The Scattering Theory Approximations

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Abstract

I offer a few derivations to compute relashioships between wavefields and scatterers that are useful for modeling and inversion in fields such as medical imaging, non–destructive testing and seismic modeling and inversion. These derivations are in no way new. I offer a few references but the list is far from being fair.

1 Introduction

Inversion problems are important in several branches of physics and engineering. We know about statistical methods that use stochastic processes and data to resolve for model parameters. In these notes I will focus in deterministic theories that help us solve inverse problems. Deterministic inversion is based mainly on posing modeling equations that are consistet with the physics and meet some basic assumptions. Inversion in scattering theory solves for model parameters from scattering data. By scattering data we will mean transmitted (forward scattering) and reflected (backward scattering) data collected when a source is activated at some point and time and the wavefiled is recorded with some space and time sampling constrains and on a given time window.

The derivations shown here are based on the wave equation and in these notes I assume an acoustic wave equation for a constant density medium.

2 General Theory

In general for perturbation theory. A linear problem is writen in two different media. A background media and a perturbed media. The background media is usually defined with constant velocity (or known velocity, which could be non–constant) and the perturbed media with a perturbed velocity. The difference of the two linear operators is a new linear operator with the same shape but where the source is now some function of the perturbation. The solution in terms of Green's functions creates a non–linear integral equation (Fredholm type in Born and Riccati type in Rytov, for example). The integral equation could be solved iteratively by starting with a linear assumption and plugging back the result into a new background field.

The standard way to solve inverse scattering problems is by formulating a linear operator

$$Lu = f \tag{2 .1}$$

where f is a source function and u is a wavefield. The operator L is written as the sum of a background operator L_0 and a perturbed operator L_1. The solution u is also written as the sum of two functions (or product of two functions in the case of the Rytov approximation, for example). In the context of the Born approximation we write

$$L = L_0 + L_1 \qquad u = u_0 + u_s \tag{2 .2}$$

We assume that we know how to solve for the background operator L_0. That is, for the case of the background field u_0 and the background operator L_0 we know

$$u_0 = L_0^{-1} f. \tag{2 .3}$$

We substitute 2 .3 and 2 .2 into 2 .1 and find

$$(L_0 + L_1)(u_0 + u_s) = f \Rightarrow L_0 u_s + L_1 u_0 + L_1 u_s = 0, \tag{2 .4}$$

from which

$$u_s = -L_0^{-1} L_1 u_0 - L_0^{-1} L_1 u_s = -L_0^{-1}(L_1 u_0 + L_1 u_s) = -L_0^{-1} L_1 (u_0 + u_s) \tag{2 .5}$$

This equation provides the scattering field u_s as a function of the incident field u_0 and the scattering field itself u_s. The Born approximation is the supression of the second term on the right, assuming that $u_s \ll u_0$.

We now translate the operator L and L_0 into wave equation operators. That is

$$L = \left(\nabla^2 + \frac{\omega^2}{c^2(\boldsymbol{r})}\right) \qquad L_0 = \left(\nabla^2 + k^2\right)$$

with $k = \omega^2/c_0^2$, from which

$$L_1 = L - L_0 = \omega^2 \left(\frac{1}{c^2(\boldsymbol{r})} - \frac{1}{c_0^2} \right) = k^2 \left(\frac{c_0}{c^2(\boldsymbol{r})} - 1 \right) = k^2 \epsilon(\boldsymbol{r}) \qquad (2\ .6)$$

Now the operator L_0^{-1} is given by the convolution of the Green's function $G(\boldsymbol{r}', \boldsymbol{r})$ such that

$$L_0 G(\boldsymbol{r}, \boldsymbol{r}') = -\delta(\boldsymbol{r} - \boldsymbol{r}').$$

and the source (whatever is on the right side). So replacing the operators accordingly in 2 .5 we find

$$u_s = -L_0^{-1} L_1 (u_0 + u_s) = k^2 \int_V d\boldsymbol{r}' G(\boldsymbol{r}, \boldsymbol{r}') \epsilon(\boldsymbol{r}')(u_0 + u_s). \qquad (2\ .7)$$

Equation 2 .7 is know as the *Lippmann-Schwinger integral equation* and it is a *Fredholm type two* integral equation. Note that the scaterring field u_s is both in the left side and inside the integral. In this way the equation is non–linear. One way to see this problem is by writing as en eigenvalue/eigenvector problem. Let us see: By defining the operator

$$T = -L_0^{-1} L_1$$

and a new source term

$$g = T u_0,$$

we might write the (in operator form) equation 2 .7 as

$$u_s = g + T u_s \qquad (2\ .8)$$

For $g = 0$ we see that 1 is an eigenvalue of the operator T and u_s is an eigvenvector. Reordering terms in 2 .8 we find

$$(1 - T) u_s = g.$$

If $(1 - T)^{-1}$ exists (and this is conditioned to the norm of the operator $\|T\| < 1$). In this case we can expand

$$(1 - T)^{-1} = \frac{1}{1 - T} = 1 + T + T^2 + \cdots$$

With the help of this series we can write

$$u_s = (1 + T + T^2 + \cdots) g. \qquad (2\ .9)$$

This expression is sometimes known as the *multiple scattering series* the *Neumann series* and the operator $(1 - T)^{-1}$ as the *resolvent* operator.

3 The Born Approximation

The linear (first term) order on the series 2 .9 is know as the Born approximation.

That is, the Born approximation can be written as

$$u_s = g = Tu_0 = -L_0^{-1}L_1 u_0 = k^2 \int_V d\boldsymbol{r}' G(\boldsymbol{r}, \boldsymbol{r}')\epsilon(\boldsymbol{r}')u_0 \tag{3 .10}$$

Note that because all the non–linear terms in equation 2 .9 were truncated, the Born approximation is good up to only single scattering. In particular single backscattering. Multiple scattering can be obtained by re–iterating on the solution 3 .10. That is, after finding u_s we can plug it back into the integral as a new u_0 and so forth. This *Picard iterative process* that will converge the final solution u_s is a fixed point of the iteration and this can be shown to exists as long as $\|T\| < 1$.

Therefore the Born approximation is best used in reflection problems (backscattering). The Born approximation 3 .10 is useful for modeling data provided something is know about the scatterers on the media or to solve for the properties perturbations in the media provided that measurements of u_s exist (inversion).

Inversion can be done in the frequency (ω) space $\boldsymbol{r}$ domain ([2]) by using the properties of the Fourier transforms or in the time t space $\boldsymbol{r}$ domain using the properties of the generalized Radon transform [1]. These inversions work that well because the kernel of the integral is a Fourier type kernel inherited by the WKBJ Green's function approximations of the wave equation operator in rectangular coordinates.

The Born approximation does not perform accurate for transmission problems, since the incident wavefield u_0 is not updated and the integral will have constructive interference along the transmission direction make it larger and larger and violating the conservation of energy principle. Hence, other approximations are provided. For example in the next section I discuss the Rytov approximation.

4 The Rytov Approximation

The strategy of the Rytov approximation [3] is a bit different form the one in Born approximations. Here Rytov writes the wave field as product of an "amplitude" term with a complex exponential "phase". Where the phase is the actual perturbation and the amplitude factor is the background field.

From equation 2 .4 we observe that

$$L_0 u_s = -L_1 u;$$

and from using equation 2 .6 and $L_0 u_0 = f$

$$L_0 u_s = L_0 u = -k^2 \epsilon(\boldsymbol{r}) u(\boldsymbol{r}) - f$$

Hence, by assuming that the extra source term is zero $f = 0$,

$$\left(\nabla^2 + k^2\right) u(\boldsymbol{r}) = -k^2 \epsilon(\boldsymbol{r}) u(\boldsymbol{r}). \tag{4 .11}$$

where

$$\epsilon(\boldsymbol{r}) = \frac{c_0^2}{c^2(\boldsymbol{r})} - 1$$

is the model perturbation (slowness square perturbation. Which is obtained by subtracting the exact from the background wave equations).

The first Rytov expression is to write the field $u(\boldsymbol{r})$ as an amplitude phase complex number that is

$$u(\boldsymbol{r}) = u_0(\boldsymbol{r}) \mathrm{e}^{\psi(\boldsymbol{r})}, \tag{4 .12}$$

where $u_0(\boldsymbol{r})$ is the background wavefield and all other junk is stacked into the phase $\psi(\boldsymbol{r})$. Then from appendix equation A.21

$$\nabla^2 u(\boldsymbol{r}) = \mathrm{e}^{\psi(\boldsymbol{r})} \left[u_0(\boldsymbol{r}) \left(\nabla^2 \psi(\boldsymbol{r}) + (\nabla \psi(\boldsymbol{r}))^2\right) + 2 \nabla u_0(\boldsymbol{r}) \cdot \nabla \psi(\boldsymbol{r}) + \nabla^2 u_0(\boldsymbol{r})\right]. \tag{4 .13}$$

From equation 4 .11

$$\left(\nabla^2 + k^2\right) u_0(\boldsymbol{r}) \mathrm{e}^{\psi(\boldsymbol{r})} = -k^2 \epsilon(\boldsymbol{r})) u_0(\boldsymbol{r}) \mathrm{e}^{\psi(\boldsymbol{r})}$$

and from 4 .13

$$u_0(\boldsymbol{r}) \left(\nabla^2 \psi(\boldsymbol{r}) + (\nabla \psi(\boldsymbol{r}))^2\right) + 2 \nabla u_0(\boldsymbol{r}) \cdot \nabla \psi(\boldsymbol{r}) + \nabla^2 u_0(\boldsymbol{r}) = -k^2 u_0(\boldsymbol{r})(\epsilon(\boldsymbol{r}) + 1)$$

and because the amplitude term $u_0(\boldsymbol{r})$ is the background wave field it satisfies the homogeneous equation (this is the second version of the wave equation)

$$\left(\nabla^2 + k^2\right) u_0(\boldsymbol{r}) = 0. \tag{4 .14}$$

we get to

$$u_0(\boldsymbol{r})\left(\nabla^2\psi(\boldsymbol{r}) + (\nabla\psi(\boldsymbol{r}))^2\right) + 2\nabla u_0(\boldsymbol{r})\cdot\nabla\psi(\boldsymbol{r}) \quad = \quad -k^2 u_0(\boldsymbol{r})\epsilon(\boldsymbol{r})$$

Moving terms around

$$2\nabla u_0(\boldsymbol{r})\cdot\nabla\psi(\boldsymbol{r}) + u_0(\boldsymbol{r})\nabla^2\psi(\boldsymbol{r}) = -u_0(\boldsymbol{r})\left[(\nabla\psi(\boldsymbol{r}))^2 + k^2\epsilon(\boldsymbol{r})\right]$$

The first two terms on the left ask for a completion of squares. That is we add the missing $\psi(\boldsymbol{r})\nabla^2 u(\boldsymbol{r})$ and see

$$\nabla^2\left(u_0(\boldsymbol{r})\psi(\boldsymbol{r})\right) = -u_0(\boldsymbol{r})\left[(\nabla\psi(\boldsymbol{r}))^2 + k^2\epsilon(\boldsymbol{r})\right] + \psi(\boldsymbol{r})\nabla^2 u_0(\boldsymbol{r}). \tag{4 .15}$$

From 4 .14

$$\psi(\boldsymbol{r})\nabla^2 u_0(\boldsymbol{r}) = -k^2 u_0(\boldsymbol{r})\psi(\boldsymbol{r}),$$

so equation 4 .15 can be written as

$$\left(\nabla^2 + k)\right) u_0(\boldsymbol{r})\psi(\boldsymbol{r}) = -u_0(\boldsymbol{r})\left[(\nabla\psi(\boldsymbol{r}))^2 + k^2\epsilon(\boldsymbol{r})\right]$$

This is the wave equation for the combined field

$$u_0(\boldsymbol{r})\psi(\boldsymbol{r})$$

and a source term

$$-u_0(\boldsymbol{r})\left[(\nabla\psi(\boldsymbol{r}))^2 + k^2\epsilon(\boldsymbol{r})\right]$$

for which the solution in terms of Green's functions is

$$u_0(\boldsymbol{r})\psi(\boldsymbol{r}) = \int_V d^3\boldsymbol{r} G(\boldsymbol{r};\boldsymbol{r}')u_0(\boldsymbol{r}')\left[(\nabla\psi(\boldsymbol{r}'))^2 + k^2\epsilon(\boldsymbol{r}')\right] \tag{4 .16}$$

with $G(\boldsymbol{r};\boldsymbol{r}')$ the Green's function of the background medium, $u_0(\boldsymbol{r}')$ and $u_0(\boldsymbol{r})$ are the incident fields at $\boldsymbol{r}'$ and $\boldsymbol{r}$ respectively.

At this point there are not yet approximations. The integral equation 4 .16 is a Riccati integral equation. If the square gradient function $(\nabla\psi(\boldsymbol{r}'))^2$ is small compared to the perturbation $k^2\epsilon(\boldsymbol{r}')$ then it can be droped from the integral. This is the:

4 .1 The Rytov Approximation

The Rytov approximation is the truncation of integral 4 .16 by dropping the gradient square function of the phase. This is

$$\psi(\boldsymbol{r}) = \frac{1}{u_0(\boldsymbol{r})} \int_V d^3\boldsymbol{r} G(\boldsymbol{r};\boldsymbol{r'}) u_0(\boldsymbol{r'}) k^2 \epsilon(\boldsymbol{r'}) \tag{4 .17}$$

assuming $\nabla\psi \ll k^2\epsilon(\boldsymbol{r'})$.

What is the meaning of $\nabla\psi \ll k^2\epsilon(\boldsymbol{r'})$? From equation 4 .12, we see that $\phi(\boldsymbol{r})$ measures the perturbation in phase. The gradient $\nabla\psi(\boldsymbol{r})$ measures the change on this perturbation. For example, a sudden change on velocity would give a bending of the ray trajectory so a phase change is correlated with changes in the direction of propagation. In this way the Rytov approximation constrains the change on angle propagation from the background to the scatterer volume. It is in this sense that the Rytov approximation is better for transmission that for reflection, since in reflection a large change on angle (and phase) is assumed. Ru–Shan Wu and friends make the following quantitative analysis. Assume that the observed total field after after interacting with the heterogeneities is nearly a plane wave:

$$u = A\mathrm{e}^{\mathrm{i}\boldsymbol{k_1}\cdot\boldsymbol{r}},$$

The incident wave is

$$u_0 = A_0\mathrm{e}^{\mathrm{i}\boldsymbol{k_0}\cdot\boldsymbol{r}}.$$

The phase ψ can be written as

$$\psi = \ln\left(\frac{A}{A_0}\right) + \mathrm{i}\left(\boldsymbol{k_1} - \boldsymbol{k_0}\right)\cdot\boldsymbol{r},$$

and

$$\nabla\psi = \nabla\ln\left(\frac{A}{A_0}\right) + \mathrm{i}\left(\boldsymbol{k_1} - \boldsymbol{k_0}\right)$$

so

$$\nabla\psi\cdot\nabla\psi = \left|\nabla\ln\left(\frac{A}{A_0}\right)\right|^2 - |\boldsymbol{k_1} - \boldsymbol{k_0}|^2 + 2\mathrm{i}(\boldsymbol{k_1} - \boldsymbol{k_0})\cdot\nabla\ln\left(\frac{A}{A_0}\right).$$

Normally wave amplitudes vary much slower than phases, so the major contribution to $\nabla\psi \cdot \nabla\psi$ is given by the term $|\boldsymbol{k_1} - \boldsymbol{k_0}|^2$. So the Rytov approximation can be stated as

$$\begin{aligned}
|\boldsymbol{k_1} - \boldsymbol{k_0}|^2 &= |k_1|^2 + |k_0^2| - 2\boldsymbol{k_1} \cdot \boldsymbol{k_0} \\
&= 2k^2 - 2k^2 \cos 2\theta \\
&= 2k^2 - 2k^2(\cos^2\theta - \sin^2\theta) \\
&= 2k^2(1 - \cos^2\theta) + 2k^2 \sin^2\theta \\
&= 4\,k^2 \sin^2\theta \ll k^2|\epsilon|
\end{aligned}$$

where $k = \omega/v$, v is the wavespeed at point $\boldsymbol{r}$, and θ is the aperture angle. Then

$$2\sin\theta \ll |\epsilon|$$

or

$$\sin\theta \ll \frac{1}{2}\sqrt{\frac{c_0^2 - c^2(\boldsymbol{r})}{c^2(\boldsymbol{r})}}.$$

If we require a small contrast between the background and the heterogeneities, then $\sin\theta \ll 1$. That is, the Rytov approximation is a small angle approximation. If the energy is backscattered, then $2\theta = \pi$ and so $\sin\theta = 1$ which violates the small angle assumption.

5 Comparison between Rytov and Born

- – In the Born approximation the wavefield is additive. That is,

  $$u(\boldsymbol{r}) = u_0(\boldsymbol{r}) + U(\boldsymbol{r})$$

  where $u(\boldsymbol{r})$ is the propagating wavefield, $u_0(\boldsymbol{r})$ is the background wavefield and $U(\boldsymbol{r})$ is the perturbed wavefield.

  – In the Rytov approximation the wavefield is multiplicative. That is,

  $$u(\boldsymbol{r}) = u_0(\boldsymbol{r})\mathrm{e}^{\psi(\boldsymbol{r})}$$

where $u(\boldsymbol{r})$ is the propagating wavefield, $u_0(\boldsymbol{r})$ is the background wavefield and $\psi(\boldsymbol{r})$ is a complex phase function that make up for the rest of the wavefield. That is this could be computed using logarithms as

$$\psi(\boldsymbol{r}) = \ln u(\boldsymbol{r}) - \ln u_0(\boldsymbol{r}).$$

A reason for the exponential function is because it fits well the case where $\psi(\boldsymbol{r}) = 0$, returning $u_0(\boldsymbol{r})$ as the solution of the background problem. Any other exponential would do, but anything more natural than the Euler constant?

The Rytov assumption about the wavefield as a multiplication of an amplitude factor times a phase factor (complex exponential) reflects in the behavior of the WKBJ Green's functions and emphasize in the approximation of the phase rather than in the whole Green's function.

- By expanding e^{ψ} into a power series we see

$$u - u_0 = u_0(\mathrm{e}^{\psi} - 1) = u_0\psi + \frac{1}{2}u_0\psi + \cdots$$

When $\psi \ll 1$, that is, when the accumulated phase change is than one radian (corresponding to about one sixth of the wave period), we can neglect high order terms in after ψ^2, so

$$u(\boldsymbol{r}) = u_0(\boldsymbol{r}) + u_0(\boldsymbol{r})\psi(\boldsymbol{r}) = u_0(\boldsymbol{r}) + k^2 \int_V d^3\boldsymbol{r} G(\boldsymbol{r}, \boldsymbol{r}')\epsilon(\boldsymbol{r}')u_0(\boldsymbol{r}'),$$

which is the Born approximation 2 .7. That is, for $\phi \ll 1$, the Rytov approximation reduces to the Born approximation. In this sense, it is customary to think that the Rytov approximation has a broader scope than the Born approximation, but this is no necessarily the case for Brown, although Oristaglio thinks his numerical evidence shows that the Rytov approximation is better and that the Born approximation is just the first order term of an infinite series of the Rytov approximation. Piperakis and friends also agree about the better approximation of the Rytov approximation over the Born approximation, for computing travel times.

- By the nature of the Born approximation and that of the Rytov approximation, as indicated in section 4 .1 we have that
  - In the Born approximation we might expect better behavior of reflected (backscattered) energy. Because the scattered wavefield with respect to the incident wavefield should be small, the propagation distance is either short (in which the incident wavefield is large) and/or the heterogeneities are weak. By observing integral **??** we see that the integral adds up the incident energy $u_0(\boldsymbol{r}')$ constructively along the transmitted energy without bounds (because the $u_0(\boldsymbol{r}')$ is not recomputed. This makes of the transmission problem for the Born approximation a poor choice.
  - As discussed in section 4 .1, the Rytov approximation has poor accuracy for backscattered energy.

6 The De Wolf Approximation

Due to the limitations of Born for forward scattering and Rytov for backward scattering some new ideas have came through. A basic idea is to include more terms in the Neumann series. However De Wolf [**?**], [**?**], introduced the idea of choosing the scattering direction so that there could any number of forward scattered energy but only one backscattered at most.

To understand De Wolf's idea we start with the Neuman series 2 .9. That is

$$\begin{aligned} u = u_0 + u_s &= u_0 + Tg + T^2 g + \cdots \\ &= u_0 + L_0^{-1} L_1 u_0 L_0^{-1} L_1 + \cdots \end{aligned}$$

7 In Between Born and Rytov

Daniel L. Marks considers an interesting idea that creates an infinite intermediate approximations bewteen the Born and the Rytov approximations. His idea is simple. The Rytov approximation is based on the assumption $u = u_0 \mathrm{e}^{\psi}$. Now from Euler's formula

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \mathrm{e}^x$$

he Marks writes

$$u = u_0 \lim_{n\to\infty} \left(1 + \frac{\psi}{n}\right)^n$$

When $n = 0$ we have that the perturbation is zero $u = u_0$, when $n = 1$ we find

$$u = u_0 + u_0\psi$$

where we identify $u_s = u_0\psi$, and when $n \to \infty$ we find the Rytov approximation. So with this formula, we can think of an infinite number of approximations in between Born and Rytov, some of which could better suited for a particular problem.

8 Conclusions

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A Appendix A

Laplacian identities. We derive the Laplacian of the product of two functions quickly. This is

$$\nabla(fg) = \nabla \cdot (\nabla fg) = \nabla \cdot (f\nabla g + g\nabla f) = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f \tag{A.18}$$

Now, let us assume that $u = u_0(\boldsymbol{r})\mathrm{e}^{\psi(\boldsymbol{r})}$, so

$$\nabla^2 u(\boldsymbol{r}) = \nabla^2 \left(u_0(\boldsymbol{r})\, \mathrm{e}^{\psi}\boldsymbol{r}\right)$$

and

$$f = u_0(\boldsymbol{r}) \qquad g = u_0\mathrm{e}^{\psi(\boldsymbol{r})}$$

Then by applying equation A.18

$$\nabla^2 \left(u_0(\boldsymbol{r})\, \mathrm{e}^{\psi(\boldsymbol{r})}\right) = u_0(\boldsymbol{r})\nabla^2\mathrm{e}^{\psi(\boldsymbol{r})} + 2\nabla u_0(\boldsymbol{r}) \cdot \nabla\mathrm{e}^{\psi(\boldsymbol{r})} + \mathrm{e}^{\psi(\boldsymbol{r})}\nabla^2 u_0(\boldsymbol{r}) \tag{A.19}$$

We are no done. Let us find

$$\nabla^2 \mathrm{e}^{\psi(\boldsymbol{r})} = \nabla \cdot \nabla \mathrm{e}^{\psi(\boldsymbol{r})} = \nabla \cdot \left(\mathrm{e}^{\psi(\boldsymbol{r})} \nabla \psi(\boldsymbol{r})\right) = \mathrm{e}^{\psi(\boldsymbol{r})} \nabla^2 \psi(\boldsymbol{r}) + \mathrm{e}^{\psi(\boldsymbol{r})} \left(\nabla \psi(\boldsymbol{r})\right)^2 \tag{A.20}$$

with $(\nabla f)^2 = (\nabla f) \cdot (\nabla f)$. We apply A.20 into A.19 and find

$$\begin{aligned} \nabla^2 \left(u_0(\boldsymbol{r})\, \mathrm{e}^{\psi(\boldsymbol{r})}\right) &= \\ &\mathrm{e}^{\psi(\boldsymbol{r})} \left[u_0(\boldsymbol{r}) \left(\nabla^2 \psi(\boldsymbol{r}) + (\nabla \psi(\boldsymbol{r}))^2\right) + 2 \nabla u_0(\boldsymbol{r}) \cdot \nabla \psi(\boldsymbol{r}) + \nabla^2 u_0(\boldsymbol{r})\right]. \end{aligned} \tag{A.21}$$

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