Elliptic Functions

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Contents

1 Introduction

I develop introductory notes about elliptic functions. Initially two topics which motivate the study are geometric insights into the rectification of the ellipse, and an introduction to the simple pendulum problem of mechanics. I feel that a stronger motivation is through the introduction of circular functions and hyperbolic functions, which I also include. The elliptic functions are motivated this time as some kind of stretched circular function starting at a circle which is stretched into an ellipse.

2 Motivation

2.1 Rectification of the Ellipse

Let us compute the arc length along an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{2.1}$$

using equation

$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Taking implicit differentiation in **??** we see

$$2\frac{x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0,$$

from which

$$\frac{dy}{dx} = -\frac{xb^2}{a^2 y}$$

and so

$$\begin{aligned} 1+\left(\frac{dy}{dx}\right)^1 &= 1+\frac{x^2b^4}{a^4y^2} \\ &= 1+\frac{x^2b^4}{a^4(b^2-b^2x^2/a^2)} \\ &= 1+\frac{x^2b^4}{a^4b^2-a^2b^2x^2} \\ &= \frac{a^4b^2-x^2b^2(a^2-b^2)}{a^2b^2(a^2-x^2)} \\ &= \frac{a^4-x^2(a^2-b^2)}{a^2(a^2-x^2)}. \end{aligned}$$

Hence

$$\begin{aligned} s &= \int_0^x \sqrt{\frac{a^4-x^2(a^2-b^2)}{a^2(a^2-x^2)}} \qquad (2.2) \\ &= \int_0^x \frac{a^4-x^2(a^2-b^2)}{\sqrt{[a^4-x^2(a^2-b^2)]\,[a^2(a^2-x^2)]}} \\ &= a^3\int_0^x \frac{dx}{\sqrt{[a^4-x^2(a^2-b^2)]\,(a^2-x^2)}} - \frac{a^2-b^2}{a}\int_0^x \frac{x^2dx}{\sqrt{[a^4-x^2(a^2-b^2)]\,(a^2-x^2)}}. \end{aligned}$$

Let us define

$$P(x) = \left[a^4 - x^2(a^2-b^2)\right](a^2-x^2).$$

which is a polynomial of four degree. The integrals

$$\int \frac{dx}{\sqrt{P(x)}}, \qquad \int \frac{x^2dx}{\sqrt{P(x)}},$$

are called elliptical integrals. More generally an integral of the form

$$\int R(x,y)dx \quad \text{with} \quad y^2 = P(x)$$

is called an elliptical integral.

Another way to see elliptical integrals is using trigonometrical functions. The eccentricity e of an ellipse is defined as the ratio between two distances. The distance between the center and a focus c, and the distance between the focus to a vertex a. The distance between the focus and a vertex is the semi–mayor axes a, and by Pythagoras theorem

$$a^2 = b^2 + c^2;$$

so

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}. \tag{2.3}$$

Figure **??** shows the figure of an ellipse with $a = 3$ and $b = 2$, so $c = \sqrt{5}$. The eccentricity is then

$$e = \frac{\sqrt{5}}{3}$$

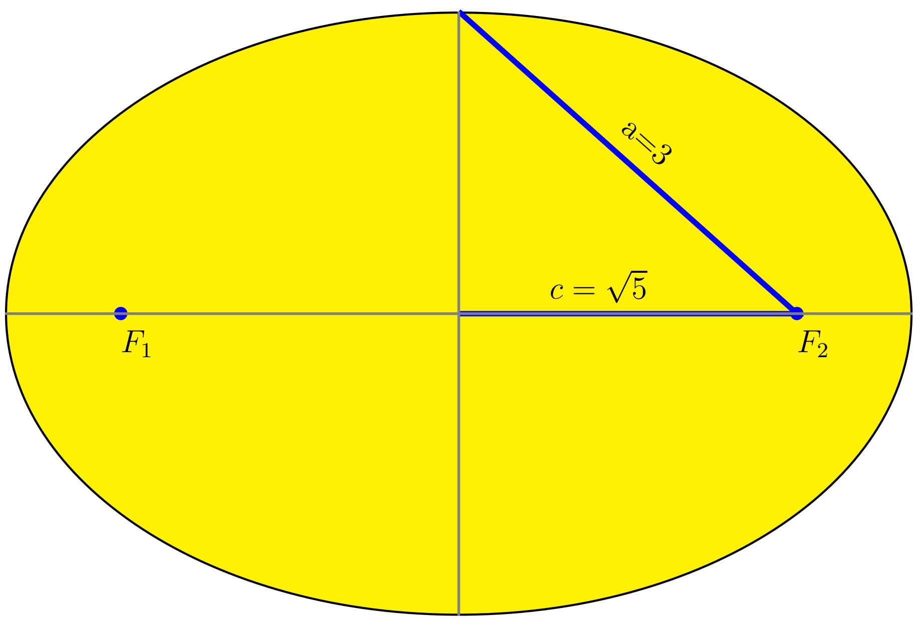

Figure 1: Ellipse with large and small semiaxes $a = 3$, $b = 2$ and eccentricity $e = \frac{\sqrt{5}}{3}$.

From **??** we see that

$$
\begin{aligned}
s &= \int_0^x \sqrt{\frac{a^4 - e^2\, x^2}{a^2 - x^2}} dx \\
&= \int_0^{\varphi} \sqrt{1 - e^2\, \sin^2\varphi}\, d\varphi \qquad \text{with} \quad x = a\sin\varphi \qquad (2.4)
\end{aligned}
$$

2.2 Simple pendulum problem

2.2.1 The precise solution of the equation of motion of a simple pendulum

Figure **??** shows a simple pendulum. The angle of inclination is θ, the length of the pendulum is l, the initial angle ($t = 0$) is α (not in the figure).

The energy conservation dictates potential energy lost from time 0 to time t, is equivalent to the kinetic energy gained (kinetic energy started at zero) in the same interval. That is

$$\frac{1}{2}ml^2\omega^2 = mgl\cos\theta - mgl\cos\alpha$$

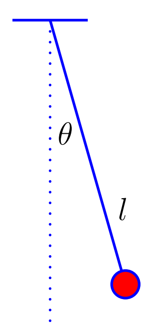

Figure 2: Simple pendulum

where $\omega = d\theta/dt = \dot{\theta}$ is the angular velocity. So

$$\begin{aligned}
\dot{\theta}^2 &= \frac{2g}{l}(\cos\theta - \cos\alpha) \\
&= \frac{2g}{l}(\cos[2(\theta/2)] - \cos[2(\alpha/2)]) \\
&= \frac{2g}{l}(\cos^2\theta/2 - \sin^2\theta/2 - \cos^2\alpha/2 + \sin^2\alpha/2) \\
&= \frac{4g}{l}(\sin^2\alpha/2 - \sin^2\theta/2) \\
&= \frac{4g\sin^2\alpha/2}{l}\left(1 - \frac{\sin^2\theta/2}{\sin^2\alpha/2}\right) \\
&= \frac{4g\sin^2\alpha/2}{l}(1 - \sin^2\varphi). \qquad (2.5)
\end{aligned}$$

We used the substitution

$$\sin\varphi = \frac{\sin\theta/2}{\sin\alpha/2}.$$

Taking derivatives

$$\frac{1}{2}\cos\frac{\theta}{2}\dot{\theta} = \sin\frac{\alpha}{2}\ \cos\varphi\dot{\varphi}$$

and so

$$\dot{\theta} = \frac{2\dot{\varphi}\ \sin\alpha/2\cos\varphi}{\cos\theta/2}.$$

Squaring

$$\begin{aligned}
\dot{\theta}^2 &= \frac{4\dot{\varphi}^2\ \sin^2\alpha/2\cos^2\varphi}{\cos^2\theta/2} \\
&= \frac{4\dot{\varphi}^2\sin^2\alpha/2\cos^2\varphi}{1 - \sin^2\theta/2} \\
&= \frac{4\dot{\varphi}^2\sin^2\alpha/2\cos^2\varphi}{1 - \sin^2\varphi\sin^2\alpha/2} \qquad (2.6)
\end{aligned}$$

Inserting **??** into **??**

$$\frac{4\dot{\varphi}^2\sin^2\alpha/2\cos^2\varphi}{1 - \sin^2\varphi\sin^2\alpha/2} = \frac{4g\sin^2\alpha/2}{l}(1 - \sin^2\varphi) = \frac{4g\sin^2\alpha/2\cos^2\varphi}{l}$$

from which

$$\dot{\varphi}^2 = \left(\frac{d\varphi}{dt}\right)\frac{g}{l}\left(1 - \sin^2\frac{\alpha}{2}\sin^2\varphi\right).$$

To integrate this equation we note that

$$dt = \sqrt{\frac{l}{g}}\ \frac{d\varphi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\varphi}}$$

and

$$t = \sqrt{\frac{l}{g}}\int \frac{d\varphi}{\sqrt{1-a^2\sin^2\varphi}}$$

with $a = \sin\alpha/2$. This is also another type of elliptical integral.

2.2.2 An approximate evaluation of the exact integral solution

The small angle assumption for φ implies

$$\sin\varphi \approx \varphi$$

and so the approximate solution becomes

$$t = \sqrt{\frac{l}{g}}\int \frac{d\varphi}{\sqrt{1-a^2\varphi^2}}$$

This integral can be evaluated analytically. Let us do the change of variables

$$\sin x = a\varphi$$

so

$$\begin{aligned} \cos x\ dx &= a\ d\varphi \quad \Rightarrow \quad d\ \varphi = \frac{\cos x\ dx}{a} \\ 1-a^2\varphi^2 &= 1-\sin^2 x = \cos^2 x \end{aligned}$$

so

$$t = \frac{1}{a}\sqrt{\frac{l}{g}}\int dx = \frac{1}{a}\sqrt{\frac{l}{g}}x,$$

or

$$t = \frac{1}{a}\sqrt{\frac{l}{g}}\sin^{-1}(a\varphi),$$

so

$$\varphi = \frac{1}{a}\ \sin\left(a\sqrt{\frac{g}{l}}t\right)$$

Going back to θ note that

$$a\varphi \approx a\sin\varphi = (\sin\alpha/2)(\sin\varphi) = \sin\theta/2 \approx \theta/2.$$

so we write

$$\theta = 2\,\sin\left(a\sqrt{\frac{g}{l}}t\right). \tag{2.7}$$

2.2.3 An approximate differential equation and its exact solution.

In the second example we show the typical pendulum problem used to teach students of first year college physics. Here the differential equation is simplified into a linear equation which is easier to solve.

Figure **??** shows an example of the simple pendulum used here.

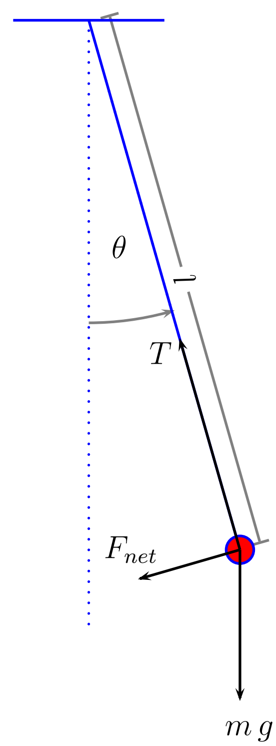

Figure 3: Simple pendulum.

Instead of using energy consideration here the static equation based on the free body plot. We choose a coordinate system in which the y component is along the tension T. The vertical component of the tension T is cancelled with the weight $m\,g$. The resultant is the component (along the x direction) is then given by

$$F_{net} = -mg\sin\theta.$$

and from Newton's second law

$$F_{net} = ma = -mg\sin\theta$$

so the acceleration a is given by

$$a = -g\sin\theta.$$

The acceleration is given by the angular acceleration times the length l. That is

$$a = \dot{\omega}L$$

The obtained differential equation is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta.$$

or

$$\frac{d^2\theta}{dt^2} + g\sin\theta = 0. \tag{2.8}$$

This is a non–linear differential equation of second order. The linearization is achieved by assuming

$$\sin\theta \approx \theta.$$

With this equation **??** turns out to be

$$\frac{d^2\theta}{dt^2} + g\theta = 0.$$

which analytic solution is

$$\theta = \alpha\sin\sqrt{\frac{g}{l}}\,t.$$

It is interesting to compare this equation with equation **??**. I do not see a way to take one into the other. The coefficient 2 in **??** is hardwired and in this equation the coefficient is α (the starting angle). The coefficient of t has a scale of $a = 0.5\sin\alpha$, which is missing here. Both solutions are a sinusoidal wave.

We will see how elliptical functions are generalization of trigonometric inverse functions. To introduce these ideas we add three little sections on

- circular functions.
- hyperbolic functions.
- elliptic functions.

We see here some similarities with the study of conics.

2.3 Inverse Circular Functions

2.3.1 Coordinate Definition, geometrical insights into the circular functions

We start this discussion with the simple description of the circular functions. We illustrate the coordinates (x, y) as the points in the unit circle with

$$x^2 + y^2 = 1,$$

and identify

$$x = \cos\varphi \qquad y = \sin\varphi,$$

with

$$\varphi = \tan^{-1}\frac{y}{x}.$$

The interpretation of φ is that of the angle made by the ray joining the point $(0, 0)$ with (x, y). Since the radius of the circle is $r = 1$, then

$$\varphi = \text{angle} = \text{arclength} = 2(\text{area swept by the angle}).$$

Actually, more than the angle, the last two attributes: arclength and twice swept area are carried naturally into the hyperbolic functions shown down below. In general in a circle

$$s = \varphi r \qquad \text{area} = \frac{s\, r}{2} = \frac{\varphi\, r^2}{2}.$$

I feel that the 2 factor is unfortunate. To have a better look to these formulas I prefer to think about φ as the ice cream cone [1] solid angle in 2D. That is, think about a right circular cone in 3D which angle between its axis and its generatrix is φ. More specifically, referring to Figure **??**, the axis of the cone is the ray from $(0, 0)$ toward the positive x axis. The generatrix of the cone is the segment of line from $(0, 0)$ to (x, y) which is rotated around the x axis 360 degrees, generating the cone. The angle φ is the solid angle of the ice cream cone and it coincides with the angle defined in the circular functions. However the area of intersection between the ice cream solid angle (the ice cream cone) and the (x, y) plane is the sum of the two areas, the light gray and the light blue. That is in this case we would have a better looking formula

$$\text{Solid ice cream cone angle area} = \text{angle} = \text{arclength}.$$

Figure **??** sketches the basic circular coordinates as well as the generating cone, the angle φ and the area of the solid angle φ (projected into the (x, y) plane.)

[1] The cone with axis coinciding with the x axis, generatrix is the segment from $(0, 0)$ to (x, y) and extended with a piece of spherical volume, with center $(0, 0, 0)$ and radius $r = 1$, so that the cone and spherical surface enclosed a convex volume: "the ice cream cone".

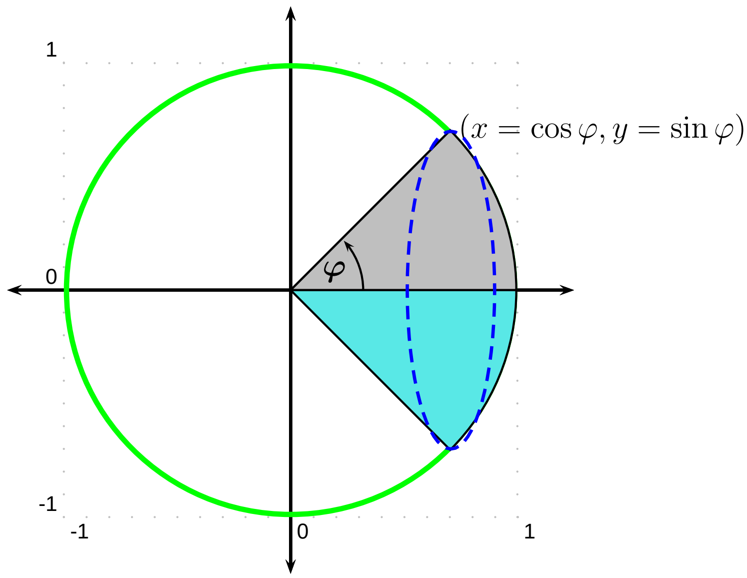

Figure 4: Unit circle $x^2 + y^2 = 1$, showing the circular coordinate functions $(\cos\varphi, \sin\varphi)$. The shaded area, angle and arclength along the circular sector have the same numerical values (angle measured in radians).

Geometrical illustration of circular functions. Figure **??** helps illustrating the definition of all circular functions as well as some basic identities. The three black dots in the picture are along the ray of angle φ. They represent

$$\begin{aligned} x &= \cos\varphi \qquad y = \sin\varphi \\ t &= \tan\varphi \qquad c = \cot\varphi \end{aligned}$$

Note that the coordinate $t = \tan\varphi$ is obtained by raising a tangent to the circle at $(1, 0)$ and finding the y intersection with the ray from $(0, 0)$ to (x, y), while the coordinate $c = \cot\varphi$ is obtained by tracing a tangent (cotangent) to the circle at point $(0, 1)$ and finding the x intersection with the ray from $(0, 0)$ to (x, y).

The secant and cosecant are found by using the Pythagoras theorem on the triangles as follows

$$\sec^2\varphi = 1 + t^2 = 1 + \tan^2\varphi;$$

which is the length of the segment of ray from $(0, 0)$ to $(1, t)$,

$$\csc^2\varphi = 1 + c^2 = 1 + \cot^2\varphi;$$

which is the length of the segment of ray from $(0,0)$ to $(c,1)$,

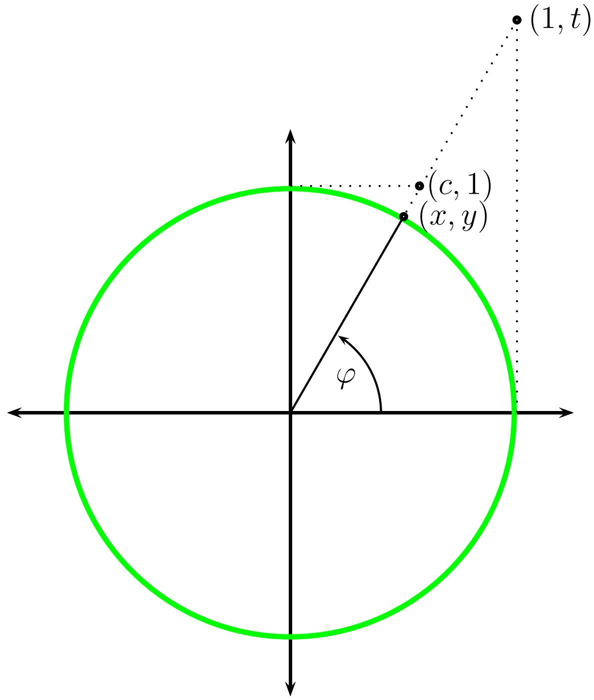

Figure 5: Unit circle $x^2 + y^2 = 1$, which serves to illustrate all circular functions and some basic identities. $x = \cos\varphi$, $y = \sin\varphi$, $t = \tan\varphi$, $c = \cot\varphi$, $\sec\varphi = 1/x = \sqrt{1+t^2} = \sqrt{1+\tan^2\varphi}$, and $\csc\varphi = 1/y = \sqrt{1+c^2} = \sqrt{1+\cot^2\varphi}$.

2.3.2 Theory

Perhaps the easiest way to find the inverse of circular functions is through the use of the the integral/derivative relationships on the forward functions. That is, given that

$$x = \sin^{-1} y \Leftrightarrow y = \sin x$$

Then

$$dy = \cos x dx,$$

and so

$$\frac{dx}{dy} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}},$$

which after integration yields

$$x = \sin^{-1} y = \int_0^y \frac{dt}{\sqrt{1-t^2}}. \tag{2.9}$$

Along the same lines the inverses of the circular functions could be computed, yielding explicit solutions in terms of integrals. For example one finds

$$x = \cos^{-1} y = -\int_0^y \frac{dt}{\sqrt{1-t^2}}. \tag{2.10}$$

On the other hand, we could evaluate the integrals using change of variables as follows.

$$\sin^{-1}\left(\frac{x}{a}\right) = \int_0^x \frac{dt}{\sqrt{a^2-t^2}} \qquad a^2 > t^2.$$

This is easily verified with the substitution $t = a\,\sin\varphi$, so $d\,t = \cos\varphi\;d\varphi$ and

$$\begin{aligned} f(x) = \int_0^x \frac{dt}{\sqrt{a^2-t^2}} &= \int_0^{\sin^{-1}(x/a)} \frac{a\,\cos\varphi\;d\varphi}{a\sqrt{1-\sin^2\varphi}} \\ &= \int_0^{\sin^{-1} x/a} d\varphi \\ &= \sin^{-1}\left(\frac{x}{a}\right). \end{aligned}$$

Likewise is easy to show similar formulas for the other trigonometrical functions. That is, for x, a reals , with $x^2 < a^2$

$$\begin{array}{ll} \sin^{-1}\left(\frac{x}{a}\right) = \int_0^x \frac{dt}{\sqrt{a^2-t^2}} & \cos^{-1}\left(\frac{x}{a}\right) = -\int_0^x \frac{dt}{\sqrt{a^2-t^2}} \\ \tan^{-1}\left(\frac{x}{a}\right) = \int_0^x \frac{a\,dt}{a^2+t^2} & \cot^{-1}\left(\frac{x}{a}\right) = \int_x^\infty \frac{a\,dt}{a^2+t^2} \\ \sec^{-1}\left|\frac{x}{a}\right| = \int_1^x \frac{a\,dt}{t\sqrt{t^2-a^2}} & \csc^{-1}\left|\frac{x}{a}\right| = -\int_0^x \frac{a\,dt}{t\sqrt{t^2-a^2}}. \end{array}$$

The inverse functions are obtained by computing the derivative on the forward functions and integrating them. That is like solving a first order ordinary differential equation. They can be checked, as shown here, by evaluating the integrals with proper change of variables.

2.4 Inverse Hyperbolic Functions

2.4.1 Definitions and Geometrical Insights

Next is a definition of circular and hyperbolic functions in terms of complex and real exponentials.

$$\begin{array}{llllll} \sin x = \frac{e^{ix}-e^{-ix}}{2i} & \cos x = \frac{e^{ix}+e^{-ix}}{2} & \tan x = \frac{\sin x}{\cos x} & \cot x = \frac{\cos x}{\sin x} & \sec x = \frac{1}{\cos x} & \csc x = \frac{1}{\sin x} \\ \sinh x = \frac{e^{x}-e^{-x}}{2} & \cosh x = \frac{e^{x}+e^{-x}}{2} & \tanh x = \frac{\sinh x}{\cosh x} & \cot x = \frac{\cosh x}{\sinh x} & \sec x = \frac{1}{\cos x} & \mathrm{csch} x = \frac{1}{\sinh x} \end{array}$$

The hyperbolic functions can be built from the hyperbola $x^2 - y^2 = 1$, where $x = \cosh\varphi$ and $y = \sinh\varphi$.

The meaning of the hyperbolic "angle" φ. To interpret the dimensions along the graphs of the hyperbolic functions I use the same model that was used in the circular function. However here the ice cream cone should be called the inverted ice cream cone, since it is formed by the volume of the cone, after carving a volume delimited by the surface of the hyperbolic function and the plane $x = M$. This volume, of course is non convex. The intersection of the volume with the plane (x, y) provides the area which matches the angle φ.

The hyperbolic function $x^2 - y^2 = 1$ as shown in Figure **??** and here the geometrical illustration of the angle and areas is in place, together with the hyperbolic coordinates (x, y).

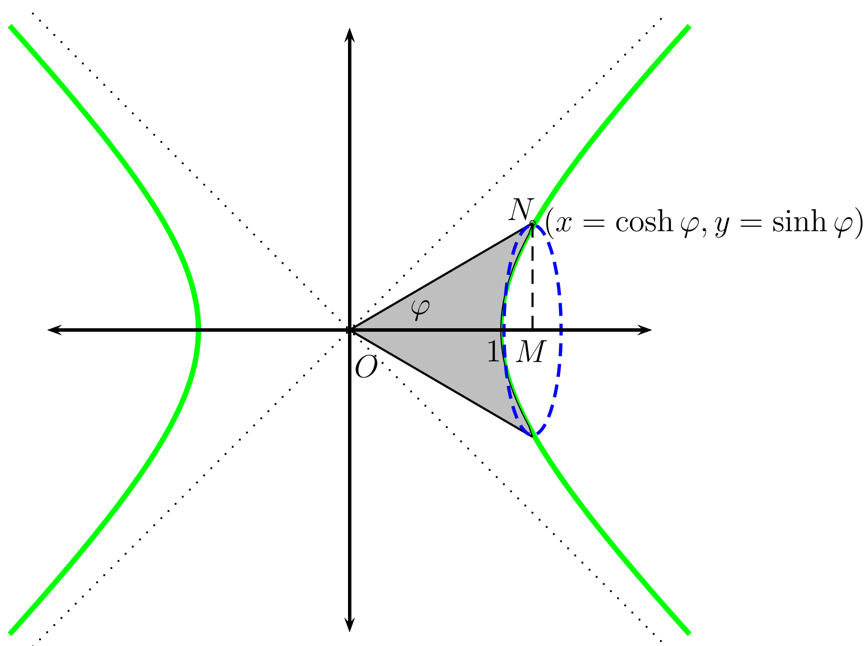

Figure 6: Canonical hyperbola $x^2 - y^2 = 1$ showing the hyperbolic coordinate functions $(\cosh\varphi, \sinh\varphi)$. Here φ is the solid angle delimited by the cone

The area of the shaded zone is that of the area of twice the triangle OMN minus the area of the curved sector from $x \in [1, M]$ under the hyperbola. This is

$$\text{Area} = My - 2\int_1^M \sqrt{x^2 - 1}dx.$$

Note that while the area in the circular functions is the integral over $\sqrt{1-x^2}$ here the area is the integral of $\sqrt{x^2-1}$. Let us evaluate the integral, which is a good exercise in calculus and I do it for practice.

by substitution $x = \sec\theta$. That is

$$dx = \tan\theta\sec\theta$$

and from $\sec^2\theta - 1 = \tan^2\theta$

$$\begin{aligned}
\int \sqrt{x^2-1}dx &= \int \tan^2\theta\sec\theta d\theta \\
&= \int \frac{\sin^2\theta}{\cos^3\theta}d\theta \\
&= \int \sin\theta\frac{\sin\theta}{\cos^3\theta}d\theta
\end{aligned}$$

Integrating by parts

$$\begin{aligned}
u = \sin\theta &\qquad dv = \frac{\sin\theta}{\cos^3\theta}d\theta \\
du = \cos\theta d\theta &\qquad v = \frac{1}{2\cos^2\theta}
\end{aligned}$$

then

$$\begin{aligned}
\int \sin\theta\frac{\sin\theta}{\cos^3\theta}d\theta &= \sin\theta\left(\frac{1}{2\cos^2\theta}\right) - \int \frac{1}{2\cos^2\theta}\cos\theta d\theta \\
& \frac{\sin\theta}{2\cos^2\theta} - \frac{1}{2}\int\frac{d\theta}{\cos\theta} \\
& \frac{\sin\theta}{2\cos^2\theta} - \frac{1}{2}\int\frac{d\theta}{\cos\theta} \\
& \frac{1}{2}\sin\theta\sec^2\theta - \frac{1}{2}\int \sec\theta d\theta. \qquad (2.11)
\end{aligned}$$

The integral of secant is, after recognizing that

$$\frac{d(\sec\theta+\tan\theta)}{d\theta} = \sec\theta\tan\theta + \sec^2\theta$$

computed as follows:

$$\begin{aligned}
\int \sec\theta\, d\theta &= \int \sec\theta\,\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}d\theta \\
&= \int \frac{\sec^2\theta\ + \sec\theta\tan\theta}{\sec\theta+\tan\theta}d\theta \\
&= \int \frac{d(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta} \\
&= \ln|\sec\theta+\tan\theta|.
\end{aligned}$$

We now return to equation **??**

$$\begin{aligned} \int \sin\theta \frac{\sin\theta}{\cos^3\theta} d\theta &= \sin\theta \sec^2\theta - \int \sec\theta d\theta \\ &= \sin\theta\sec^2\theta - \ln|\sec\theta + \tan\theta|. \end{aligned}$$

Returning to the original problem, remembering that $x = \sec\theta$, we can write the previous expressions only in term of secant as follows,

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{x^2}}$$
$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{1-x^2}$$

and so

$$2\int \sqrt{x^2-1} = x^2\sqrt{1-\frac{1}{x^2}} - \ln|x+\sqrt{x^2-1}| = x\sqrt{x^2-1} - x\,\ln|x+\sqrt{x^2-1}|$$

and the net area (subtraction of this from the triangular area) is given by

$$\begin{aligned} \text{Area} &= My - \int_1^M \sqrt{x^2-1}dx \\ &= My - M\sqrt{M^2-1} + \frac{1}{2}\ln|M+\sqrt{M^2-1}| \end{aligned}$$

Finally, from the figure $M = x$, and $y = \sqrt{x^2-1} = \sqrt{M^2-1}$ so

$$\text{Area} = xy - xy + \ln|x+y| = \ln|x+y|.$$

This is

$$\text{Area} = \begin{cases} \ln(x+y), & \text{if} \quad x+y \geq 0 \\ -\ln(x+y) = \frac{1}{2}\ln\frac{1}{x+y} & \text{if} \quad x+y \geq 0 \end{cases}$$

and since

$$\frac{1}{x+y} = \frac{1}{x+\sqrt{x^2-1}} = x - \sqrt{x^2-1} = x - y,$$

then we find the system of two equations for (x, y)

$$\begin{aligned} \text{Area} &= \ln(x+y) \\ -\text{Area} &= \ln(x-y). \end{aligned} \tag{2.12}$$

The reason why the sign was reversed in the second equation, is because for $x + y < 0$ the hyperbolic sheet is in the two left quadrants and a reverse on coordinates changes the sign of the area under the integral.

The solution of equation **??** for (x, y) in terms of Area is simply, after adding/subtracting and taking exponentials in both sides, both equations,

$$\begin{aligned} x &= \frac{\mathrm{e}^{\mathrm{Area}} + \mathrm{e}^{-\mathrm{Area}}}{2} \\ y &= \frac{\mathrm{e}^{\mathrm{Area}} - \mathrm{e}^{-\mathrm{Area}}}{2} \end{aligned}$$

We see then by definition that

$$\begin{aligned} x &= \cosh{(\mathrm{Area})} \\ y &= \sinh{(\mathrm{Area})}, \end{aligned}$$

and this justifies the statement that the argument of the hyperbolic functions is the area of intersection of the inverted ice cream cone and the (x, y) plane. See Figure **??**. We should not think that φ is an actual angle and that is why we do not draw the angular arc with an arrow head as we did in the case of the circular functions. Indeed φ is the area of the intersection of the inverted ice cream cone, with the (x, y) plane.

Geometrical illustration of hyperbolic functions. I only know about the geometrical interpretation for $\sinh\varphi$, $\cosh\varphi$, and $\tanh\varphi$. We already saw how the (x, y) coordinates represent the pair of functions $(\cosh\varphi, \sinh\varphi)$. To see the interpretation of $\tanh\varphi$ let us observe in Figure **??**

That the triangles OST and OMN are similar so

$$\frac{ST}{MN} = \frac{OS}{OM}$$

but the lengths MN, OS, and OM are

$$MN = y = \sinh\varphi, \qquad OS = 1, \qquad \text{and,} \quad OM = x = \cosh\varphi,$$

hence

$$ST = \frac{\sinh\varphi}{\cosh\varphi} = \tanh\varphi.$$

That is, $\tanh\varphi$ can be interpreted as the length of the vertical segment from the x axis through the point $(1, 0)$ and intersected with the segment that joins $(0, 0)$ with (x, y). Note that the segment is actually tangent to the hyperbola at the point $(1, 0)$.

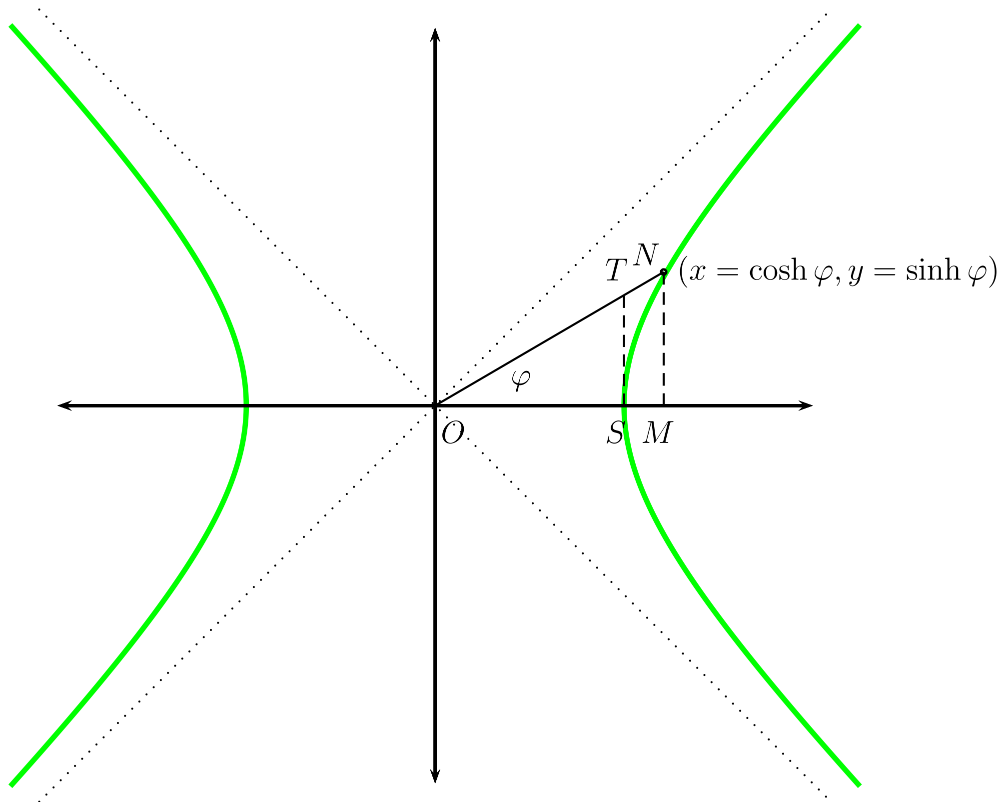

Figure 7: Canonical hyperbola $x^2 - y^2 = 1$ showing the hyperbolic coordinate functions $(\cosh\varphi, \sinh\varphi)$. Here φ is the solid angle delimited by the cone

Rectification of the hyperbola: The length of a segment of a canonical hyperbola does not behave as well as in a circle. After the previous analysis is natural to think that the nice properties of circular functions will translate one–to–one into nice properties of the hyperbola. We show below that the arc length along a segment of hyperbola produces and elliptic integral which can not be solved analytically.

To compute the arc length we use the parametric representation $x = \cosh\theta, y = \sinh\theta$ and compute the integral

$$\begin{aligned} s &= \int_1^\theta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\ &= \int_1^\theta \sqrt{\sinh^2\theta + \cosh^2\theta}\, d\theta \\ &= \int_1^\theta \sqrt{2\cosh^2\theta - 1}\, d\theta \end{aligned}$$

The eccentricity of the hyperbola is defined by

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$

and since here $a = b = 1$, then $e = \sqrt{2}$. We can then write

$$s = \int_1^\theta \sqrt{e^2 \cosh^2 \theta - 1}\, d\theta.$$

This is an elliptical integral with no analytical solution. Hence we can not easily say that $s = \varphi$ as we did in the unit circle. In a way we could expect this since the circle is the only closed curve with constant radius of curvature. Note the parallel between this equation and equation **??**.

Two interesting facts about hyperbolic cosine $\cosh \varphi$

- The function $\cosh x$ satisfies that its area under the curve in a given interval, coincides with the arclength over the same interval. The proof is simple

$$\begin{aligned}
\text{area} &= \int_a^b \cosh x dx \\
&= \int_a^b \sqrt{1 + \sin^2 x} dx \\
&= \int_a^b \sqrt{1 + \left(\frac{d}{dx}\cosh x\right)^2} dx \\
&= \int_a^b \sqrt{1 + \left(\frac{d}{dx}\cosh x\right)^2} dx \\
&= \text{arc length}
\end{aligned}$$

Description of the catenary The hyperbolic cosine function $\cosh \varphi$ serves to describe the catenary, which is a curve that a rope, string, bridge, chain [2] etc., makes when hanging from two fixed points. The equation of the catenary is given by

$$y = \frac{1}{a} \cosh ax \tag{2.13}$$

where a is a scaling factor, such that $y(0) = 1/a$, and depends only on the tension at the bottom of the string (T_0) , and the linear density of the string μ. Figure **??** shows a catenary with $a = 1$.

[2]The word “catenary” comes from the Latin word “catena” which means “chain”

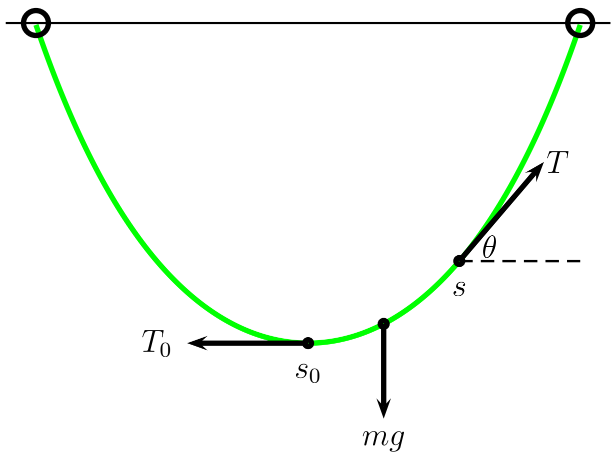

Figure 8: Catenary $y = \cosh\varphi$.

Derivation of the Catenary Equation The derivation of the catenary equation is simple and I include it here. We assume that the string has constant linear density μ. The free body diagram (**??**) shows three forces along the string. We pick the bottom (vertex) as the reference point (the zero point (0,0)). The arclength s is measured away from this zero point (and in the figure, to right of it). The three forces are: the two extrema of the segment on discussion , that is where the labels $s_0 = 0$ and s are located. In the bottom we have the tension T_0 and up where the label s is, we have the tension T. The weight (mg) is applied at the center of mass [3]

Let us assume that the catenary is a function $y = f(x)$ which we want to find.

From statics

$$\begin{aligned} T\cos\theta &= T_0 \\ T\sin\theta &= s\mu g \end{aligned}$$

where $m = s\,\mu$, T_0 is the tension at the bottom (vertex), T is the tension at some distance s along the string and θ is the angle that the tension vector T makes with the horizontal line.

To cancel the dependency on T we divide these equations, so that

$$\tan\theta = \frac{s\mu g}{T_0}$$

We know that

$$\frac{dy}{dx} = \tan\theta$$

[3]which in the picture is only a poor approximation, but this is irrelevant for the problem since by assuming that the masses at the points labeled as s and s_0 are assume negligible and the mass at the center of mass will not generate any torque, all torques are zero.

so

$$\frac{dy}{dx} = \frac{s\mu g}{T_0}. \tag{2.14}$$

We define the constant a as

$$a = \frac{\mu\, g}{T_0}$$

so a only depends on the tension at the bottom and the linear density (assuming g is constant and known). Equation **??** turns out to be

$$\frac{dy}{dx} = as.$$

By applying one more derivative

$$\frac{d^2y}{dx^2} = a\frac{ds}{dx} = a\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{2.15}$$

where we used the fact that the arclength satisfies the equation

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Equation **??** is a differential equation for y as a function of x. This can be solve initially by introducing the auxiliary variable $p = dy/dx$, so

$$\frac{dp}{dx} = a\,\sqrt{1+p^2},$$

and

$$\frac{dp}{a\,\sqrt{1+p^2}} = dx,$$

from which

$$x(p) = \int_0^p \frac{dt}{a\,\sqrt{1+t^2}} = \frac{1}{a}\,\sinh^{-1} p$$

This integral is shown in the next section. See equation **??**.

Inverting the previous equation we find

$$p = \sinh ax,$$

that is

$$\frac{dy}{dx} = \sinh a\,x,$$

which after one more integration yields

$$y(x) = \frac{1}{a}\cosh x + C$$

where C is a constant. We can choose C such that $y(0) = a$ and so $C = 0$. So the catenary is given by the equation

$$y(x) = \frac{1}{a}\cosh x.$$

2.4.2 Inverse Functions

Based on the definition of the hyperbolic functions as real functions of exponential character, we can use a real algebraic method to invert for them directly. Here there is an example for the function $\sinh^{-1} x$. From the definition

$$y = \sinh x = \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2},$$

we can easily invert for x in terms of y as follows.

$$\begin{aligned} 2y &= \mathrm{e}^{x} - \mathrm{e}^{-x} \\ 2\,y\,\mathrm{e}^{\mathrm{x}} &= \mathrm{e}^{2x} - 1. \end{aligned}$$

So, we need to solve the quadratic function

$$\mathrm{e}^{2x} - 2\,y\,\mathrm{e}^{x} - 1 = 0,$$

for e^{x}, which yields

$$\mathrm{e}^{x} = y \pm \sqrt{y^2 + 1}.$$

Since e^x is always positive and $y^2 + 1 > y^2$, then the minus "-" sign should be withdrawn. That is, we find

$$e^{x} = y + \sqrt{y^2 + 1}.$$

The inversion of this equation is simple using natural log.

$$x = \sinh^{-1}(y) = \ln(y + \sqrt{y^2 + 1})$$

Let us swap the variables x and y and rewrite this equation as

$$y = \sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$$

To find an integral representation, we observe from the definition of the hyperbolic functions in terms of exponentials that, if $\sinh y = x$ then

$$\frac{d\sinh y}{dx} = \cosh y \qquad \text{and} \qquad \cosh^2 y - \sinh^2 y = 1,$$

(hence the name hyperbolic). So taking derivatives in $\sinh y = x$,

$$\begin{aligned} \cosh y dy &= dx \\ \sqrt{1+\sinh^2 y}\, dy &= dx \\ \sqrt{1+x^2} dy &= dx \end{aligned}$$

so integrating

$$y = \sinh^{-1} x = \int_0^x \frac{d\,t}{\sqrt{1+t^2}} = \ln(x + \sqrt{x^2+1}). \tag{2.16}$$

It is interesting to observe that the integral representation of $\sinh^{-1} x$ can be obtained by changing the real variable t in the integral representation of $\sin^{-1}(x)$ (see equation **??**) by $t \to \mathrm{i}t$, and then multiplying the resulting expression by $-\mathrm{i}$. Note that from the exponential definition

$$\begin{aligned} \sinh y &= \frac{\mathrm{e}^y - \mathrm{e}^{-y}}{2} \\ &= \mathrm{i}\frac{\mathrm{e}^{-\mathrm{i}(\mathrm{i}y)} - \mathrm{e}^{\mathrm{i}(\mathrm{i}y)}}{2\mathrm{i}} \\ &= -\mathrm{i}\sin \mathrm{i}y \end{aligned} \tag{2.17}$$

That is, we say

$$\sinh^{-1} x = -\mathrm{i}\sin^{-1} \mathrm{i}x$$

Let us move to the next function $\cosh^{-1} x$. We first note that, from the real exponential definition, $\cosh y \geq 1$. The inverse relation is given by

$$y = \cosh^{-1} x \Leftrightarrow x = \cosh y,$$

From the real exponential definition

$$x = \cosh y = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2},$$

we can easily invert for y in terms of x as follows.

$$\begin{aligned} 2x &= \mathrm{e}^y + \mathrm{e}^{-y} \\ 2\,x\,\mathrm{e}^{\mathrm{y}} &= \mathrm{e}^{2y} + 1. \end{aligned}$$

So, we have to do is to solve the quadratic function

$$\mathrm{e}^{2y} - 2\,x\,\mathrm{e}^{y} + 1 = 0,$$

for e^{y}, which yields

$$\mathrm{e}^{y} = x \pm \sqrt{x^2 - 1}.$$

Since e^y is always positive and $x \geq 1$, then the minus "-" sign should be withdrawn. So,

$$e^{y} = x + \sqrt{x^2 - 1}.$$

The inversion of this equation is simple using natural log. That is

$$y = \cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1}).$$

Let us now find the inverse hyperbolic cosine in terms of its integral representation. As we did before, we first note that

$$\frac{d \cosh y}{dx} = \sinh y.$$

So taking derivatives in $\cosh y = x$,

$$\begin{aligned} \sinh y dy &= dx \\ \sqrt{\cosh^2 y - 1}\, dy &= dx \\ \sqrt{x^2 - 1} dy &= dx \end{aligned}$$

so integrating, from 1 to x,

$$y = \cosh^{-1}(x) = \int_1^x \frac{d\,t}{\sqrt{t^2 - 1}} = \ln(x + \sqrt{x^2 - 1}).$$

Along the same lines, as shown in equation **??**, let us see the relationship between $\cos y$ and $\cosh y$.

$$\begin{aligned} \cosh y &= \frac{\mathrm{e}^{y} + \mathrm{e}^{-y}}{2} \\ &= \frac{\mathrm{e}^{-\mathrm{i}(\mathrm{i}y)} + \mathrm{e}^{\mathrm{i}(\mathrm{i}y)}}{2} \\ &= \cos \mathrm{i}y. \end{aligned}$$

Hence the inversion of $\cosh y$ produces y while that of $\cos \mathrm{i}y$, produces $\mathrm{i}y$. This means that

$$\cosh^{-1} x = \mathrm{i}\ \cos^{-1} x.$$

Let us verify this equality using equation **??**.

$$\begin{aligned}
\cosh^{-1} x &= \mathrm{i}\cos^{-1} y \\
&= -\mathrm{i}\int_0^y \frac{dt}{\sqrt{1-t^2}} \\
&= \frac{1}{\mathrm{i}}\int_0^y \frac{dt}{\sqrt{1-t^2}} \\
&= \int_0^y \frac{dt}{\sqrt{t^2-1}}
\end{aligned}$$

Hence it is only necessary to compute either the inverse circular functions or the inverse hyperbolic functions but not both. Here are the relations that can be used when one of the function types are known.

$$\begin{array}{lll}
\sinh^{-1} x = -\mathrm{i}\sin^{-1}\mathrm{i}x & \cosh^{-1} x = \mathrm{i}\cos^{-1}\mathrm{i}x & \tanh^{-1} x = -\mathrm{i}\tan^{-1}\mathrm{i}x \\
\mathrm{sech}^{-1} x = \mathrm{i}\sec^{-1}\mathrm{i}x & \mathrm{csch}^{-1} x = \mathrm{i}\csc^{-1}\mathrm{i}x & \coth^{-1} x = -\mathrm{i}\coth^{-1}\mathrm{i}x
\end{array}$$

The following identities are the corresponding hyperbolic inverses in terms of integrals.

$$\begin{array}{ll}
\sinh^{-1}\left(\frac{x}{a}\right) = \int_0^x \frac{dt}{\sqrt{a^2+t^2}} & \cosh^{-1}\left(\frac{x}{a}\right) = \int_1^x \frac{dt}{\sqrt{t^2-a^2}} \\
\tanh^{-1}\left(\frac{x}{a}\right) = \int_0^x \frac{a\,dt}{a^2-t^2}, \quad t^2 > a^2 & \coth^{-1}\left(\frac{x}{a}\right) = \int_0^x \frac{a\,dt}{a^2-t^2}, \quad t^2 < a^2 \\
\mathrm{sech}^{-1}\left|\frac{x}{a}\right| = -\int_0^x \frac{a\,dt}{t\sqrt{a^2-t^2}} & \mathrm{csch}^{-1}\left|\frac{x}{a}\right| = -\int_0^x \frac{a\,dt}{t\sqrt{a^2+t^2}}.
\end{array}$$

2.5 Elliptic Integrals as generalizations of inverse trigonometrical functions

We describe the Jacobi elliptical functions based on an ellipse rather than in a circle as a natural extension of circular functions.

Let us start with the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{2.18}$$

Figure **??** shows such an ellipse.

Next we should define the "angle" of the elliptic functions. For circular functions such angle is φ. We saw that in this case the angle coincides with the area of the circular sector which is also equal to the arc length. In the case of hyperbolic functions the argument is not the angle but the area of the intersection of the inverse ice cream cone with the (x, y)

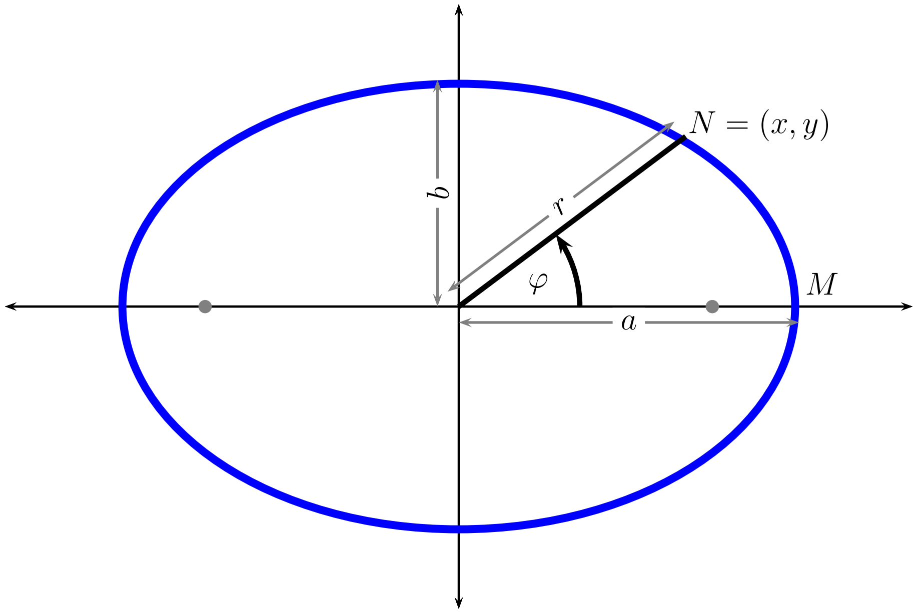

Figure 9: Ellipse with large and small semiaxes $a = 3$, $b = 2$ and eccentricity $e = \frac{\sqrt{5}}{3}$.

plane. Interestingly for elliptic functions the argument is no the angle φ, no the area under the swept angle and neither the arc length but defined as

$$u = \int_M^N r d\varphi,$$

If $a = b = 1$ then we are back to the circle and $\varphi = s =$ Area, but of course we are interested in the case $a \neq 1$.

The first three elliptic Jacobi definitions are

$$\text{sn } u = \frac{y}{b}, \qquad \text{cs } u = \frac{x}{a}, \qquad \text{dn } u = \frac{r}{a},$$

Here sn u corresponds with $\sin\varphi$ in the circular functions, cs u corresponds with $\cos\varphi$ in the circular functions, and we get the additional function dn u which exist because r is changing along the ellipse. Of course when $a = b = 1$, they coincide with the circular functions and dn $u = 1$. Given that there is also a dependence on a and b, this dependence should be taken into account in the definition and here is how it is done. We know that the eccentricity of the ellipse e is given by equation **??** that we rewrite here as

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

so indeed each elliptical function is not only a function of its argument u but on the ellipticity e. In the literature for the Jacobi functions the symbol k is used instead of e. We write

$$\mathrm{sn} = \mathrm{sn}(u,k), \qquad \mathrm{cn} = \mathrm{cn}(u,k), \qquad \mathrm{dn} = \mathrm{dn}(u,k)$$

The three inverses of these functions are defined as

$$\mathrm{ns}(u,k) = \frac{1}{\mathrm{sn}(u,k)}; \qquad \mathrm{nc}(u,k) = \frac{1}{\mathrm{cn}(u,k)}; \qquad \mathrm{nd}(u,k) = \frac{1}{\mathrm{dn}(u,k)}; \tag{2.19}$$

Now, as in the circular functions we consider quotients of the three basic functions with their inverses. This generates a total of $3 \times 3 = 9$ quotients. However from those 9 quotients three of them are equal to 1 so they are not interesting, the 6 remaining quotients are

$$\begin{aligned} \mathrm{sc}(u,k) &= \frac{\mathrm{sn}(u,k)}{\mathrm{cn}(u,k)} \qquad \mathrm{dc}(u,k) = \frac{\mathrm{dn}(u,k)}{\mathrm{cn}(u,k)} \qquad \mathrm{cs}(u,k) = \frac{\mathrm{cn}(u,k)}{\mathrm{sn}(u,k)} \\ \mathrm{ds}(u,k) &= \frac{\mathrm{dn}(u,k)}{\mathrm{sn}(u,k)} \qquad \mathrm{sd}(u,k) = \frac{\mathrm{sn}(u,k)}{\mathrm{dn}(u,k)} \qquad \mathrm{cd}(u,k) = \frac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)} \end{aligned} \tag{2.20}$$

For a total of 12 Jacobi functions. With so many functions, we want to have some kind of nemotechnic trick. We have four letters s,c,d, and n. We should permute them in groups of 2 with . This gives a total of

$$(4)_2 = 4 \times 3 = 12,$$

possible permutations. The primitive elliptic functions "sn" and "cn" should be obvious for its similarity with their circular counterparts sin and cos. The new primitive function "dn" comes because 'd' comes after 'c'. The inverse of these three primitive functions reverse the order on the letters, as defined in equation **??**. Now for the quotients; all 6 quotients should have the letter 'n' as the second index both in the numerator and the denominator. The name of the elliptical function comes from the first index on the numerator followed by the first index in the denominator. A quick look to equations **??** should make this argument clear. In general we can write the short hand notation

$$\mathrm{pq}(u,k) = \frac{\mathrm{pr}(u,k)}{\mathrm{qr}(u,k)}$$

where each p, q, and r is any of the letters s, c, d, and n, with the understanding that $ss = cc = dd = nn = 1$. According to Wikipedia [4] "this notation is due to Gudermann and Glaisher and is not Jacobi's original notation"

This first observation is that the identity

$$\mathrm{cn}^2(u,k) + \mathrm{sn}^2(u,k) = 1,$$

[4]http://en.wikipedia.org/wiki/Jacobi_elliptic_functions

which is the restatement of the ellipse **??**. This corresponds with the circular functions identity $\cos^2\varphi + \sin^2\varphi = 1$. Since $x^2 + y^2 = r^2$, then

$$\begin{aligned}
a^2\text{cn}^2(u,k) + b^2\,\text{sn}^2(u,k) &= r^2, \\
\text{cn}^2(u,k) + \frac{b^2}{a^2}\,\text{sn}^2(u,k) &= \frac{r^2}{a^2}, \\
\text{cn}^2(u,k) + (1-e^2)\,\text{sn}^2(u,k) &= \frac{r^2}{a^2},
\end{aligned}$$

this is

$$\text{cn}^2(u,k) + k^2\text{sn}^2(u,k) = \text{dn}^2(u,k).$$

In the case of a circle where the eccentricity is 0 and the radius is $r = a = b$, then this equation also reduces to the basic identity $\sin^2\varphi + \cos^2\varphi = 1$.

We now strive for the angle φ.

3 Theory

3.1 General Framework

An elliptic integral is any integral of the general form

$$f(x) = \int \frac{A(x) + B(x)}{C(x) + D(x)\sqrt{S(x)}}dx$$

where $A(x), B(x), C(x)$, and $D(x)$ are polynomials in x and $S(x)$ is a polynomial of degree 3 or 4.