Differential Forms

Herman Jaramillo

May 10, 2016

1 Introduction

I use the Wikepedia source as a main reference for this article, as well as this link from Donu Arapura's notes about differential forms. Some text is taken directly from the sources. It is not my plan at this time to publish these notes but to use them only as a learning excercise.

Differential forms are part of the larger field of *Differential geometry* (mainly Analysis on Manifolds) Munkres [2]. Munkres does a great job explaining the concept of *Differential forms* but, to get there, I would need to build up work up to and including the theory of manifolds. Hence, I picked Donu Arapura's path because, although is not as clean, as Munkres' path, it is quick.

The main motivation that I had to write these notes is in my search for the most general form of the Stoke's theorem in n–dimensional real spaces. This is Newton's Fundamental Theorem of Calculus generalized. Under certain hypothesis it is possible to express a hyper–volume integral in terms of its hyper–surface integral. These type of problems are related to Green's and Stoke's theorems and relations between some form of differentiations, mainly divergence and curl and the relationship with their parent functions. I got started looking in this direction after trying to find the connection of Cauchy's integral theorem on the complex plane and the Green's theorem in the plane. (which relates an integral of an area over the integral over the boundary of that area).

Differential forms are an approach to multivariable calculus that is coordinate independent. In this way differential forms are tensors. They are a unified approach to define integrands over curves, surfaces, volumes, and higher dimensional *manifolds* .

The simplest example of a differential form is the 1–form $f(x)dx$ that is ready for integration under the symbol

$$\int_a^b f(x)dx.$$

Higher order forms are natural integrands over surfaces and volumes (hyper–surfaces).

The concept or linear operators, differential operators, tangent spaces are all related. From calculus, the differential is a linear operator that is defined at each point of the domain and for each direction vector $\boldsymbol{v}$ in tangent plane of the object we are studying (a curve, a surface, ... , a manifold).

In the simplest form in the one–dimensional space $\mathbb{R}$, the differential is defined by the equation

$$dy = f'(x)dx, \tag{1.1}$$

where $f'(x)$ is the derivative of f with respect to x (assuming its existence), and dx is an additional real variable, this suggests why Leibniz uses the notation

$$f'(x) = \frac{dy}{dx}, \quad \text{for} \quad y = f(x).$$

Observe that equation 1.1 defines a line that goes through zero and has slope $f'(x)$ in the variables dy and dx (we see the tangent space here as having the origin $dx = dy = 0$ at the given point (x, y)). In this way the differential is a linear transformation. Likewise for a scalar field, if the gradient exists, we can write

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i$$

This represents a plane through the origin (in the dx^i, df coordinates) which is orthogonal to the gradient $\nabla_x f$.

In general, for a vector field

$$\begin{aligned} F : A \subseteq \mathbb{R}^n &\rightarrow \mathbb{R}^m \\ \boldsymbol{x} &\mapsto F(\boldsymbol{x}) \end{aligned}$$

if we can associate a linear operator D such that

$$\lim_{\boldsymbol{h}\to 0} \frac{\|f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - D_{\boldsymbol{x}}(\boldsymbol{h})\|}{\|\boldsymbol{h}\|} = 0,$$

then we say that f is differentiable in $\boldsymbol{x}$ and the operator $D_{\boldsymbol{x}}(\boldsymbol{h})$ is called the Fréchet derivative. We redefine the Frèchet derivative in equation 3.2. The norm $\|.\|$ operator is applied in the corresponding space. [1]

The linear operator $D_{\boldsymbol{x}}$ serves to define a differential form. It can be explicitly written in therms of the m $\times$ n matrix of partial derivatives $\partial F_i/\partial x_j$.

1.1 Motivations from Physics

Edwards [1] presents good illustrations where the of work and flow directly lead to the definition about differential forms. The amount of work incremented on a particle displaced along the vector (dx, dy, dz), in a constant force field (F_x, F_y, F_z) is given by

$$dW = F_x dx + F_y dy + F_z dz.$$

This is the natural definition of a 1–form. Generalization to high order forms is given through the concept of flow.

2 History

According to Wikipedia, the modern treatment of differential forms was pioneered by Élie Cartan in his paper *Sur certaines expressions diffèrentielles le problème de Pfaff.*

3 The Fréchet derivative and the Gâteaux derivative

Before going into more detail about differential forms two basic definitions of derivative are important. The relation between these type of derivatives and differential forms will be clear later on in the text.

[1] $\mathbb{R}^m$ for the numerator and $\mathbb{R}^n$ for the denominator. The concept of Fréchet derivative can be extended to Banach spaces.

In general, the definition of directional (Gâteaux) derivative is given by

$$(\partial_{\boldsymbol{v}} f)(\boldsymbol{p}) = \lim_{t\to 0} \frac{f(\boldsymbol{p}+t\boldsymbol{v}) - f(\boldsymbol{p})}{t} = \frac{d}{dt} f(\boldsymbol{p}+t\boldsymbol{v})\bigg|_{t=0}.$$

For the moment let us think that both $\boldsymbol{v}$ and $\boldsymbol{p}$ are n–dimensional vectors ($\in \mathbb{R}^n$) If we fix $\boldsymbol{v}$ then we have all the derivative of f at each point in the fixed direction $\boldsymbol{v}$. If we fix $\boldsymbol{p}$, then we freeze the function at that $\boldsymbol{p}$ and look at all possible directions $\boldsymbol{v}$ where we can take derivatives.

As a function of $\boldsymbol{v}$ it could happens that this operator is linear, that is

$$(\partial_{\boldsymbol{v}+\alpha \boldsymbol{w}})(\boldsymbol{p}) = (\partial_{\boldsymbol{v}} f)(\boldsymbol{p}) + \alpha(\partial_{\boldsymbol{w}} f)(\boldsymbol{p}).$$

for any scalar α and vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, at each fixed point $\boldsymbol{p}$. This motivates the definition

$$df_{\boldsymbol{p}}(\boldsymbol{v}) = (\partial_{\boldsymbol{v}} f)(\boldsymbol{p})$$

as a linear opertator. If this differential operator is linear then the derivative is called Frêchet derivative More formally the Fréchet derivative is a linear operator $df_{\boldsymbol{p}}(\boldsymbol{v})$ such that

$$\lim_{\boldsymbol{v}\to 0} \frac{\|f(\boldsymbol{p}+\boldsymbol{v}) - f(\boldsymbol{p}) - df_{\boldsymbol{p}}(\boldsymbol{v})\|}{\|\boldsymbol{h}\|} = 0, \tag{3.2}$$

The operator $df_{\boldsymbol{p}}$ is called the *Fréchet derivative* of f at $\boldsymbol{p}$, from the space of “tangent” vectors $\mathbb{R}^n$ into the scalars. Note that f could be extended to a vector field F to $\mathbb{R}^m$, and in this case the derivative would be a linear map

$$\begin{aligned} DF_{\boldsymbol{p}} : A \subseteq \mathbb{R}^n &\rightarrow \mathbb{R}^m. \\ \boldsymbol{v} &\mapsto DF_{\boldsymbol{p}}(\boldsymbol{v}) = (\partial_{\boldsymbol{v}} F)(\boldsymbol{p}) \end{aligned}$$

with A the space of all possible directions in which the differential (derivative) exists. More general the Fréchet derivative can be extended to Banach spaces.

If all the Gâteaux differentials are continuous functions of $\boldsymbol{p}$ at $\boldsymbol{p} = \boldsymbol{p}_0$, then the differential is linear and it coincides with the Fréchet derivative.

Kevin Long provides good examples of functions which are Gâteaux differentiable but not Fréchet differentiable. In particular $f(x) = |x|$. To be Fréchet differentiable, the differential should be the same in a given point for all possible directions of approach to that point. This is why Fréchet differentiation is stronger than Gâteaux differentiation.

4 1–Forms

Given an open set $U \in \mathbb{R}^2$, a *differential form* (or a 1–form) is an expression $F(x,y)dx + G(x,y)dy$ where F and G are real valued functions in U.

For example if $f(x,y)$ is C^1 in U, then its total differential (or exterior derivative) is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Then df is a differential 1–form. Similarly, a differential 1–form in an open subset $U \in \mathbb{R}^3$ is an expression of the form $F(x,y,z)dx + G(x,y,z)dy + H(x,y,z)dz$ where F, G, and H are real valued. If $f(x,y,z)$ is C^1, then its total differential is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

This notation is powerful. If dividing by dt,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

We can re–interpret the form as a vector field as follows:

$$F\,\mathbf{i} + G\,\mathbf{j} + H\,\mathbf{j} \leftrightarrow F dx + G dy + H dz$$

under this mapping ∇f corresponds with df.

In general for $f(x_1, x_2, \cdots, x_n)$, is C^1 in U then we can write

$$\begin{aligned} df &= \sum_i \frac{\partial f}{\partial x_i}dx_i, \\ \frac{df}{dt} = \nabla f \cdot \dot{\boldsymbol{x}} &= \sum_i \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} \end{aligned}$$

and we can think of the mapping ∇f to df. The space of gradient vectors has as a dual the space of functionals which are built with the inner product of these gradient vectors with the differential vectors dx_i.

5 Exactness in $\mathbb{R}^2$

Assume $Fdx + Gdy$ is a differential on $\mathbb{R}^2$ with C^1 coefficients. We will say that it is *exact* if one can find a C^2 function $f(x,y)$ with $df = Fdx + Gdy$. Most differential forms are not exact. To see why, note that

$$
\begin{aligned}
df = Fdx + Gdy \quad &\Leftrightarrow \quad \frac{df}{dt} = F\frac{dx}{dt} + G\frac{dy}{dt} \\
&\Leftrightarrow \quad \left(F = \frac{\partial f}{\partial x}\right) \wedge \left(G = \frac{\partial f}{\partial y}\right) \\
&\Leftrightarrow \quad \frac{\partial F}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial G}{\partial x}.
\end{aligned}
$$

For example take

$$df = y\,dx,$$

here $F = y$ and $G = 0$, so

$$\frac{\partial F}{\partial y} = 1 \neq 0 = \frac{\partial G}{\partial x}.$$

A differential is *closed* if $\partial F/\partial y = \partial G/\partial x$. So a differential exact has to be closed.

The exact differential is found in the method of characteristics in the solution of partial differential equations. A differential equation

$$\frac{dy}{dx} = \frac{F(x,y)}{G(x,y)}$$

can be written as

$$Fdx - Gdy = 0.$$

If $Fdx - Gdy$ is eact and equal, for example to df then, the curves $f(x,y) = c$ give solutions to the equation. This concepts arise in physics. For example, Assume vector field $\boldsymbol{F} = F_1\mathbf{i} + F_2\mathbf{j}$ representing a force. There is a vector $P(x,y)$ called the *potential,* such that $\boldsymbol{F} = -\nabla P$. The force is called *conservative* if it has a potential. That is, $\boldsymbol{F}$ is conservative if and only if $F_1dx + F_2dy$ is exact.

6 Parametric curves

Before moving to line integrals let us define a *parametric curve* C as a vector valued function

$$C : [a, b] \in \mathbb{R} \to \mathbb{R}^2.$$

Let us now assume that C is C^1. Then we can define the velocity tangent vector $\boldsymbol{v} = (dx/dt, dy/dt)$. Assume that the particle travels without stopping, that is $\boldsymbol{v} \neq 0$. Then $\boldsymbol{v}$ gives a direction C, which we refer to as its *orientation.* If C is given by

$$x = f(t),\ y = g(t), \quad a \leq t \leq b$$

then

$$x = f(-u),\ y = g(-u), \quad -b \leq u \leq -a$$

is called $-C$. Suppose that C is given depending on some parameter t,

$$x = f(t), \quad y = g(t)$$

and that t depends in turn on a new parameter $t = h(u)$ such that $dt/du \neq 0$. Then we can get a new parametric curve C' such that

$$x = f(h(u)), \quad y = g(h(u)).$$

If the derivative $dt/du > 0$, everywhere we want to view the oriented curves C and C' as the *equivalent* . If this derivative is negative everywhere, then $-C$ and C' are *equivalent* . For example, the curves

$$\begin{aligned} C : x = \cos\theta, &\quad y = \sin\theta \quad 0 \leq \theta \leq 2\pi \\ C' : x = \sin\theta, &\quad y = \cos\theta \quad 0 \leq \theta \leq 2\pi \end{aligned}$$

are counterclockwise and clockwise respectively. So C' is equivalent to $-C$. This can be seen easily by the change of variables $\theta = \pi/2 - t$. We can allow piecewise C^1 curves, and think about them as union of C^1 curves, where one starts where the previous one ends. These concepts can be generalized to $\mathbb{R}^n$ along the same lines.

7 Line Integrals

One the motivations that I had to write (copy) this document is to understand better the Cauchy theorems of complex variables. They can be seen coming from this context. We want to understand more about what it means for $Fdx + Gdy$ to be exact. We know that if a form is exact then it is closed. We will show that if $F(x,y)dx + G(x,y)dy$ is a closed form on all of $\mathbb{R}^2$ with C^1 coefficients, then it is exact.

The requirement to be C^1 in all of $\mathbb{R}^2$ is necessary. For example the form

$$-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dx$$

is closed in $\mathbb{R}^2 - \{(0,0)\}$ but is not exact (we show this later).

To show the relashionship between exactness and closeness we need the concept of line integrals. Let us assume that a curve γ is C^1 and can be parameterized by a parameter t. That is $\gamma = (x(t), y(t))$. Then

$$\int_\gamma Fdx + Gdy = \int_a^b \left[F((x(t), y(t))\frac{dx}{dt} + G(x(t), y(t))\frac{dy}{dt} \right] dt.$$

From the chain rule we find

$$\int_{-\gamma} Fdx + Gdy = -\int_\gamma Fdx + Gdy$$

and, if γ and γ' are equivalent, then

$$\int_\gamma Fdx + Gdy = \int_{\gamma'} Fdx + Gdy$$

In general if γ is a curve C^1 defined as

$$\begin{aligned} \gamma : [a,b] \in \mathbb{R} &\rightarrow \mathbb{R}^n \\ t &\mapsto (x_1(t), x_2(t), \cdots, x_n(t)) \end{aligned}$$

then we can define the line integral

$$\begin{aligned} \int_\gamma F_1dx_1 + F_2dx_2 + \cdots + F_ndx_n &= \int_a^b \Big[F_1((x_1(t), x_2(t), \cdots, x_n(t))\frac{dx_1}{dt} + \\ &+ F_2(x_1(t), x_2(t), \cdots, x_n(t))\,\frac{dx_2}{dt} + \cdots \\ &+ F_n(x_1(t), x_2(t), \cdots, x_n(t))\frac{dx_n}{dt} \Big] dt \end{aligned}$$

From the chain rule we find

$$\int_{-\gamma} F_1 dx_1 + F_2 dx_2 + \cdots + F_n dx_n = -\int_\gamma F_1 dx_1 + F_2 dx_2 + \cdots + F_n dx_n,$$

and, if γ and γ' are equivalent, then

$$\int_\gamma F_1 dx_1 + F_2 dx_2 + \cdots F_n dx_n = \int_{\gamma'} F_1 dx_1 + F_2 dx_2 + \cdots + F_n dx_n.$$

We say that the differential form $F_1 dx_1 + F_2 dx_2 + \cdots + F_n dx_n$ on $\mathbb{R}^n$ with C^1 coefficients is *exact* if one can find a C^2 funtion $f(x_1, x_2, \cdots, x_n)$ with $df = F_1 dx_1 + F_2 dx_2 + \cdots + F_n dx_n$.

Another way to characterize exactness is through line integrals. It is common to see the notation ω for a differential form. If ω is exact and γ_1 and γ_2 are two parametrized curves with the same endpoints then

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$

This is a consequence of the chain rule and the fundamental theorem of calculus. That is,

$$\int_{\gamma_1} df = \int_a^b \frac{df}{dt} dt = f(\gamma_1(b)) - f(\gamma_1(a))$$

which only depends on the end points, and since $\gamma_1(b) = \gamma_2(b)$ and $\gamma_1(a) = \gamma_2(a)$, and from

$$\int_{\gamma_2} df = \int_a^b \frac{df}{dt} dt = f(\gamma_2(b)) - f(\gamma_2(a))$$

we see that the line integral is path independent, up to the end points. If a curve γ is closed then $\gamma(a) = \gamma(b)$ and so the line integral is zero. Then, if a form is exact then the line integral along a closed curve is zero.

We now have the elements to proof that if a differential form is closed in all $\mathbb{R}^2$ then it is exact. Let us see:

We define the following function

$$f(x, y) = \int_\gamma F dx + G dy \tag{7.3}$$

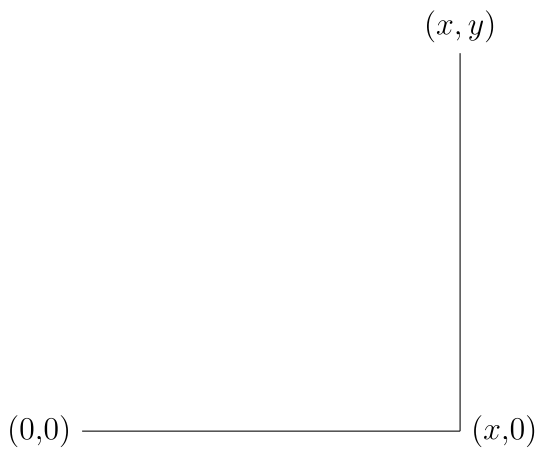

Figure 1: Path for curve γ

where γ is the curve defined in Figure 7

We parameterize both line segments separately by $x = t$, $y = 0$ and $x = x(constant)$, $y = t$, and sum to get

$$f(x, y) = \int_0^x F(t, 0)dt + \int_0^y G(x, t)dt$$

and show that $df = Fdx + Gdy$. To see this we differentiate 7.3 and find

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int_0^y G(x, t)dt = G(x, y).$$

and

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} \int_0^x F(t, 0)dt + \frac{\partial}{\partial x} \int_0^y G(x, t)dt \\
&= F(x, 0) + \int_0^y \frac{\partial G(x, t)}{\partial x} dt \\
&= F(x, 0) + \int_0^y \frac{\partial F(x, t)}{\partial t} dt \quad \text{from the definition of closed} \\
&= F(x, 0) + F(x, y) - F(x, 0) \\
&= F(x, y).
\end{aligned}$$

So indeed, the differential form $\omega = Fdx + Gdy$ is closed in all $\mathbb{R}^2$ then ω is exact. In fact it is only necessary that the form is closed in a simply connected open region of the space $\mathbb{R}^n$. Cirilo, Regis and Lutz show that in $\mathbb{R}^2$, if each 1–closed form is exact then the domain is simply connected. Cirilo, Regis and Lutz show counter–examples in $\mathbb{R}^3$ and $\mathbb{R}^4$ where the connected hypothesis fails.

We know that if ω is exact, then its integral around a closed curve should be zero. We now go back to the counter example:

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

We integrate along the unit circle

$$\gamma : x(t) = \cos t \quad y(t) = \sin t. \quad t \in [0, 2\pi].$$

Then

$$\int \omega = -\int_0^{2\pi} \sin t \; d\cos t dt + \int_0^{2\pi} \cos t \; d\sin t dt = \int_0^{2\pi} (\cos^2 t + \sin^2 t)dt = 2\pi.$$

Since the integral is not zero, then ω can not be exact. The reason is that, ω is not closed in $(x, y) = (0, 0)$ and the closedeness requirement is in all of $\mathbb{R}^2$.

Work: A direct application of line integrals is the concept of work. By definition the work by a force $\boldsymbol{F}$ in along a path γ physics is defined by the line integral

$$W = -\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s}$$

8 Green's theorem on a rectangle

Let D be a rectangle in the xy–plane with vertices $(0, 0), (a, 0), (a, b), (0, b)$. Let C be the boundary curve of the rectangle oriented counter clockwise. Given C^1 functions $P(x, y)$, $Q(x, y)$ on D, the fundamental theorem of calculus yields

$$
\begin{aligned}
\int_D \frac{\partial Q}{\partial x} dxdy &= \int_0^b \left[Q(a,y) - Q(0,y)\right] dy \\
&= \int_0^a Q(x,0)dy + \int_0^b Q(a,y)dy - \int_0^a Q(x,b)dy - \int_0^b Q(0,y)dy \\
&= \int_C Q(x,y)dy. \qquad (8.4)
\end{aligned}
$$

Note that the first and third integrals in the second line are zero, since the integrating element is along the y–axis (dy) and the integrating path is along the x–axis.

Along the same lines

$$
\begin{aligned}
\int_D \frac{\partial P}{\partial y} dydx &= \int_0^a \left[P(x,b) - P(x,0)\right] dx \\
&= \int_0^b P(0,y)dx + \int_0^a P(x,b)dx - \int_0^b P(a,y)dx - \int_0^a P(x,0)dx \\
&= -\int_C P(x,y)dx. \qquad (8.5)
\end{aligned}
$$

and from 8.4 and 8.5

$$\int_C P(x,y)dx + Q(x,y)dy = \int_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dxdy$$

which is Green's theorem for D.

We want to extend the result to n dimensions.

9 2–forms

The goal in this section is to understand the operation

$$Pdx + Qdy \mapsto \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

in a more direct way. Therefore we should understand better how to work with expressions of the form $Fdxdy$. We introduce the wedge symbol $dx \wedge dy$ which carries more information, such as the sense of direction.

The cross product of vectors $\boldsymbol{u} \times \boldsymbol{v}$ is a very useful operation in 3 dimensional geometry. Its length gives the area of the parallelogram spanned by $\boldsymbol{u}$, $\boldsymbol{v}$, and it determines the plane containing this parallelogram. There is no direct analogue of the cross product in higher dimensions (actually I believe there is, and this is in my tensor document). The idea behind the wedge product is that has the following attributes. The magnitude is the area given by the two vectors that form it [2] The direction is orthogonal to the plane having the two vectors and directed according to the right hand rule. We have the following simple properties

$$\begin{aligned} \boldsymbol{v} \wedge \boldsymbol{u} &= -\boldsymbol{u} \wedge \boldsymbol{v} \\ \boldsymbol{u} \wedge \boldsymbol{u} &= 0, \quad \text{and} \\ c\,(\boldsymbol{u} \wedge \boldsymbol{v}) &= (c(\boldsymbol{u})) \wedge \boldsymbol{v} = \boldsymbol{u} \wedge (c\boldsymbol{v}) \end{aligned}$$

for any scalar c.

It can be shown that the space of wedge products is a vector space. Rephrasing, The wedge products can be seen geometrical as oriented paralellograms. Parallelograms are to wedge vectors as regular vectors are to lines. The wedge vector has (as any vector) magnitude (the area fo the paralellogram expanded by the two vectors) and direction. The direction is normal to the paralellogram and follows the right hand side rule for orientation.

In addition we have the distributive and anti–conmutative laws

$$\begin{aligned} \boldsymbol{u} \wedge (\boldsymbol{v} \wedge \boldsymbol{w}) &= \boldsymbol{u} \wedge \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{w} \quad \text{distributive} \\ \boldsymbol{u} \wedge \boldsymbol{v} &= -\boldsymbol{v} \wedge \boldsymbol{u} \quad \text{anti} - \text{conmutative}. \end{aligned}$$

Wedge products are also known as 2–vectors.

A 2–form is like a 2–vector but using forms. On $\mathbb{R}^3$, this would be an expression:

$$F(x, y, z)dx \wedge dy + G(x, y, z)dy \wedge dz + H(x, y, z)dz \wedge dx$$

where F, G and H are functions defined in an open subset of $\mathbb{R}^3$. Any wedge products of two 1–forms can be put in this format. For example using the

[2] think about the area as the product of the magnitudes of the vectors times the $\sin\theta$ where θ is the angle between them and

$$\cos\theta = \frac{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}{\boldsymbol{u} \cdot \boldsymbol{v}}$$

above rules we can see that

$$(3dx + dy) \wedge (dx + 2dy) = 6dx \wedge dy + dy \wedge dx = 5dx \wedge dy.$$

The real significance of 2–forms will come later when we do surface integrals. A 2–form will be an expression that can be integrated over a surface in the same way a 1–form can be integrated over a curve.

To make the comparison with traditional calculus, we note that we can convert vector fields to 2–forms and back

$$F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k} \leftrightarrow F_1 dy \wedge dz + F_2\, dz \wedge dx + F_3 dx \wedge dy.$$

We showed the mapping

$$F_1\,\boldsymbol{i} + F_2\,\boldsymbol{j} + F_3\,\boldsymbol{i} \leftrightarrow F_1 dx + F_2 dy + F_3 dz. \leftrightarrow$$

A new operator, the *Hodge star* operator is defined as

$$*(F_1 dx + F_2 dy + F_3 dz) = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy$$
$$*(F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy) = F_1 dx + F_2 dy + F_3 dz.$$

Next, we explore the definition for a derivative of a 1–form $\omega = F(x, y, z)dx + G(x, y, z)dy + H(x, y, z)dz$, which is $d\omega$ We will show that derivative of a 1–form is a 2–form. The following rules should be in place:

$$\begin{aligned} d(\alpha + \beta) &= d\alpha + d\beta \\ d(f\alpha) &= (df) \wedge \alpha + f d\alpha \quad \text{and} \\ d(dx) &= d(dy) = d(dz) = 0 \end{aligned}$$

where α and β are 1–forms and f is a function.

References

[1] H.M. Edwards. *Advanced calculus: a differential forms approach.* Birkhäuser, 1994.

[2] J.R. Munkres. *Analysis On Manifolds.* Advanced Book Classics. Basic Books, 1997.

10 Appendix A

An appendix