-2} = r\sin\phi_0 \sin\phi_1 \cos\phi_2,$$

When the projection of the vector $\boldsymbol{x}$ falls into a three dimensional space, the expression for x_3 inductively would be

$$x_3 = r\sin\phi_0 \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{n-4} \cos\phi_{n-3}$$

with all angles in the interval $[0, \pi]$ except for ϕ_1 which satisfies $0 \leq \phi_1 \leq 2\pi$.

That is,

$$x_j = r\cos\phi_{n-j} \prod_{i=0}^{n-j-1} \sin\phi_i \quad , \quad j = 3 \cdots n.$$

The last step is to project $\boldsymbol{x}$ into the plane spanned by axis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. That is,the unit vector projection is

$$\boldsymbol{u}_{n-2} = \frac{x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2}{\sqrt{x_1^2 + x_2^2}}$$

and so

$$\begin{aligned} \sin\phi_{n-2} &= \langle \boldsymbol{u}_{n-2}, \boldsymbol{e}_2 \rangle = \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \\ \cos\phi_{n-2} &= \langle \boldsymbol{u}_{n-2}, \boldsymbol{e}_1 \rangle = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \end{aligned} \tag{A.1.3}$$

At this point we note that the angle ϕ_{n-2} can be in the interval $[0, 2\pi]$ and with the radius r there is a unique mapping between (x_1, x_2) and ρ, ϕ_{n-2}, where ρ is the length of the projected $\boldsymbol{x}$ vector along the $x_1 - x_2$ plane.

Now,

$$\begin{aligned}
x_1^2 + x_2^2 &= r^2 - x_n^2 - x_{n-1}^2 - \cdots - x_3^2 \\
&= r^2 \Bigg(1 - \cos^2\phi_0 - \cos^2\phi_1 \sin^2\phi_0 - \cos^2\phi_2 \sin^2\phi_0 \sin^2\phi_1 - \cdots \\
&\qquad - \cos^2\phi_{n-3} \prod_{i=0}^{n-4} \sin^2\phi_i \Bigg) \\
&= r^2 \Bigg(\sin^2\phi_0 - \cos^2\phi_1 \sin^2\phi_0 - \cos^2\phi_2 \sin^2\phi_0 \sin^2\phi_1 - \cdots \\
&\qquad - \cos^2\phi_{n-3} \prod_{i=0}^{n-4} \sin^2\phi_i \Bigg) \\
&= r^2 \sin^2\phi_0 \Bigg(1 - \cos^2\phi_1 - \cos^2\phi_2 \sin^2\phi_1 - \cdots - \cos^2\phi_{n-3} \prod_{i=1}^{n-4} \sin^2\phi_i \Bigg) \\
&= r^2 \sin^2\phi_0 \Bigg(\sin^2\phi_1 - \cos^2\phi_2 \sin^2\phi_1 - \cdots - \cos^2\phi_{n-3} \prod_{i=1}^{n-4} \sin^2\phi_i \Bigg) \\
&= r^2 \sin^2\phi_0 \sin^2\phi_1 \Bigg(1 - \cos^2\phi_2 - \cdots - \cos^2\phi_{n-3} \prod_{i=2}^{n-4} \sin^2\phi_i \Bigg) \\
&\vdots \\
&= r^2 \prod_{i=0}^{n-3} \sin^2\phi_i,
\end{aligned}$$

so

$$\sqrt{x_1^2 + x_2^2} = r \prod_{i=0}^{n-3} \sin\phi_i,$$

and from A.1.3

$$x_1 = r \prod_{i=0}^{n-2} \sin\phi_i, \qquad x_2 = r\ \cos\phi_{n-2} \prod_{i=0}^{n-3} \sin\phi_i,$$

In summary

$$x_j = \begin{cases} r \cos\phi_{n-j} \prod_{i=0}^{n-j-1} \sin\phi_i & , \quad j = 2 \cdots n \\ r \prod_{i=0}^{n-2} \sin\phi_i & \quad j = 1 \end{cases}$$

The product $\prod$ is assumed to be 1 if the upper index is smaller than the lower index. The angle θ is the polar angle between 0 and π, the last azimuthal angle ϕ_{n-2} between

0 and 2π and rest of the angles ϕ_i are polar angles in their own domain, between 0 and π.

We can rewrite equation A.1.4 in a more compact form by calling $\phi_{n-1} = 0$.

$$x_j = r\cos\phi_{n-j} \prod_{i=0}^{n-j-1} \sin\phi_i \quad , \quad j = 1 \cdots n \tag{A.1.4}$$

I have seen notations that are in the reverse order with respect to the notation here. See Wikipedia, for example.

A.2 The Jacobian

The Jacobian is, by definition, the determinant

$$J = \begin{vmatrix} \frac{\partial x_1}{dr} & \frac{\partial x_1}{d\phi_0} & \cdots & \frac{\partial x_1}{d\phi_{n-2}} \\ \frac{\partial x_2}{dr} & \frac{\partial x_2}{d\phi_0} & \cdots & \frac{\partial x_2}{d\phi_{n-2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{dr} & \frac{\partial x_n}{d\phi_0} & \cdots & \frac{\partial x_n}{d\phi_{n-2}} \end{vmatrix}$$

In order to compute the Jacobian J we need to estimate the partial derivatives. This is done in the next section.

A.2.1 Partial Derivatives

We split the new variables in two groups.

- The variable r.

  From equation A.1.4

$$\frac{\partial x_j}{\partial r} = \cos\phi_{n-j} \prod_{i=0}^{n-j-1} \sin\phi_i \quad , \quad j = 1 \cdots n$$

- The angle ϕ_k, $\quad 0 \le k \le n-2$

$$\frac{\partial x_j}{\partial \phi_k} = \begin{cases} -r\sin\phi_{n-j} \prod_{i=0}^{n-j-1} \sin\phi_i & \text{if} \quad k = n-j \\ r\cos\phi_{n-j}\cot\phi_k \prod_{i=0}^{n-j-1} \sin\phi_i & \text{if} \quad k < n-j \\ 0 & \text{if} \quad k > n-j \end{cases}$$

For the differential matrix we will understand a matrix where its j column is given by the partial derivatives $(x_i)_{,k}$, for $x_i = 1\cdots n$, $k = 1, n$. Here the value $k = 1$ corresponds to the radius r, the value $k = 2$ corresponds to the theta angle ϕ_0, and for any other value $k > 2$, the variable of differentiation is ϕ_{k-2}. Here is the differential matrix, (recall that $\cos\phi_{n-1} = 1$)

$$D = \begin{pmatrix} \prod_{i=0}^{n-2} \sin\phi_i & r\cos\phi_0 \prod_{i=1}^{n-2} \sin\phi_i & \cdots & r\cos\phi_{n-3} \prod_{i\neq n-3}^{n-2} \sin\phi_i & r\cos\phi_{n-2} \prod_{i\neq n-2}^{n-2} \sin\phi_i \\ \cos\phi_{n-2} \prod_{i=0}^{n-3} \sin\phi_i & r\cos\phi_0 \cos\phi_{n-2} \prod_{i=1}^{n-3} \sin\phi_i & \cdots & r\cos\phi_{n-3}\cos\phi_{n-2} \prod_{i\neq n-3}^{n-3} \sin\phi_i & -r\sin\phi_{n-2} \prod_{i=0}^{n-3} \sin\phi_i \\ \cos\phi_{n-3} \prod_{i=0}^{n-4} \sin\phi_i & r\cos\phi_0 \cos\phi_{n-3} \prod_{i=1}^{n-4} \sin\phi_i & \cdots & -r\sin\phi_{n-3} \prod_{i=0}^{n-4} \sin\phi_i & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sin\phi_0 \cos\phi_1 & r\cos\phi_0 \cos\phi_1 & \cdots & 0 & 0 \\ \cos\phi_0 & -r\sin\phi_0 & \cdots & 0 & 0 \end{pmatrix}$$

The next step is to find $\det D$. First, let us see that r is in each column after the second, and so by properties of the determinant $\det D = r^{n-1} \det D'$ where D' is the same matrix D except that it does not have any r factor.

Next we pull out $\prod_{i=0}^{n-2} \sin\phi_i$ from the first raw, $\prod_{i=0}^{n-3} \sin\phi_i$ from the second raw, and so forth until $\prod_{i=0}^{0} \sin\phi_i$ from the $n-1$–th row. Then we get

$$\det D = r^{n-1} \prod_{i=0}^{n-2} \sin\phi_i \prod_{i=0}^{n-3} \sin\phi_i \cdots \prod_{i=0}^{0} \sin\phi_i \ \det G \tag{A.2.5}$$

where the matrix G is represented as follows:

$$G = \begin{pmatrix} 1 & \cot\phi_0 & \cdots & \cot\phi_{n-3} & \cot\phi_{n-2} \\ \cos\phi_{n-2} & \cot\phi_0 \cos\phi_{n-2} & \cdots & \cot\phi_{n-3}\cos\phi_{n-2} & -\sin\phi_{n-2} \\ \cos\phi_{n-3} & \cot\phi_0 \cos\phi_{n-3} & \cdots & -\sin\phi_{n-3} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cos\phi_1 & \cot\phi_0 \cos\phi_1 & \cdots & 0 & 0 \\ \cos\phi_0 & -\sin\phi_0 & \cdots & 0 & 0 \end{pmatrix}$$

We, working on G, now pull out $\cot\phi_0$ from the second column, $\cot\phi_1$ from the third column and so up to $\cot\phi_{n-2}$ from the last (n–the) column to obtain

$$\det G = \prod_{i=0}^{n-2} \cot\phi_i \det G_1 \tag{A.2.6}$$

with

$$G_1 = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \cos\phi_{n-2} & \cos\phi_{n-2} & \cdots & \cos\phi_{n-2} & -\frac{\sin^2\phi_{n-2}}{\cos\phi_{n-2}} \\ \cos\phi_{n-3} & \cos\phi_{n-3} & \cdots & -\frac{\sin^2\phi_{n-3}}{\cos\phi_{n-2}} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \cos\phi_1 & \cos\phi_1 & \cdots & 0 & 0 \\ \cos\phi_0 & -\frac{\sin^2\phi_0}{\cos\phi_0} & \cdots & 0 & 0 \end{pmatrix}$$

Now, working on G_1 we pull out $\cos\phi_{n-2}$ from the second raw, $\cos\phi_{n-3}$ from the third raw, and so forth until $\cos\phi_0$ from the last raw to find

$$\det G_1 = \prod_{i=0}^{n-2} \cos\phi_i \det H_1, \tag{A.2.7}$$

with

$$H_1 = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & -\tan^2\phi_{n-2} \\ 1 & 1 & \cdots & -\tan^2\phi_{n-3} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 0 \\ 1 & -\tan^2\phi_0 & \cdots & 0 & 0 \end{pmatrix}$$

To evaluate $\det H$ we use Gaussian elimination. That is

$$
\begin{aligned}
\det H_1 &= \cot^2\phi_{n-2}\det\begin{pmatrix}
\tan^2\phi_{n-2} & \tan^2\phi_{n-2} & \cdots & \tan^2\phi_{n-2} & \tan^2\phi_{n-2}\\
1 & 1 & \cdots & 1 & -\tan^2\phi_{n-2}\\
1 & 1 & \cdots & -\tan^2\phi_{n-3} & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
1 & 1 & \cdots & 0 & 0\\
1 & -\tan^2\phi_0 & \cdots & 0 & 0
\end{pmatrix}\\
&= \cot^2\phi_{n-2}\det\begin{pmatrix}
\tan^2\phi_{n-2} & \tan^2\phi_{n-2} & \cdots & \tan^2\phi_{n-2} & \tan^2\phi_{n-2}\\
\sec^2\phi_{n-2} & \sec^2\phi_{n-2} & \cdots & \sec^2\phi_{n-2} & 0\\
1 & 1 & \cdots & -\tan^2\phi_{n-3} & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
1 & 1 & \cdots & 0 & 0\\
1 & -\tan^2\phi_0 & \cdots & 0 & 0
\end{pmatrix}\\
&= \sec^2\phi_{n-2}\det\begin{pmatrix}
1 & 1 & \cdots & 1 & 1\\
1 & 1 & \cdots & 1 & 0\\
1 & 1 & \cdots & -\tan^2\phi_{n-3} & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
1 & 1 & \cdots & 0 & 0\\
1 & -\tan^2\phi_0 & \cdots & 0 & 0
\end{pmatrix}\\
&= (-1)^n\sec^2\phi_{n-2}\det H_2
\end{aligned}
$$

with H_2 the $n-1\times n-1$ matrix

$$
H_2 = \begin{pmatrix}
1 & 1 & \cdots & 1\\
1 & 1 & \cdots & -\tan^2\phi_{n-3}\\
\vdots & \vdots & \ddots & \vdots\\
1 & 1 & \cdots & 0\\
1 & -\tan^2\phi_0 & \cdots & 0
\end{pmatrix}.
$$

We now that H_2 has the same structure as H_1, but instead of being an $n\times n$ matrix it is an $n-1\times n-1$, so we have the following recursion

$$
\begin{aligned}
\det H_1 &= (-1)^n\sec^2\phi_{n-2}\det H_2\\
&= (-1)^n(-1)^{n-1}\sec^2\phi_{n-2}\sec^2\phi_{n-3}\det H_3 = (-1)^1\sec^2\phi_{n-2}\sec^2\phi_{n-3}\det H_3\\
&\vdots\\
&= (-1)^{n-1}\prod_{i=1}^{n-2}\sec^2\phi_i\det H_{n-1},
\end{aligned}
$$

with

$$
H_{n-1} = \begin{pmatrix} 1 & 1\\ 1 & -\tan^2\phi_0\end{pmatrix},
$$

and

$$\det H_1 = (-1)^{n-2} \prod_{i=0}^{n-2} \sec^2 \phi_i. \tag{A.2.8}$$

We will ignore the sign in what follows. The sign is related to, if the system is right or left hand oriented. I assume it is right hand oriented and ignore the "minus" sign.

We are now ready to put all the pieces together, by invoking equations A.2.5, A.2.6, A.2.7, and A.2.8. That is

$$\begin{aligned}
\det D &= r^{n-1} \prod_{i=0}^{n-2} \sin \phi_i \prod_{i=0}^{n-3} \sin \phi_i \cdots \prod_{i=0}^{0} \sin \phi_i \;\det G \\
&= r^{n-1} \prod_{i=0}^{n-2} \sin \phi_i \prod_{i=0}^{n-3} \sin \phi_i \cdots \prod_{i=0}^{0} \sin \phi_i \prod_{i=0}^{n-2} \cot \phi_i \det G_1, \\
&= r^{n-1} \prod_{i=0}^{n-2} \sin \phi_i \prod_{i=0}^{n-3} \sin \phi_i \cdots \prod_{i=0}^{0} \sin \phi_i \prod_{i=0}^{n-2} \cot \phi_i \prod_{i=0}^{n-2} \cos \phi_i \det H_1 \\
&= r^{n-1} \prod_{i=0}^{n-2} \sin \phi_i \prod_{i=0}^{n-3} \sin \phi_i \cdots \prod_{i=0}^{0} \sin \phi_i \prod_{i=0}^{n-2} \cot \phi_i \prod_{i=0}^{n-2} \cos \phi_i \prod_{i=0}^{n-2} \sec^2 \phi_i \\
&= r^{n-1} \frac{\cancel{\prod_{i=0}^{n-2} \sin \phi_i} \prod_{i=0}^{n-3} \sin \phi_i \cdots \prod_{i=0}^{0} \sin \phi_i}{\cancel{\prod_{i=0}^{n-2} \sin \phi_i}} \\
&= r^{n-1} \prod_{i=0}^{n-3} \sin \phi_i \cdots \prod_{i=0}^{0} \sin \phi_i \\
&= r^{n-1} \prod_{i=1}^{n-2} \sin^{n-i-1} \phi_{i-1}.
\end{aligned}$$

A.3 The Volume and Surface Area

With the help of the Jacobian we can now compute the volume and the surface of a sphere of radius a. The formula for the volume is given by

$$\begin{aligned}
V_n &= \int_S dV = \int_S r^{n-1} \sin^{n-2} \phi_0 \, \sin^{n-3} \phi_1 \, \cdots \sin \phi_{n-3} \sin \phi_1 dr \, d\phi_0 \, d\phi_1 \cdots d\phi_{n-2} \\
&= 2\pi \int_S r^{n-1} \sin^{n-2} \phi_0 \, \sin^{n-3} \phi_1 \, \cdots \sin \phi_{n-3} \sin \phi_1 dr \, d\phi_0 \, d\phi_1 \cdots d\phi_{n-3},
\end{aligned}$$

with $S = [0, a] \prod_{i=0}^{n-3} [0, \pi/2]$ Given that both r and $\sin \phi_i$ are bounded functions in the integration domains, we can apply Fubini's rule and separate the integrals in decoupled variables as follows:

$$
\begin{aligned}
V_n &= 2\pi \int_0^a r^{n-1} dr \int_0^\pi \sin^{n-2}\phi_0 d\phi_0 \int_0^\pi \sin^{n-3}\phi_1 d\phi_1 \cdots \int_0^\pi \sin\phi_{n-3} d\phi_{n-3} \\
&= \frac{2\pi\, r^n}{n} \int_0^\pi \sin^{n-2}\phi_0 d\phi_0 \int_0^\pi \sin^{n-3}\phi_1 d\phi_1 \cdots \int_0^\pi \sin\phi_{n-3} d\phi_{n-3}
\end{aligned}
$$

Let us test this equation with simple cases. In 2D, $n = 2$ and

$$V_2 = \frac{2\pi a^2}{2} = \pi a^2.$$

In 3D

$$V_3 = \frac{2\pi r^3}{3} \int_0^\pi \sin\phi_0 d\phi_0 = 2\frac{2\pi a^3}{3} = \frac{4\pi a^3}{3}.$$

For higher dimensions the problem is more interesting. We have the recursion

$$
\begin{aligned}
V_{n+1} &= \frac{2\pi\, r^{n+1}}{n+1} \int_0^\pi \sin^{n-1}\phi_0 d\phi_0 \int_0^\pi \sin^{n-2}\phi_1 d\phi_1 \cdots \int_0^\pi \sin\phi_{n-2} d\phi_{n-2} \\
&= \frac{nr}{n+1} \int_0^\pi \sin^{n-1}\phi_0 d\phi_0 \\
&\quad \left(\frac{2\pi r^n}{n} \int_0^\pi \sin^{n-2}\phi_1 d\phi_1 \int_0^\pi \sin^{n-3}\phi_1 d\phi_1 \cdots \int_0^\pi \sin\phi_{n-2} d\phi_{n-2} \right) \\
&= \frac{nr}{n+1} V_n \int_0^\pi \sin^{n-1}\phi_0 d\phi_0,
\end{aligned}
$$

and after using equation 4.1.4 where

$$a_{n-1} = \int_0^{\pi/2} \sin^{n-1}\phi_0 d\phi_0 = \frac{1}{2} \int_0^\pi \sin^{n-1}\phi_0 d\phi_0,$$

we find the recursion:

$$
\begin{aligned}
V_{n+1} = \frac{nr(2\,a_{n-1})}{n+1} V_n &= \frac{nr}{n+1} \frac{\Gamma\left(\frac{n}{2}\right)\ \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} V_n \\
&= \frac{nr}{n+1} \frac{\Gamma\left(\frac{n}{2}\right)\ \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \frac{(n-1)r}{n} \frac{\Gamma\left(\frac{n-1}{2}\right)\ \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} V_{n-1}
\end{aligned}
$$

where instead of writing a we use the symbol r for the radius. To easy on this recursion we will separate member factors as follows.

- We will stop at $V_2 = \pi r^2$.
- The r factors will go up to r^{n-1}

- The fractions with n–like factors will be arranged as follows

$$\frac{\cancel{n}}{n+1}\frac{\cancel{n-1}}{\cancel{n}}\frac{\cancel{n-2}}{\cancel{n-1}}\cdots\frac{2}{\cancel{3}}=\frac{2}{n+1}.$$

- Finally, the Gamma function factors go like

$$\frac{\cancel{\Gamma\left(\frac{n}{2}\right)}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\frac{\cancel{\Gamma\left(\frac{n-1}{2}\right)}\Gamma\left(\frac{1}{2}\right)}{\cancel{\Gamma\left(\frac{n}{2}\right)}}\cdots\frac{\cancel{\Gamma\left(\frac{2}{2}\right)}\Gamma\left(\frac{1}{2}\right)}{\cancel{\Gamma\left(\frac{3}{2}\right)}}=\frac{\Gamma^{n-1}\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$$

Putting all pieces togheter we find

$$V_{n+1}=\pi r^{n+1}\frac{2}{n+1}\frac{\Gamma^{n-1}\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}=\frac{2\pi r^{n+1}\Gamma^{n-1}\left(\frac{1}{2}\right)}{(n+1)\Gamma\left(\frac{n+1}{2}\right)}$$

or

$$\begin{aligned}V_n &= \frac{\pi r^n\Gamma^{n-2}\left(\frac{1}{2}\right)}{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)}\\ &= \frac{\pi^{n/2}r^n}{\Gamma\left(\frac{n+2}{2}\right)}\end{aligned}$$

which agrees with equation 4.1.5 for the volume of an n–dimensional ball.

The surface area is easily found by taking the derivative of the volume formula. So the n–dimensional surface area formula is

$$S_n=\frac{dV_n(r)}{dr}=\frac{nr^{n-1}\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)}$$

which is a repeat of what was done in equation 4.1.10.

Appendix B

Derivations of wave equations

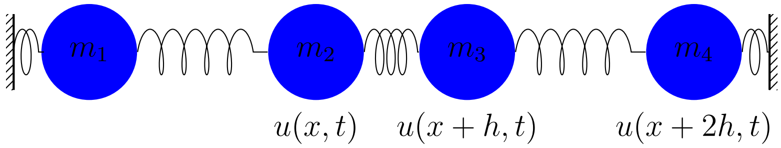
Which waves? Waves are everywhere. Waves are acoustic or elastic or electromagenetic etc. Waves are studied in 1D, 2D, 3D and n–dimensional spaces. Waves are studied in homogeneous/hetereogeneous, isotropic/anisotropic media. Waves carry energy and the manifestation of that energy is motion of particles which is registred as for example, particle displacement, particle velocity along some direction, pressure, acceleration along certain direction, voltage etc. Among the characteristics of wave propagation are frequency, wave speed, wavelength, phase etc. These characteristics depend on the medium where they travel, for example if it is a string, they depend on the string length, linear density and tension of the string. If it is a system of springs, it depends on the Hook's constant k and the mass. If it is an acoustic wave in water or the air, it depends on density and compressibility parameter K. Because of acoustic waves we can hear, because of electromagnetic waves we can see. Elastic waves let us see through the earth's interior with the use of seismology. Sometimes waves are characterized according to the direction of particle vibration. For example, if the particle vibration is aligned with the energy displacement direction, they are called compressional, while if they are orthogonal to that direction (or quasi–orthogonal in the case of anisotropic media) they are called transverse or shear. In 1D, the spring is a simple example of a generator of compressional waves, while a guitar string is a simple example of transverse or shear waves. This appendix show some simple derivations of these two types of waves.

All wave equation derivations in elastic solids are the result of two basic laws from continuum mechancis:

- Hook's Law
- Newton's second law.

In the case of a string only Newton's second law is needed. In fluids the continuity equation and Newton's second law are needed. So, in general, all elastic or acoustic wave equations are consequences of Newton second's law. In the case of electromagnetic waves, they can be derived directly from Maxwell equations.

Figure B.1: System of springs and masses



B.1 One dimensional compressional wave equation

I present two derivations. The first derivation illustrates the case of a one–dimensional compressional wave in a system of springs. It is interesting because it shows the transicion between the discrete to the continuum. The second derivation is a more classical derivation based on continuum mechanic concepts.

B.1.1 From a system of springs to a bar

Figure B.1 shows a system of springs with masses. Let us for the moment focus on the mass m_3 and study the interaction of this mass with its two neighbors m_2 and m_4. For simplicity we assume that all springs have the same Hook's constant k and mass m. The quantity $u(x,t)$ represents the displacement of the mass at the location x from its center of equilibrium (where the force is zero).

Then by Newton second law mass m_3 is moving with some force $F_H = ma$ where

$$a = \frac{\partial^2 u(x+h,t)}{\partial t^2}$$

The force F_H is the resultant of two forces by its neighbor masses, that is

$$\begin{aligned} F_H &= k[u(x+2h,t) - u(x+h,t)] - k[u(x+h,t) - u(x,t)] \\ &= k\,[u(x+2h,t) - 2u(x+h,t) + u(x,t)] \end{aligned}$$

so

$$m\frac{\partial^2 u(x+h,t)}{\partial t^2} = k\,[u(x+2h,t) - 2u(x+h,t) + u(x,t)]$$

or

$$\frac{\partial^2 u(x+h,t)}{\partial t^2} = \frac{k}{m}\,[u(x+2h,t) - 2u(x+h,t) + u(x,t)] \qquad \text{(B.1.1)}$$

We are ready to pass to the limit, that sends us from a discrete system of springs into the contiuum. Assume we have an infinite number of masses between springs and that $h \to 0$. That is, we are converting an infinite system of springs into a bar with density ρ and cross–section area A_0.

We now use the relationship between the Hook's constant and the Young's modulus E, that is

$$k = \frac{EA_0}{h}$$

with A_0 the cross section of the string. We see that

$$m = \rho V = \rho A_0 h$$

and so

$$\frac{k}{m} = \frac{EA_0}{\rho A_0 h^2} = \frac{E}{\rho h^2}$$

Now taking the limit in B.1.1 we find

$$\frac{\partial^2 u(x,h)}{\partial t^2} = \frac{E}{\rho}\frac{\partial u(x,t)}{\partial x^2}. \tag{B.1.2}$$

For this particular wave equation, the wave speed is given by

$$v = \sqrt{\frac{E}{\rho}}.$$

B.1.2 From continuum mechanics tools

Assume a homogenous bar with stress σ distributions along the x direction as shown in figure B.2. Here

$$\sigma = \frac{P}{A}$$

is the axial stress and A is the cross–section area of the bar. As shown in the figure, the stress in two close points is approximately given by σ at the point x and $\sigma + \frac{\partial \sigma}{\partial x}\Delta x$ at $x + \Delta x$.

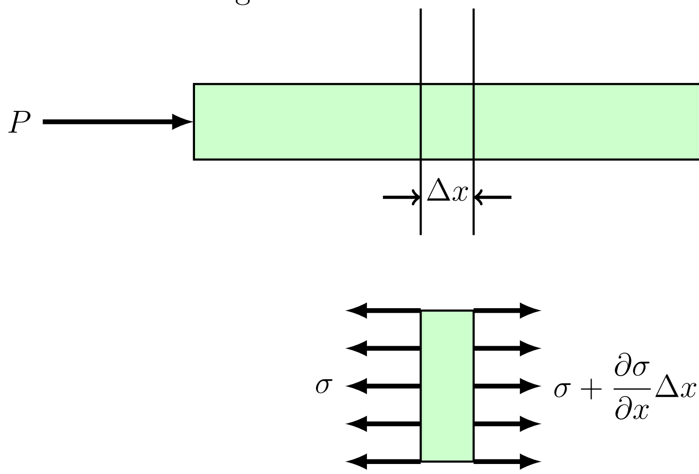
Now, from Newton's second law the total the total force resulting from the stresses on left and right is

$$-\sigma A + (\sigma + \frac{\partial \sigma}{\partial x}\Delta x)A = m\rho\frac{\partial^2 u}{\partial t^2}.$$

Now since $m = A\Delta x$ we find

$$-\sigma + (\sigma + \frac{\partial \sigma}{\partial x}\Delta x) = \rho\frac{\partial^2 u}{\partial t^2}\Delta x.$$

Figure B.2: A solid bar



or

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}. \tag{B.1.3}$$

We now apply Hook's law, that is

$$\sigma = E\epsilon,$$

where E is the Young's modulus and $\epsilon = \dfrac{\partial u}{\partial x}$, so equation B.1.3 becomes

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E}\frac{\partial^2 u}{\partial t^2},$$

which agrees with equation B.1.2.

B.2 One dimensional transverse wave equation

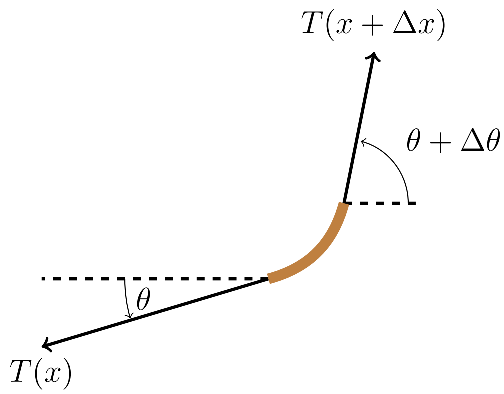
This derivation is taken from Armstead and Karls article.

Figure B.3 shows a piece of string with two tension vectors at the ends. The angles and bending of the string are exagerated to make the picture easy to visualize.

We assume that the motion is pure transverse. That is, that the energy propagates along the horizontal while the vibration is taken place along the vertical. So, since there is not horizontal vibration the horizontal sum of forces is zero. For this to happen the horizontal tension should be constant. That is

$$T(x)\cos\theta = T(x+\Delta x)\cos(\theta+\Delta\theta) = T. \tag{B.2.4}$$

Figure B.3: A piece of string (brown). The piece of string represents the displacement function $u(x,t)$.



For the vertical direction we apply Newton's second law. That is,

$$-T(x)\sin\theta + T(x+\Delta x)\sin(\theta+\Delta\theta) = m\frac{\partial^2 u}{\partial t^2}(x,t) \tag{B.2.5}$$

where $u(x,t)$ measures the displacement in the vertical direction.

Now from B.2.4

$$T(x) = \frac{T}{\cos\theta} \quad \text{and} \quad T(x+\Delta x) = \frac{T}{\cos(\theta+\Delta\theta)}$$

and plugging this into B.2.5 we find

$$-T\tan\theta + T\tan(\theta+\Delta\theta) = \mu\Delta x\frac{\partial^2 u}{\partial t}(x,t). \tag{B.2.6}$$

Note that we use $m=\mu\Delta x$, where μ is the linear density of the string. Since $\tan\theta$ and $\tan(\theta+\Delta\theta)$ are the tangents at each end of the string piece, they measure the slope of the displacement function $u(x,t)$. That is

$$\tan\theta = \frac{\partial u}{\partial x}(x,t) \quad \text{and} \quad \tan(\theta+\Delta\theta) = \frac{\partial u}{\partial x}(x+\Delta x,t);$$

We now sustitue this last equation in B.2.6, and find

$$-T\,\frac{\partial u}{\partial x}(x,t) + T\,\frac{\partial u}{\partial x}(x+\Delta x,t) = \rho\,\Delta x\,\frac{\partial^2 u}{\partial t^2}(x,t),$$

from which, after dividing by $T, \Delta x$ is

$$\frac{\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)}{\Delta x} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}(x,t).$$

The final step is to take the limit ad $\Delta x \to 0$, so

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial^2 t}.$$

The wave speed for this particular wave is

$$v = \sqrt{\frac{T}{\rho}}.$$

B.3 Elastic Waves in 3D Anisotropic Media

This derivation is taken from Aki and Richards [1] The idea of the derivation of the wave equation is again based on

- Hook's Law, and
- Newton's second law.

The general point in the three–dimensional space is noted by $\boldsymbol{x}$. The function $u_i(\boldsymbol{x}, t)$ in the three–dimensional space and time is the particle displacement function along the i–direction, ($i = 1, 2, 3$, for x_i coordinate axis)

In general strain is a tensor (a matrix) defined as

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

where the shorthand notation $u_{i,j}$ means

$$u_{i,j} = \frac{\partial u_i(\boldsymbol{x}, t)}{\partial x_i}.$$

The total forces acting on a volume consist of

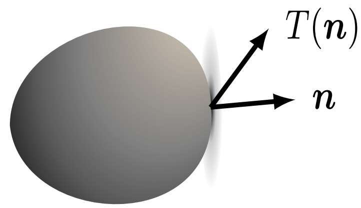
- Body Forces: for example gravity and magnetic forces
- Surface Forces: these are contact forces, for example shear and normal forces along the surface of a body.

Figure B.3 illustrates a vector of a surface force acting on a piece of volume associated with a normal vector $\boldsymbol{n}$.

A force f acting on a point $\boldsymbol{\xi}$ and in an given specific time τ and along a given direction i can be described with the help of the Dirac and Kronercker delta symbols as follows

$$f_i(\boldsymbol{x}, t) = A\delta(\boldsymbol{x} - \boldsymbol{\xi})\delta(t - \tau)\delta_{in}.$$

Figure B.4: A body and a surface force acting on it



Newton's second law applied to the body, indicates that the total force on the body is equal to ma, where a is the acceleration of the body's center of mass. If we consider the body as a continuum of particles and on each particle we assume that Newton's second law is applied, the total forces are integrated by

$$\underbrace{\int_V \rho \frac{\partial^2 u_i}{\partial t^2}\, dV}_{\text{Mass times acceleration}} = \underbrace{\int_V \boldsymbol{f}\, dV}_{\text{Total body forces}} + \underbrace{\int_S T(\boldsymbol{s})\, dS}_{\text{Total surface forces}} . \tag{B.3.7}$$

We want to study the Cauchy stress tensor, which from definition is

$$T(\boldsymbol{n}) = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S}$$

where $\boldsymbol{n}$ denotes the normal to the surface element. An infinite number of traction vectors can act on a point with arbitrary directions. We want to prove the Cauchy's law that shows how any arbitrary stress tensor can be written in term of some basis functions which are stress tensors along the main coordinate directions, but before we can prove the Cauchy's law, we first prove the Cauchy's lemma.

B.3.1 Cauchy's Lemma

Cauchy's lemma states that traction vectors acting on opposite sides of a surface are equal and opposite. This is

$$T(\boldsymbol{n}) = -T(-\boldsymbol{n}). \tag{B.3.8}$$

This is equivalent to the third Newton's law of action and reaction.

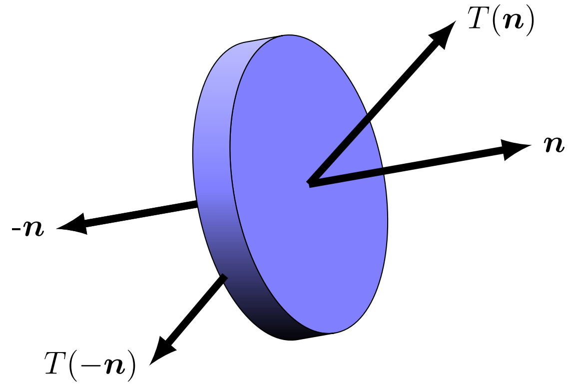
Initially we want to take a volume slab around a particle of interest as shown in figure B.5, with an infinitesimal thickness δ.

We treat each term on equation B.3.7 individually, and use the mean value theorem for integrals.

The first integral,

$$\int_{\Delta V} \rho \frac{\partial^2 u}{\partial t^2} dV = \overline{\boldsymbol{a}} \Delta m$$

Figure B.5: A small disc within a stressed medium.



where the last equality comes from the mean value theorem for integrals. Here $\overline{\boldsymbol{a}}$ is the acceleration vector at some point inside the volume.

The body force acting in the volume is, similarly

$$\int_{\Delta V} \boldsymbol{f} dV = \overline{\boldsymbol{f}} \Delta V.$$

where $\overline{\boldsymbol{f}}$ is a force vector inside the body. The total mass is

$$\Delta m = \int_{\Delta V} \rho dV = \overline{\rho} \Delta V$$

where $\overline{\rho}$ is the density at some point interior to the volume.

The resultant force from the two tractions (last integral) is $T(\boldsymbol{n})\,\Delta S + T(-\boldsymbol{n})\,\Delta S$, where ΔS is the area of the slab.

So we find

$$\overline{\boldsymbol{a}} \Delta m = \overline{\boldsymbol{f}} \Delta V + T(\boldsymbol{n})\,\Delta S + T(-\boldsymbol{n})\,\Delta S.$$

We divide by ΔS,

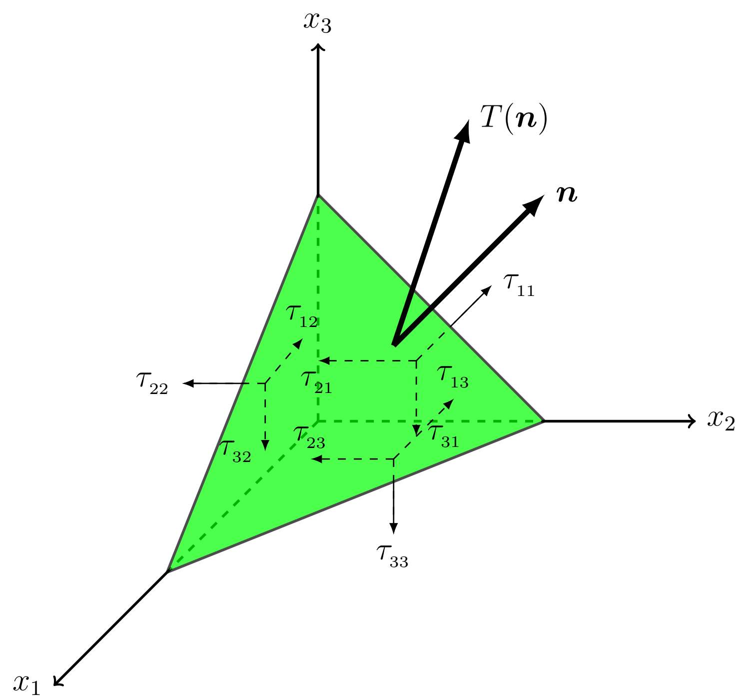
$$\frac{\overline{\boldsymbol{a}}\,\rho\,\Delta V}{\Delta S} = \overline{\boldsymbol{f}} \frac{\Delta V}{\Delta S} + T(\boldsymbol{n}) + T(-\boldsymbol{n}),$$

now, clearly

$$\lim_{\delta \to 0} \frac{\Delta V}{\Delta S} = 0.$$

since $\Delta V = \Delta S \delta$, so we find the Cauchy lemma

$$T(\boldsymbol{n}) = -T(-\boldsymbol{n}).$$



B.3.2 Cauchy's Law

The Cauchy's law states that there exists a Cauchy stress tensor $\boldsymbol{\tau}$ which maps the normal to a surface to the traction vector acting on that surface according to

$$T = \boldsymbol{\tau n},$$

or in short hand index notation

$$T_i = \tau_{ij} n_j,$$

where we assume the Einstein's notation of sum over repeated indices. This is a very useful formula because each traction can be computed along arbitrary normal directions with the help of the Cauchy stress tensor $\boldsymbol{\tau}$. It is similar to what we do in calculus when we know the gradient and use it to compute directional derivatives, only that here we have one more dimension. The gradient is a tensor of order 1 and the Cauchy stress tensor is a tensor of order 2.

Bibliography

[1] K. Aki and R. P. *Quantitative Seismology: Theory and Methods*, volume I, pages 156–158. W. H. Freeman and Company, 1980.

[2] N. Bleistein. *Mathematical methods for wave phenomena*. Academic Press, Inc, 1984.

[3] A. Komech and K. A. *Principles of Partial Differential Equations*. Springer-Verlag, 2012.