The Huygens Fresnel–Principle

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1 Introduction

These notes are taken from the classical optics book by Born and Wolf [1]. My intention is to rewrite them in my own style, more as an exercise to learn about Fresnel bridge between Huygens intuition and his own mathematical skills, that allowed him to put it in quantitative formula with great precision.

2 The Principle

Referring to figure 1 the Huygens principle states that the energy recorded at point of a propagating wave, is due to secondary sources all located to a previous wavefront (the blue wavefront). Fresnel went a bit further and used the interference principle (superposition) to add the contributions in a mathematical way to obtain an integral representation.

We assume that the initial wave is a pure point source with a given spectrum \( F(\omega) \) and at any point in a distance from the source we have the wavefield described by the Green’s function

\[
W_0 = A \exp(-i\omega t_0) F(\omega) \frac{\exp(ikr_0)}{r_0}
\]  

(2.1)

The factor \( A \) is a constant due to the source strength, and we will omit \( \exp(-i\omega t_0) F(\omega) \) because it will not be used for any of the computations here. This could be inserted at the end of all the computations.

Now the share of this source in the total wavefield in \( P \) is given by

\[
W_0 K(\chi) \frac{\exp(iks)}{s}
\]
Figure 1: A wave starts at a point source $P$. After sometime, a wavefront is located in the blue sphere (only a main cross-section in the $xy$ plane is shown). A disturbance at point $P$, according to Huygens principle is understood as the superposition of all disturbances starting at the wavefront.

where $W_1$ given by equation 2.1 is the strength of the secondary source and $K(\chi)$ is an angle dependent weight factor that is initially assumed unknown. This is the so-called obliquity factor.

The integral becomes

$$U(P) = \int_S W_0 K(\chi) \frac{\exp(iks)}{s} dS$$

$$= \int_S K(\chi) \frac{\exp(ikr_0)}{r_0} \frac{\exp(iks)}{s} dS$$

$$= A \frac{\exp(ikr_0)}{r_0} \int_S K(\chi) \frac{\exp(iks)}{s} dS.$$ 

Fresnel decided to integrate the effect of all sources in zones as shown in the figure. The zone $Z_1$ is the piece of sphere which is between the nearest wavefront point and half wavelength $\lambda/2$ farther. In the figure these are the
points along the sphere between distances \( b \) and \( b + \lambda/2 \) from \( P \), with \( b = CP \) being the the distance between the point \( P \) and the wavefront. The second zone \( Z_2 \) contains the peace of wavefront which is between the distances \( b + \lambda/2 \) and \( b + \lambda \). The constructions continues in this way so that the zone \( Z_i \) is between the distances \( b + (i - 1)(\lambda/2) \) and \( b + i\lambda/2 \). The partition in these type of slices is quite interesting as we will observe.

One assumption for the evaluation of the integral is that in each slice the obliquity factor remains constant. Hence the integral for the \( j\)-th slice is

\[
U_j(P) = A \frac{\exp ikr_0}{r_0} K(\chi) \int_{Z_j} \frac{\exp (iks)}{s} \frac{r_0^2}{s} \sin \theta d\theta d\phi.
\]

where we use the transformation into polar spherical coordinates with

\[
dxdy = Jd\theta d\phi,
\]

with \( J = r_0^2 \sin \theta \), \( \theta \) is the polar angle and \( \phi \) is the azimuthal angle. Since the integral is azimuthally invariant

\[
U_j(P) = 2\pi A \frac{\exp ikr_0}{r_0} K(\chi) \int_{Z_j} \frac{\exp (iks)}{s} \frac{r_0^2}{s} \sin \theta d\theta d\phi.
\]

From the cosine law we see that

\[
s^2 = r_0^2 + (r_0 + b)^2 - 2r_0(r_0 + b) \cos \theta,
\]

and so

\[
sds = r_0(r_0 + b) \sin \theta d\theta,
\]

so

\[
\sin \theta d\theta = \frac{sds}{r_0(r_0 + b)},
\]

and integral 2.3 becomes

\[
U_j(P) = 2\pi A \frac{\exp ikr_0}{r_0} K_j(\chi) \int_{Z_j} \frac{\exp (iks)}{s} \frac{r_0^2}{s} \frac{sds}{r_0(r_0 + b)}
\]

\[
= 2\pi \frac{Ar_0}{r_0 + b} \frac{\exp ikr_0}{r_0} K_j(\chi) \int_{Z_j} \frac{\exp (iks)}{s} ds
\]

\[
= 2\pi A \frac{\exp ik(r_0 + b)}{k} \frac{r_0 + b}{r_0 + b} K_j(\chi) \exp (ikj\lambda/2)[1 - \exp (-i\lambda/2)].
\]
Since \( k \lambda = 2\pi \), the last two factors reduce to

\[
    \exp (ikj\lambda/2)[1 - \exp (-ik\lambda/2)] = \exp (ij\pi)(1 - \exp [-(i\pi)]) = 2(-1)^j
\]

so

\[
    U_j(P) = 2i \lambda(-1)^{j+1} A \frac{\exp ik(r_0 + b)}{r_0 + b} K_j(\chi). \tag{2.3}
\]

From now on, to simplify notation, I write \( K_j \) instead of \( K_j(\chi) \). The total field is the sum of all contributions \( U_j(P) \) with alternating signs. That is

\[
    U(P) = \sum_{j=1}^{n} U_j(P) = 2i \lambda(-1)^{j+1} A \frac{\exp ik(r_0 + b)}{r_0 + b} \sum_{j=1}^{n} K_j. \tag{2.4}
\]

What is \( n \)? This is a source of controversy. Fresnel claimed that all he needed is to cover the forward (I mean to the right in the figure) propagating front. That is \( 0 \leq \chi \leq \pi/2 \). This was put in question for several scientists since the waves propagating to the left also send pulses to the receiver and since the receiver is static they should arrive at some time and interference should occur at that time. For example this link [2] discusses in detail the issue of the “left propagating” wavefront. With that said, the number \( n \) should be such that \( n \lambda/2 \geq r_0 \) so that all sphere is covered.

We will evaluate the series

\[
    \sigma = \sum_{j=1}^{n} (-1)^{j+1} K_j. \tag{2.5}
\]

Born and Wolf [1] use a method due to A. Schuster [1] to evaluate this sum. They first write 2.3 as follows

\[
    \sigma = \frac{K_1}{2} + \left( \frac{K_1}{2} - K_2 + \frac{K_3}{2} \right) + \left( \frac{K_3}{2} - K_4 + \frac{K_5}{2} \right) + \cdots + \text{Tail} \tag{2.6}
\]

The \( \text{Tail} \) is

\[
    \text{Tail} = \begin{cases} 
        \frac{K_n}{2} & \text{if } n \text{ if odd} \\
        \frac{K_{n-1}}{2} - K_n & \text{if } n \text{ if even}
    \end{cases}
\]

\[1\] A. Schuster, *Phil. Mag.* (5), 31 (1891), p. 77. At this time I do not know what Fresnel did because he died in 1827, so obviously he did not know A. Schuster work
Born and Wolf refer to a law that says that “the directional variation is such that $K_j$ is greater than the arithmetic mean of its two neighbors $K_{j-1}$ and $K_{j+1}$”. I have no clue where this law comes from, but the $\cos \chi$ ($0 \leq \chi \leq \pi$) function satisfies such a condition as it does any concave down function, since the average is a linear interpolation and a line is under any concave down curve. If this is the case then any partial sum in parenthesis in (2.6) is negative, so

$$\sigma \leq \begin{cases} 
\frac{K_1 + K_n}{2} & \text{for } n \text{ odd} \\
\frac{K_1}{2} + \frac{K_{n-1}}{2} - K_n & \text{for } n \text{ even}
\end{cases}$$

We can also write (2.5) in the form

$$\sigma = K_1 - \frac{K_2}{2} - \left(\frac{K_2}{2} - K_3 + \frac{K_4}{2}\right) - \left(\frac{K_4}{2} - K_5 + \frac{K_6}{2}\right) - \cdots + \text{Tail}$$

with

$$\text{Tail} = \begin{cases} 
-\frac{K_{n-1}}{2} + K_n & \text{for } n \text{ odd} \\
-\frac{K_n}{2} & \text{for } n \text{ even}
\end{cases}$$

Hence

$$\sigma \geq \begin{cases} 
K_1 - \frac{K_2}{2} - \frac{K_{n-1}}{2} + K_n & \text{for } n \text{ odd} \\
K_1 - \frac{K_2}{2} - \frac{K_n}{2} & n \text{ for even}
\end{cases}$$

We found then that

$$K_1 - \frac{K_2}{2} - \frac{K_{n-1}}{2} + K_n \leq \sigma \leq \frac{K_1}{2} + \frac{K_n}{2} \quad \text{for } n \text{ odd}$$

$$K_1 - \frac{K_2}{2} - \frac{K_n}{2} \leq \sigma \leq \frac{K_1}{2} + \frac{K_{n-1}}{2} - K_n \quad \text{for } n \text{ even}$$
Next, is a kind of high frequency assumption. If the wavenumber is large, then \( \lambda \ll 1 \) and \( K_i \approx K_{i+1} \), so we can approximate

\[
\sigma = \begin{cases} 
\frac{K_1}{2} + \frac{K_n}{2} & \text{for } n \text{ odd} \\
\frac{K_1}{2} - \frac{K_n}{2} & \text{for } n \text{ even}
\end{cases}
\]

Born and Wolf proceed to weaken the argument about the convexity of the directivity factor (\( \cos \chi \) like behavior) by saying that even if the directivity function is concave up the conclusion would be the same and further that even if the difference of a \( K_j \) with the average of its neighbors oscillate the argument will still apply. I do not think there is much point here. An approximation is an approximation, an assumption is an assumption. We know from the more rigorous mathematical (Kirchhoff–Sommerfeld using Helmholtz equation) approach that the directivity is a \( \cos \chi \)-like function. The purpose of the analysis shown here is more historical and to see how the big scientists approached the problem. With this last result, applied to equations 2.4 and 2.3 we find

\[ U(P) = i\lambda A \exp \left[ i k (r_0 + b) \right] \left( K_1 \pm K_n \right) = \frac{1}{2} \left[ U_1(P) + U_n(P) \right] \]  

(2.7)

where the “+” is for odd \( n \) and the “-” is for even \( n \).

The last \( K_n \) is such that \( \chi = \pi/2 \). In this case the ray path is tangent to the sphere and it can not reach \( P \), so \( K_n = 0 \). Then from equation 2.7

\[ U(P) = i\lambda K_1 A \exp \left[ i k (r_0 + b) \right] \frac{1}{r_0 + b} = \frac{1}{2} U_1(P) \]

A major finding here is that "the total disturbance at \( P \) is equal to half the disturbance due to the first zone\(^2\)."

The next step on the development is crucial and it will show that the 90 degrees phase shift is needed to correctly predict the Huygens principle according to the wave equation. To validate Huygens principle we should have

\[ K_1 = -\frac{i}{\lambda} = \frac{\exp \left[ -i\pi/2 \right]}{\lambda} \]

\(^2\)Now known as first Fresnel zone
If this is the case, then

\[ U(P) = \frac{A \exp [ik(r_0 + b)]}{r_0 + b}, \]

which is exactly what the wavefield would be for a spherical wave propagating in free space. This validates the Huygens principle. This is reaffirmed by using the method of stationary phase in my notes with title “The Huygens Principle” (2013).

Note that these does not provide information about what the wavefield above or below \( P \), and the point \( P \) should be such that the intersecting spheres from \( P \) to the wavefront \( S \) should have \( P_0P \) as the main axis. That is, there is totally symmetry on the figure with respect to the angle \( \theta \).

The Fresnel zone partition provides a way to know how much contribution to the total wavefield we would obtain by putting obstructing circular zones of different radii along the \( P_0P \) axis.

For example, if all zones are present, except the first zone \( Z_1 \), then

\[ U(P) = \frac{2i\lambda K_1 A \exp [ik(r_0 + b)]}{r_0 + b} = 2 \frac{A \exp [ik(r_0 + b)]}{r_0 + b}, \]

and the intensity \( I(P) = |U(P)|^2 \) is four times larger than if the screen were absent.

Now, if we allow light to go through the first two zones \( Z_1, Z_2 \) and obstruct the other zones, the amplitude should be almost zero, since \( K_1 \approx K_2 \) and they have opposite signs. The amplitude will grow again (but not that much) when letting three zones illuminate the object in \( P \) (that is zones \( Z_1, Z_2, Z_3 \) and will oscillate as more zones are added.

Here is a prediction that make a strong impression in Fresnel contemporaries. What if only the first zone is obstructed and all other are open for light through. Then we would have

\[ U(P) = \frac{2i\lambda A \exp [ik(r_0 + b)]}{r_0 + b} \left[ -K_2 + K_3 - K_4 + \cdots \right], \]

and since the missing \( K_1 \) is approximately half of the alternating sum of the rest of its \( K_j \) partners, with \( K_1 = 1/i\lambda \) then

\[ U(P) = \frac{A \exp [ik(r_0 + b)]}{r_0 + b} \]
as if there would not be any block to light. The important thing about this contribution is that it validated the wave theory of light which was rejected by Newton about 200 years before that. Here is the footnote in Born and Wolf’s book: “That a bright spot should appear at the centre of the shadow of a small disc was deduced from Fresnel’s theory by S. D. Poisson in 1818. Poisson who was a member of the committee of the French Academy which reviewed Fresnel’s prize memoir, appears to have considered this conclusion contrary to experiment and so refuting Fresnel’s theory. However, Arago, another member of the committee, performed the experiment and found that the surprising prediction was correct. A similar observation had been made a century earlier by Maraldi but had been forgotten.”

References
