13 Chapter

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Maxwell's Equations and Electromagnetic Waves

13.1 The Displacement Current

In Chapter 9, we learned that if a current-carrying wire possesses certain symmetry, the magnetic field could be obtained by using Ampere's law:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} \,. \qquad (13.1.1)$$

The equation states that the line integral of a magnetic field around an arbitrary closed loop is equal to $\mu_0 I_{\text{enc}}$, where I_{enc} is the flux of the charge carries passing through the surface bound by the closed path. In addition, we also learned in Chapter 10 that, as a consequence of Faraday's law of induction, an electric field is associated with a changing magnetic field, according to

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d}{dt}\iint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} \,. \qquad (13.1.2)$$

One might then wonder whether or not the converse could be true, namely, a magnetic field is associated with a changing electric field. If so, then the right-hand side of Eq. (13.1.1) will have to be modified to reflect such "symmetry" between electric and magnetic fields, $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$.

To see how magnetic fields are associated with a time-varying electric field, consider the process of charging a capacitor. During the charging process, the electric field strength increases with time as more charge is accumulated on the plates. The conduction current that carries the charges also produces a magnetic field. In order to apply Ampere's law to calculate this field, let us choose curve C_1 as the Amperian loop, shown in Figure 13.1.1.

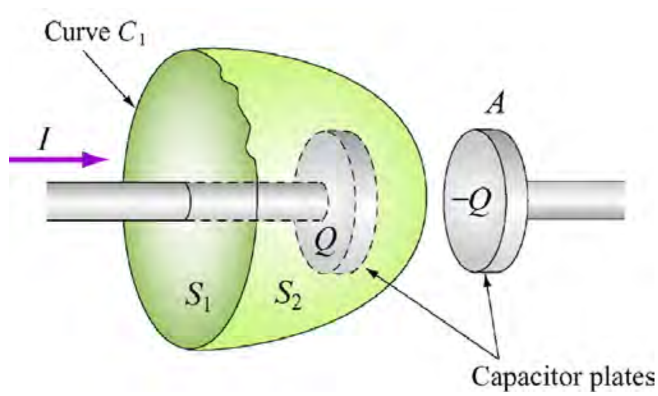


Figure 13.1.1 Surfaces S_1 and S_2 bound by curve C_1.

If the surface bounded by the path is the flat surface S_1, then the enclosed current is $I_{\text{enc}} = I$. On the other hand, if we choose S_2 to be the surface bounded by the curve, then $I_{\text{enc}} = 0$ because there is no current through S_2. Thus, we see that there exists an ambiguity in choosing the appropriate surface bounded by the curve C_1.

Maxwell resolved this ambiguity by adding to the right-hand side of Ampere's law an extra term

$$\mu_0 I_d = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}. \tag{13.1.3}$$

He called the quantity I_d the *displacement current*,

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt}. \tag{13.1.4}$$

The displacement current is proportional to the change in electric flux. The generalized Ampere's (or the Ampere-Maxwell) law now reads

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 (I + I_d). \tag{13.1.5}$$

The origin of the displacement current can be understood as follows:

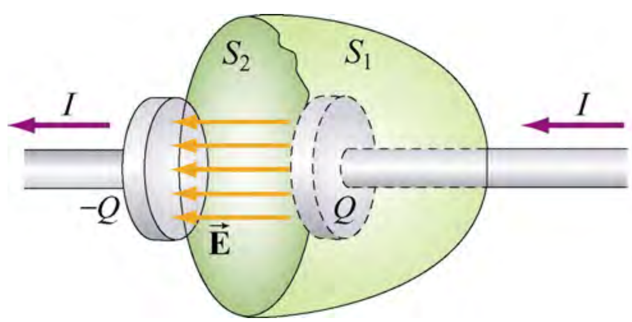


Figure 13.1.2 Displacement through S_2

In Figure 13.1.2, we have assumed that the electric field is uniform between the capacitor plates and zero outside. Recall that the magnitude of the electric field for such a configuration is $E = Q / A\varepsilon_0$, where A is the area of the capacitor plates. Then the electric flux through the surface S_2 is given by

$$\Phi_E = \oiint_{S_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = \frac{Q}{\varepsilon_0}. \tag{13.1.6}$$

From Eq. (13.1.4), the displacement current I_d is equal to the rate of increase of charge on the plate

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \frac{dQ}{dt}, \qquad (13.1.7)$$

which is equal to the conduction current, I. Thus, we conclude that the conduction current that passes through S_1 is precisely equal to the displacement current that passes through S_2, namely $I = I_d$. With the Ampere-Maxwell law, the ambiguity in choosing the surface bound by the Amperian loop is removed.

13.2 Gauss's Law for Magnetism

We have seen that Gauss's law for electrostatics states that the electric flux through a closed surface is proportional to the charge enclosed (Figure 13.2.1a). The electric field lines originate from the positive charge (source) and terminate at the negative charge (sink). One would then be tempted to write down the magnetic equivalent as

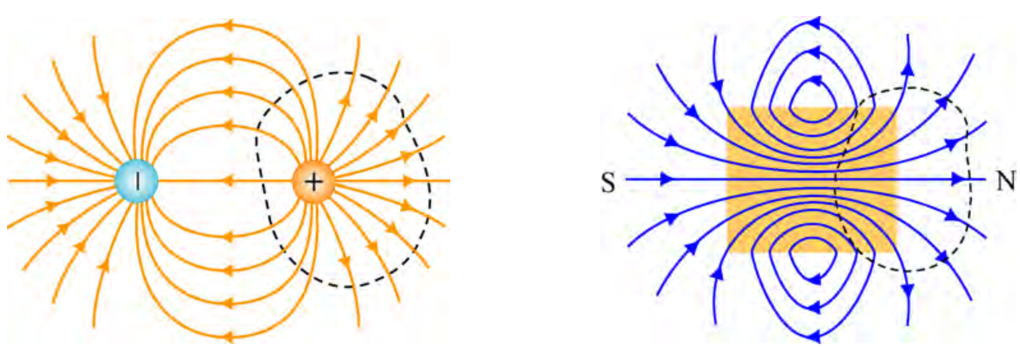
$$\Phi_B = \oint\!\!\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = \frac{Q_m}{\mu_0}, \qquad (13.2.1)$$

where Q_m is the magnetic charge (*magnetic monopole*) enclosed by the Gaussian surface S. However, despite intense experimental searches, no isolated magnetic monopole has ever been observed. Hence,

$$Q_m = 0, \qquad \text{(experimentally)}. \qquad (13.2.2)$$

Gauss's law for magnetism becomes

$$\boxed{\Phi_B = \oint\!\!\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0}. \qquad (13.2.3)$$



(a) (b)

Figure 13.2.1 Gauss's law for (a) electrostatics, and (b) magnetism.

This implies that the number of magnetic field lines entering a closed surface is equal to the number of field lines leaving that surface. There is no magnetic source or sink. In addition, the lines must be continuous with no starting or end points. In fact, as shown in Figure 13.2.1(b) for a bar magnet, the field lines that emanate from the north pole to the south pole outside the magnet return within the magnet and form closed loops.

13.3 Maxwell's Equations

We now have four equations that form the foundation of electromagnetic phenomena:

Law	Equation	Physical Interpretation
Gauss's law for $\vec{\mathbf{E}}$	$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{\varepsilon_0}$	Electric flux through a closed surface is proportional to the charge enclosed
Faraday's law	$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}$	Changing magnetic flux is associated with an electric field
Gauss's law for $\vec{\mathbf{B}}$	$\oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$	The total magnetic flux through a closed surface is zero
Ampere – Maxwell law	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$	Electric current and changing electric flux is associated with a magnetic field

Collectively they are known as **Maxwell's Equations**. The above equations may also be written in differential form as

$$\begin{aligned} \nabla \cdot \vec{\mathbf{E}} &= \frac{\rho}{\varepsilon_0}, \\ \nabla \times \vec{\mathbf{E}} &= -\frac{\partial \vec{\mathbf{B}}}{\partial t}, \\ \nabla \cdot \vec{\mathbf{B}} &= 0, \\ \nabla \times \vec{\mathbf{B}} &= \mu_0 \vec{\mathbf{J}} + \mu_0 \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}. \end{aligned} \tag{13.3.1}$$

where ρ and $\vec{\mathbf{J}}$ are the free charge and the conduction current densities, respectively.

In the absence of charged sources, $Q = 0$ and $I = 0$, the above integral equations become

$$
\begin{aligned}
&\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 0, \\
&\oint \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}, \\
&\oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0, \\
&\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}.
\end{aligned} \tag{13.3.2}
$$

An important consequence of Maxwell's equations, as we shall see below, is the prediction of the existence of electromagnetic waves that travel with the speed of light $c = 1/\sqrt{\mu_0\varepsilon_0}$. The reason is due to the fact that a changing electric field is associated with a magnetic field and vice versa, and the coupling between the two fields leads to the generation of electromagnetic waves. In 1887, H. Hertz confirmed this prediction.

13.4 Plane Traveling Electromagnetic Waves

To examine the properties of the electromagnetic waves, let's consider an electromagnetic wave propagating in the $+x$-direction, with a uniform electric field $\vec{\mathbf{E}}$ pointing in the $+y$-direction and a uniform magnetic field $\vec{\mathbf{B}}$ in the $+z$-direction. At any instant both $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are uniform over any yz-plane perpendicular to the direction of propagation. This means that for any value of x, the electric and magnetic fields are the same at all points yz-plane perpendicular to that value of x. The electric and magnetic field are independent of the (y,z) coordinates. Therefore the electric and magnetic fields are only functions of the (x,t) coordinates, $\vec{\mathbf{E}}(x,t) = E_y(x,t)\hat{\mathbf{j}}$ and $\vec{\mathbf{B}}(x,t) = B_z(x,t)\hat{\mathbf{k}}$.

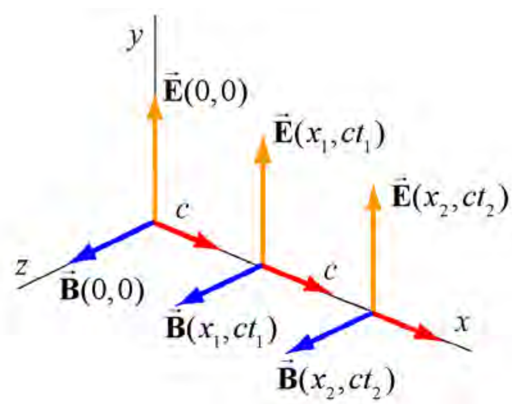


Figure 13.4.1 Electric and magnetic fields at a few selected points along the x-axis associated with a plane electromagnetic wave.

In the representation shown in Figure 13.4.1, at time $t=0$, consider the vectors $\vec{\mathbf{E}}(0,0)$ and $\vec{\mathbf{B}}(0,0)$, corresponding to the electric and magnetic fields on the plane $x=0$. We show two additional pairs of electric and magnetic vectors representing $\vec{\mathbf{E}}(x_1,ct_1)$ and $\vec{\mathbf{B}}(x_1,ct_1)$ on the plane $x_1=ct_1$, and $\vec{\mathbf{E}}(x_2,ct_2)$ and $\vec{\mathbf{B}}(x_2,ct_2)$ on the plane $x_2=ct_2$.

This (non-physical) electric and magnetic field is called a *plane wave* because at any instant both $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are uniform over any plane perpendicular to the direction of propagation. In addition, the wave is *transverse* because both fields are perpendicular to the direction of propagation, which points in the direction of the cross product $\vec{\mathbf{E}}\times\vec{\mathbf{B}}$.

Using Maxwell's equations, we may obtain the relationship between the magnitudes of the fields and their derivatives. To see this, consider a rectangular loop that lies in the xy-plane, with the left side of the loop at x and the right at $x+\Delta x$. The bottom side of the loop is located at y and the top of the loop is located at $y+\Delta y$ as shown in Figure 13.4.2. Let the unit vector normal to the loop be in the positive z-direction, $\hat{\mathbf{n}}=\hat{\mathbf{k}}$.

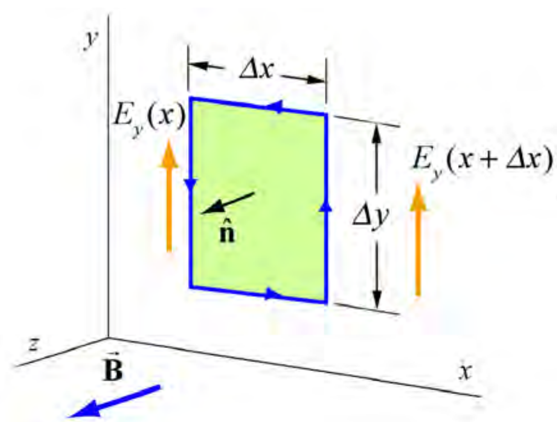


Figure 13.4.2 Spatial variation of the electric field $\vec{\mathbf{E}}$

Recall Faraday's law

$$\oint \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}} = -\frac{d}{dt}\iint \vec{\mathbf{B}}\cdot d\vec{\mathbf{A}}\,. \tag{13.4.1}$$

In order to evaluate the left-hand-side of Eq. (13.4.1), we integrate counterclockwise around the closed path shown in Figure 13.4.2,

$$\oint \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}} = E_y(x+\Delta x)\Delta y - E_y(x)\Delta y \tag{13.4.2}$$

We can use the Taylor expansion to approximate

$$E_y(x+\Delta x) = E_y(x) + \frac{\partial E_y}{\partial x}\Delta x + \cdots. \quad (13.4.3)$$

Then left-hand-side of Faraday's law becomes

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} \simeq \frac{\partial E_y}{\partial x}\Delta x.\Delta y\,. \quad (13.4.4)$$

We assume that Δx and Δy are very small such that the time derivative of the z-component of the magnetic field is nearly uniform over the area element. Then the rate of change of magnetic flux on the right-hand-side of Eq. (13.4.1) is given by

$$-\frac{d}{dt}\iint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = -\frac{\partial B_z}{\partial t}\Delta x\,\Delta y\,. \quad (13.4.5)$$

Equating the two sides of Faraday's Law and dividing through by the area $\Delta x \Delta y$ yields

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}\,. \quad (13.4.6)$$

Eq. (13.4.6) result indicates that at each point in space a time-varying magnetic field is associated with a spatially varying electric field.

The second condition on the relationship between the electric and magnetic fields may be deduced by using the Ampere-Maxwell equation:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \varepsilon_0 \frac{d}{dt}\iint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \quad (13.4.7)$$

Consider a rectangular loop in the xy-plane depicted in Figure 13.4.3, with a unit normal $\hat{\mathbf{n}} = \hat{\mathbf{j}}$.

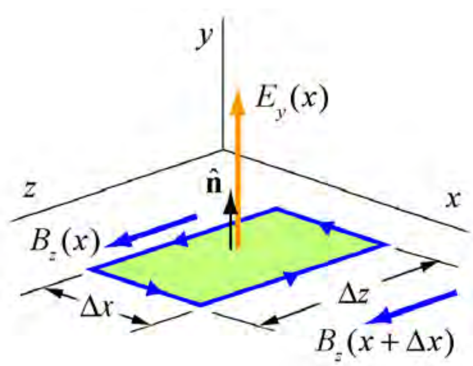


Figure 13.4.3 Spatial variation of the magnetic field $\vec{\mathbf{B}}$

We begin by evaluating the line integral of the magnetic field counterclockwise around the closed path shown in Figure 13.4.3,

$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = B_z(x)\Delta z - B_z(x+\Delta x)\Delta z\,. \tag{13.4.8}$$

We now use the Taylor expansion to approximate

$$B_z(x+\Delta x) = B_z(x) + \frac{\partial B_z}{\partial x}\Delta x + \cdots. \tag{13.4.9}$$

Then left-hand-side of the Maxwell-Ampere law then becomes

$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = -\frac{\partial B_z}{\partial x}\Delta x \Delta z\,. \tag{13.4.10}$$

We assume that Δx and Δz are very small such that the time derivative of the y-component of the electric field is nearly uniform over the area element. Then the rate of change of electric flux on the right-hand-side of Eq. (13.4.7) is given by

$$\mu_0\varepsilon_0 \frac{d}{dt}\iint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \mu_0\varepsilon_0 \frac{\partial E_y}{\partial t}\Delta x\,\Delta z\,. \tag{13.4.11}$$

Equating the two sides of the Maxwell-Ampere law and dividing by $\Delta x \Delta z$ yields

$$-\frac{\partial B_z}{\partial x} = \mu_0\varepsilon_0 \frac{\partial E_y}{\partial t}\,. \tag{13.4.12}$$

Eq. (13.4.12) result indicates that at each point in space a time-varying electric field is associated by a spatially varying magnetic field.

Eqs. (13.4.6) and (13.4.12) are coupled differential equations. To uncouple them, we first take another partial derivative of Eq. (13.4.6) with respect to x,

$$\frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial x}\right) \tag{13.4.13}$$

We have assumed that the field B_z is sufficiently well behaved such that the partial derivatives are interchangeable,

$$\frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial x}\right) \qquad (13.4.14)$$

Then substitute Eq. (13.4.12) into Eq. (13.4.13) yielding

$$\frac{\partial^2 E_y}{\partial x^2}=\mu_0\varepsilon_0\frac{\partial^2 E_y}{\partial t^2}. \qquad (13.4.15)$$

Eq. (13.4.15) is called the *one-dimensional wave equation, which* refers to the fact that the only spatial variation of our plane wave is in the x-direction. This equation involves second partial derivatives in space and in time. By a dimensional analysis the quantity $1/\mu_0\varepsilon_0$ has the same dimensions as speed squared.

We can repeat the argument to find a one-dimensional wave equation satisfied by the z-component of the magnetic field. We start by taking $\partial/\partial x$ of Eq. (13.4.12).

$$-\frac{\partial^2 B_z}{\partial x^2}=\mu_0\varepsilon_0\frac{\partial}{\partial x}\frac{\partial E_y}{\partial t}=\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\frac{\partial E_y}{\partial x}\right). \qquad (13.4.16)$$

Now we substitute Eq. (13.4.6) into Eq. (13.4.16), yielding a one-dimensional wave equation satisfied by the z-component of the magnetic field,

$$\frac{\partial^2 B_z}{\partial x^2}=\mu_0\varepsilon_0\frac{\partial^2 B_z}{\partial t^2}. \qquad (13.4.17)$$

The general form of a one-dimensional wave equation is given by

$$\frac{\partial^2\psi(x,t)}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2\psi(x,t)}{\partial t^2}, \qquad (13.4.18)$$

where v is the *speed of propagation* and $\psi(x,t)$ is the *wave function*. Both E_y and B_z satisfy the wave equation and propagate with speed

$$v=\frac{1}{\sqrt{\mu_0\varepsilon_0}}. \qquad (13.4.19)$$

We can rewrite Eq. (13.4.19) as

$$\varepsilon_0=\frac{1}{\mu_0 v^2}. \qquad (13.4.20)$$

Recall from (Eq. 2.2.3), that ε_0 is defined to be exactly equal to

$$\varepsilon_0 = \frac{1}{\mu_0 c^2}, \tag{13.4.21}$$

Comparing Eqs. (13.4.21) and (13.4.20), we conclude that the speed of propagation is exactly equal to the speed of light,

$$v = c = 2997092458 \text{ m}^1 \cdot \text{s}^{-1}. \tag{13.4.22}$$

Thus, we conclude that Maxwell's Equations predict that electric and magnetic field can propagate through space at the speed of light. The spectrum of electromagnetic waves is shown in Figure 13.4.4.

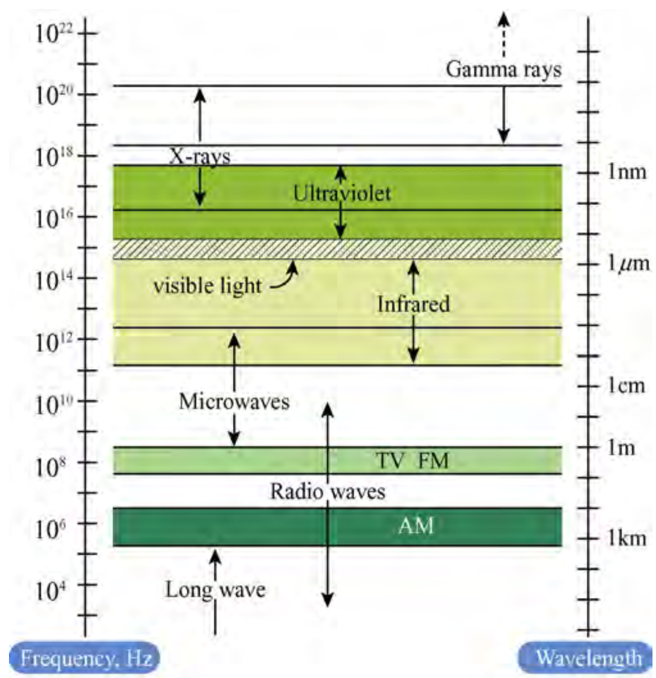


Figure 13.4.4 Electromagnetic spectrum

13.4.1 One-Dimensional Wave Equation

We shall now explore the properties of wave functions $\psi(x,t)$ that are solutions to the one-dimensional wave equation, Eq. (13.4.18). Many types of physical phenomena can be described by wave functions. We have already seen in the previous section that the plane transverse electric and magnetic fields, $E_y(x,t)$ and $B_z(x,t)$, propagating at the speed of light satisfy a one-dimensional wave equations. The transverse displacement of a stretched string, $y(x,t)$ will propagate along a string oriented along the x-direction at s speed dependent on the material properties and tension of the string. In each of these instances, the wave function is a function of both position and time. The propagation of the wave a constant speed v is a means of translating the wave function from a position and time, (x_1,t_1), to a position and time (x_2,t_2). We shall now explore wave functions that solve the one-dimensional wave equation, Eq. (13.4.18). We shall first begin with an example.

13.4.2 Gaussian One Dimensional Wave Pulse

Consider the function

$$y(x,t) = y_0 e^{-(x-vt)^2/a^2} \quad (13.4.23)$$

The variables (x,t) appear together as $x - vt$. At $t_1 = 0$,

$$y(x,0) = y_0 e^{-x^2/a^2} \quad (13.4.24)$$

In Figure 13.4.5(a) we show the position of the wave function given in Eq. (13.4.23) for three times, $t_1 = 0$, $t_2 = (2\text{ m})/v$, and $t_3 = (4\text{ m})/v$. The wave function is propagating in the positive x-direction. Figure 13.4.5(b) shows the propagation of a wave function given by

$$y(x,t) = y_0 e^{-(x+vt)^2/a^2} \quad (13.4.25)$$

When variables (x,t) appear together as $x + vt$, as in

$$y(x,t) = y_0 e^{-(x+vt)^2/a^2} \quad (13.4.26)$$

The wave function is propagating in the negative x-direction. Figure 13.4.5(b) shows the propagation of the wave function given in Eq. (13.4.26).

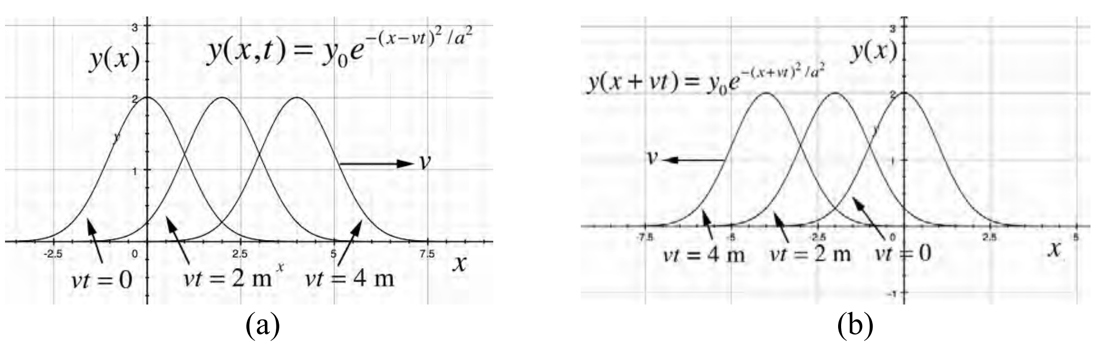


Figure 13.4.5 (a) Wave function propagating in the (a) positive x-direction, (b) negative x-direction.

Let's show by direct computation that Eq. (13.4.23) satisfies the wave equation,

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}. \tag{13.4.27}$$

The left-hand-side of Eq. (13.4.27) is

$$\begin{aligned}\frac{\partial^2 y(x,t)}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} y_0 e^{-(x-vt)^2/a^2}\right) = \frac{\partial}{\partial x}\left((-2(x-vt)/a^2) y_0 e^{-(x-vt)^2/a^2}\right) \\ &= (-2/a^2) y_0 e^{-(x-vt)^2/a^2} + (-2(x-vt)/a^2)^2 y_0 e^{-(x-vt)^2/a^2}\end{aligned}. \tag{13.4.28}$$

The right-hand-side of Eq. (13.4.27) is

$$\begin{aligned}\frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} &= \frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} y_0 e^{-(x-vt)^2/a^2}\right) = \frac{1}{v^2}\frac{\partial}{\partial t}\left((2v(x-vt)/a^2) y_0 e^{-(x-vt)^2/a^2}\right) \\ &= \frac{1}{v^2}\left((-2v^2/a^2) y_0 e^{-(x-vt)^2/a^2} + (2v(x-vt)/a^2)^2 y_0 e^{-(x-vt)^2/a^2}\right) \\ &= (-2/a^2) y_0 e^{-(x-vt)^2/a^2} + (2(x-vt)/a^2)^2 y_0 e^{-(x-vt)^2/a^2}\end{aligned}. \tag{13.4.29}$$

A comparison of Eqs. (13.4.28) and (13.4.29) shows that our wave function (Eq. (13.4.23) is a solution to the one-dimensional wave equation (Eq. (13.4.27)).

Example 13.4.2 illustrates the general property that any function of the form $\psi(x+vt)$ or $\psi(x-vt)$ satisfies the one-dimensional wave equation, Eq. (13.4.18). We shall now demonstrate this property. Let $x' = x \pm vt$. The space and time partial derivatives are

$$\partial x' / \partial x = 1, \tag{13.4.30}$$

$$\partial x' / \partial t = \pm v. \tag{13.4.31}$$

Using the chain rule, the first partial derivatives with respect to x is

$$\frac{\partial \psi(x')}{\partial x} = \frac{\partial \psi}{\partial x'}\frac{\partial x'}{\partial x}. \tag{13.4.32}$$

Substitute Eq. (13.4.30) into Eq. (13.4.32) yielding

$$\frac{\partial \psi(x')}{\partial x} = \frac{\partial \psi}{\partial x'}. \tag{13.4.33}$$

We now repeat this calculation for the second partial derivative,

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x'}\right) = \frac{\partial^2\psi}{\partial x'^2}\frac{\partial x'}{\partial x} = \frac{\partial^2\psi}{\partial x'^2}. \qquad (13.4.34)$$

Similarly, the partial derivatives in t are given by

$$\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial x'}\frac{\partial x'}{\partial t} = \pm v\frac{\partial\psi}{\partial x'}, \qquad (13.4.35)$$

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial}{\partial t}\left(\pm v\frac{\partial\psi}{\partial x'}\right) = \pm v\frac{\partial^2\psi}{\partial x'^2}\frac{\partial x'}{\partial t} = v^2\frac{\partial^2\psi}{\partial x'^2}. \qquad (13.4.36)$$

Comparing Eq. (13.4.34) with Eq. (13.4.36) results in

$$\frac{\partial^2\psi}{\partial x'^2} = \frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}, \qquad (13.4.37)$$

which shows that $\psi(x \pm vt)$ satisfies the one-dimensional wave equation.

The wave equation is an example of a linear differential equation, which means that if $\psi_1(x,t)$ and $\psi_2(x,t)$ are solutions to the wave equation, then $\psi_1(x,t) \pm \psi_2(x,t)$ is also a solution.

Let's now return to our electromagnetic fields. One possible solution to the one-dimensional wave equations for the electric and magnetic field is

$$\begin{aligned} \vec{\mathbf{E}} &= E_y(x,t)\hat{\mathbf{j}} = E_0 \sin\left(\frac{2\pi}{\lambda}(x-ct)\right)\hat{\mathbf{j}}, \\ \vec{\mathbf{B}} &= B_z(x,t)\hat{\mathbf{k}} = B_0 \sin\left(\frac{2\pi}{\lambda}(x-ct)\right)\hat{\mathbf{k}}, \end{aligned} \qquad (13.4.38)$$

where the fields are sinusoidal, with amplitudes E_0 and B_0.

The electric field, $E_y(x,t=0)$, at time $t=0$, as a function of position x is given by the expression

$$E_y(x,t=0) = E_0 \sin\left(\frac{2\pi}{\lambda}x\right). \qquad (13.4.39)$$

In Figure 13.4.6, we show a plot $E_y(x,t=0)$ as a function of x. On the plane $x=0$, the electric field is zero, $E_y(0,0)=0$. On the plane $x=\lambda$, the electric field has completed one spatial oscillation, $E_y(\lambda,0) = E_0 \sin(2\pi) = 0$. The distance λ is called the *wavelength.*

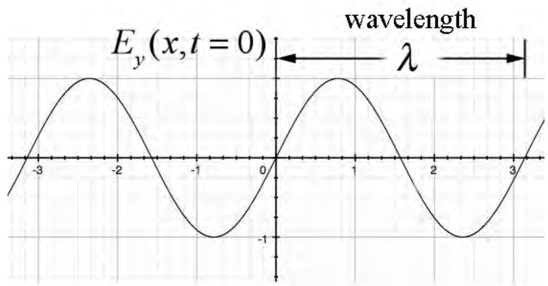


Figure 13.4.6 Wavelength

The *wave number* a is defined as the inverse of the wavelength,

$$a \equiv \frac{1}{\lambda}. \tag{13.4.40}$$

The (*angular*) *wave number* k is defined by

$$k \equiv 2\pi a = \frac{2\pi}{\lambda}, \tag{13.4.41}$$

and so has SI units $[\text{rad} \cdot \text{m}^{-1}]$. In practice, k is usually called the wave number and the symbol a and terminology have disappeared from use.

The electric field, $E_y(x=0,t)$, on the plane $x=0$, as a function of time t, is given by the expression

$$E_y(x=0,t) = E_0 \sin\left(-\frac{2\pi}{\lambda}ct\right) = -E_0 \sin\left(\frac{2\pi}{\lambda}ct\right). \tag{13.4.42}$$

In Figure 13.4.7, we show a plot of the electric field, $E_y(x=0,t)$, as a function of time t.

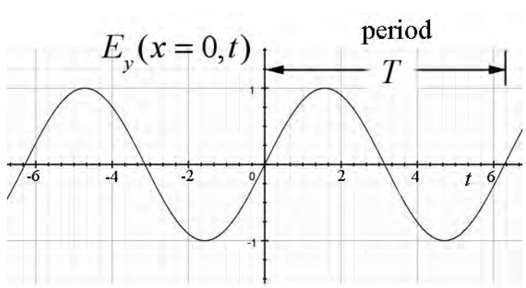


Figure 13.4.7 Period

At time $t = 0$, $E_y(0,0) = 0$. Let T be the time that it takes the electric field to complete one oscillation on the plane $x = 0$. Then $E_y(0,T) = -E_0 \sin(2\pi c T / \lambda) = 0$. This condition is satisfied when

$$T = \frac{\lambda}{c}, \tag{13.4.43}$$

then $E_y(0,T) = -E_0 \sin(2\pi) = 0$. The time T is called the *period.* The *frequency* f is defined as the inverse of the period,

$$f \equiv \frac{1}{T}. \tag{13.4.44}$$

The *angular frequency* ω is defined by

$$\omega \equiv 2\pi f = \frac{2\pi}{T} = \frac{2\pi c}{\lambda}. \tag{13.4.45}$$

The SI units of angular frequency are $[\text{rad} \cdot \text{s}^{-1}]$.

With these definitions we can rewrite our electric and magnetic fields as

$$\begin{aligned} \vec{\mathbf{E}} &= E_y(x,t)\hat{\mathbf{j}} = E_0 \sin(kx - \omega t)\hat{\mathbf{j}} \\ \vec{\mathbf{B}} &= B_z(x,t)\hat{\mathbf{k}} = B_0 \sin(kx - \omega t)\hat{\mathbf{k}} \end{aligned}. \tag{13.4.46}$$

In empty space, the wave propagates at the speed of light c. The characteristic behavior of the sinusoidal electromagnetic wave is illustrated in Figure 13.4.8.

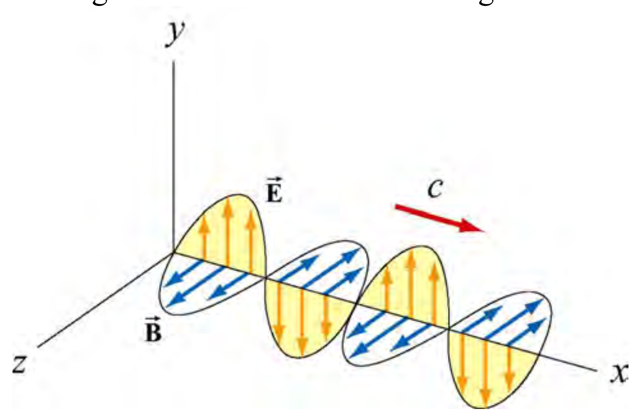


Figure 13.4.8 Plane electromagnetic wave propagating in the $+x$-direction.

We see that the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields are always in phase (attaining maxima and minima at the same time.) To obtain the relationship between the field amplitudes E_0 and B_0, we make use of Eqs. (13.4.6) and (13.4.12). We first calculate the following the partial derivatives

$$\frac{\partial E_y}{\partial x} = kE_0 \cos(kx - \omega t), \tag{13.4.47}$$

$$\frac{\partial B_z}{\partial t} = -\omega B_0 \cos(kx - \omega t), \tag{13.4.48}$$

Eq. (13.4.6) then implies that $E_0 k = \omega B_0$, which we rewrite as

$$\boxed{\frac{E_0}{B_0} = \frac{\omega}{k} = c}. \tag{13.4.49}$$

Because the sinusoidal behavior of the fields are the same, the magnitudes of the fields at any instant are related by

$$\frac{E}{B} = c. \tag{13.4.50}$$

Let us summarize the important features of electromagnetic waves described in Eq. (13.4.38):

1. The wave is transverse since both $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields are perpendicular to the direction of propagation, which points in the direction of the cross product $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$.

2. The $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields are perpendicular to each other. Therefore, their dot product vanishes, $\vec{\mathbf{E}} \cdot \vec{\mathbf{B}} = 0$.

3. The speed of propagation in vacuum is equal to the speed of light, and

$$c = 1/\sqrt{\mu_0 \varepsilon_0}.$$

4. The ratio of the magnitudes and the amplitudes of the fields is

$$c = \frac{E}{B} = \frac{E_0}{B_0} = \frac{\omega}{k} = \frac{\lambda}{T}.$$

13.5 Standing Electromagnetic Waves

Let us examine the situation where there are two sinusoidal plane electromagnetic waves, one traveling in the positive x-direction, with

$$E_{1y}(x,t) = E_{10} \sin(k_1 x - \omega_1 t), \quad B_{1z}(x,t) = B_{10} \sin(k_1 x - \omega_1 t) \tag{13.5.1}$$

and the other traveling in the negative x-direction, with

$$E_{2y}(x,t) = +E_{20}\sin(k_2 x + \omega_2 t), \quad B_{2z}(x,t) = -B_{20}\sin(k_2 x + \omega_2 t). \tag{13.5.2}$$

For simplicity, we assume that these electromagnetic waves have the same amplitudes, $E_0 \equiv E_{10} = E_{20}$ and $B_0 \equiv B_{10} = B_{20}$, and wavelengths, $\lambda \equiv \lambda_1 = \lambda_2$. Using the superposition principle, the electric field and the magnetic fields can be written as

$$E_y(x,t) = E_{1y}(x,t) + E_{2y}(x,t) = E_0[\sin(kx - \omega t) + \sin(kx + \omega t)], \tag{13.5.3}$$

$$B_z(x,t) = B_{1z}(x,t) + B_{2z}(x,t) = B_0[\sin(kx - \omega t) - \sin(kx + \omega t)]. \tag{13.5.4}$$

Using the identities

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{13.5.5}$$

The above expressions may be rewritten as

$$\begin{aligned} E_y(x,t) &= E_0[(\sin(kx)\cos(\omega t) - \cos(kx)\sin(\omega t)) + (\sin(kx)\cos(\omega t) + \cos(kx)\sin(\omega t))] \\ &= 2E_0\sin(kx)\cos(\omega t), \end{aligned} \tag{13.5.6}$$

and

$$\begin{aligned} B_z(x,t) &= B_0[(\sin(kx)\cos(\omega t) + \cos(kx)\sin(\omega t)) - (\sin(kx)\cos(\omega t) - \cos(kx)\sin(\omega t))] \\ &= 2B_0\cos(kx)\sin(\omega t), \end{aligned} \tag{13.5.7}$$

One may verify that the total fields $E_y(x,t)$ and $B_z(x,t)$ still satisfy the wave equation stated in Eqs. (13.4.15) and (13.4.17) even though they no longer have the form of functions of $k(x \pm ct) = (kx \pm \omega t)$. The waves described by Eqs. (13.5.6) and (13.5.7) are *standing waves*, which do not propagate but simply oscillate independently in space and time.

Let's first examine the spatial dependence of the fields. Eq. (13.5.6) shows that the total electric field remains zero at all times if $\sin(kx) = 0$. This occurs on the planes

$$x = \frac{n\pi}{k} = \frac{n\pi}{2\pi/\lambda} = \frac{n\lambda}{2}, \quad n = 0,1,2,\ldots \text{ (nodal planes of } \vec{\mathbf{E}}). \tag{13.5.8}$$

The planes that contain these points are called the *nodal planes* of the electric field. On the planes where $\sin(kx) = \pm 1$,

$$x = \left(n + \frac{1}{2}\right)\frac{\pi}{k} = \left(n + \frac{1}{2}\right)\frac{\pi}{2\pi/\lambda} = \left(\frac{n}{2} + \frac{1}{4}\right)\lambda, \quad n = 0,1,2,\ldots \text{ (anti-nodal planes of } \vec{\mathbf{E}}),$$

(13.5.9)

the amplitude of the field is at its maximum $2E_0$. The planes that contain these points are the *anti-nodal planes* of the electric field. Note that in between two nodal planes, there is an anti-nodal plane, and vice versa.

For the magnetic field, the nodal planes must contain points that meet the condition $\cos(kx)=0$. This occurs on the planes

$$x=\left(n+\frac{1}{2}\right)\frac{\pi}{k}=\left(\frac{n}{2}+\frac{1}{4}\right)\lambda,\quad n=0,1,2,\ldots \text{ (nodal planes of } \vec{\mathbf{B}}\text{)}. \tag{13.5.10}$$

Similarly, the anti-nodal planes for $\vec{\mathbf{B}}$ contain points that satisfy $\cos(kx)=\pm 1$, which occurs on the planes

$$x=\frac{n\pi}{k}=\frac{n\pi}{2\pi/\lambda}=\frac{n\lambda}{2},\quad n=0,1,2,\ldots \text{ (anti-nodal planes of } \vec{\mathbf{B}}\text{)}. \tag{13.5.11}$$

Thus, we see that a nodal plane of $\vec{\mathbf{E}}$ corresponds to an anti-nodal plane of $\vec{\mathbf{B}}$, and vice versa.

For the time dependence, Eq. (13.5.6) shows that the electric field is zero everywhere when $\cos(\omega t)=0$. This occurs at the times

$$t=\frac{(2n+1)\pi}{2\omega}=\frac{(2n+1)\pi}{4\pi/T}=\frac{(2n+1)T}{4},\quad n=0,1,2,\ldots \tag{13.5.12}$$

where $T=1/f=2\pi/\omega$ is the period. However, this is precisely the maximum condition for the magnetic field, i.e. when $\sin(\omega t)=1$. Thus, unlike the traveling electromagnetic wave in which the electric and the magnetic fields are always in phase, in standing electromagnetic waves, the two fields are $90°$ out of phase.

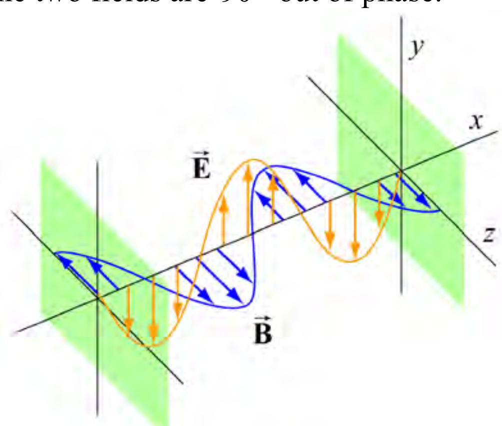


Figure 13.4.9 Formation of standing electromagnetic waves using two perfectly reflecting conductors.

Standing electromagnetic waves can be formed by confining the electromagnetic waves within two perfectly reflecting conductors, as shown in Figure 13.4.8.

13.6 Poynting Vector

In Chapters 5 and 11 we had seen that electric and magnetic fields store energy. Energy can also be transported by electromagnetic waves that consist of both fields. Consider a plane electromagnetic wave passing through a small volume element of area A and thickness dx, as shown in Figure 13.6.1.

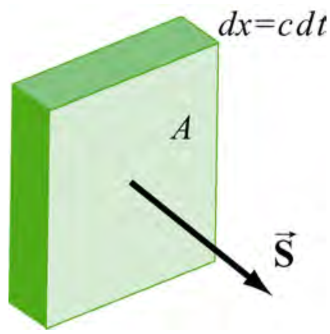


Figure 13.6.1 Electromagnetic wave passing through a volume element

The total energy stored in the electromagnetic fields in the volume element is given by

$$dU = uA\,dx = (u_E + u_B)A\,dx = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)A\,dx\,, \tag{13.6.1}$$

where

$$u_E = \frac{1}{2}\varepsilon_0 E^2, \qquad u_B = \frac{B^2}{2\mu_0}\,. \tag{13.6.2}$$

are the energy densities associated with the electric and magnetic fields. Because the electromagnetic wave propagates with the speed of light c, the amount of time it takes for the wave to move through the volume element is $dt = dx / c$. Thus, one may obtain the rate of change of energy per unit area, denoted by the symbol S, as

$$S = \frac{dU}{A\,dt} = \frac{c}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right). \tag{13.6.3}$$

The SI unit of S is $[\mathrm{W\cdot m^{-2}}]$. Recall that the magnitude of the fields satisfy $E = cB$ and $c = 1/\sqrt{\mu_0\varepsilon_0}$. Therefore Eq. (13.6.3) may be rewritten as

$$S=\frac{c}{2}\left(\varepsilon_0 E^2+\frac{B^2}{\mu_0}\right)=\frac{cB^2}{\mu_0}=c\varepsilon_0 E^2=\frac{EB}{\mu_0}. \tag{13.6.4}$$

We can turn this energy flow into a vector by assigning the direction as the direction of propagation. The rate of energy flow per unit area is called the *Poynting vector* $\vec{\mathbf{S}}$ (after the British physicist John Poynting), and defined by the vector product

$$\vec{\mathbf{S}}=\frac{1}{\mu_0}\vec{\mathbf{E}}\times\vec{\mathbf{B}}. \tag{13.6.5}$$

For our plane transverse electromagnetic waves, the fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are perpendicular, and the magnitude of $\vec{\mathbf{S}}$ is

$$\left|\vec{\mathbf{S}}\right|=\frac{\left|\vec{\mathbf{E}}\times\vec{\mathbf{B}}\right|}{\mu_0}=\frac{EB}{\mu_0}=S. \tag{13.6.6}$$

As an example, suppose the electric field associated with a plane sinusoidal electromagnetic wave is $\vec{\mathbf{E}}=E_0\cos(kx-\omega t)\hat{\mathbf{j}}$. The corresponding magnetic field is $\vec{\mathbf{B}}=B_0\cos(kx-\omega t)\hat{\mathbf{k}}$, and the direction of propagation is the positive x-direction. The Poynting vector is then

$$\vec{\mathbf{S}}=\frac{1}{\mu_0}(E_0\cos(kx-\omega t)\hat{\mathbf{j}})\times(B_0\cos(kx-\omega t)\hat{\mathbf{k}})=\frac{E_0B_0}{\mu_0}\cos^2(kx-\omega t)\hat{\mathbf{i}} \tag{13.6.7}$$

As expected, $\vec{\mathbf{S}}$ points in the direction of wave propagation (Figure 13.6.2).

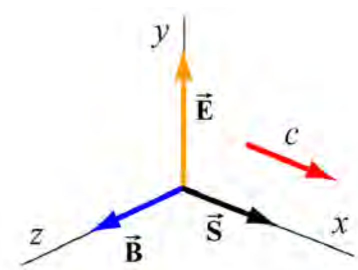


Figure 13.6.2 Electric and magnetic fields, and the Poynting vector for a plane wave on the plane $x=0$.

The *intensity* of the wave, I, is defined as the time-average of S, and is given by

$$I=\langle S\rangle=\frac{E_0B_0}{\mu_0}\langle\cos^2(kx-\omega t)\rangle=\frac{E_0B_0}{2\mu_0}=\frac{E_0^2}{2c\mu_0}=\frac{cB_0^2}{2\mu_0}, \tag{13.6.8}$$

where we have used the time average

$$\left\langle \cos^2(kx-\omega t)\right\rangle = \frac{1}{2}\,. \tag{13.6.9}$$

To relate intensity to the energy density, we first note the equality between the electric and the magnetic energy densities

$$u_B = \frac{B^2}{2\mu_0} = \frac{(E/c)^2}{2\mu_0} = \frac{E^2}{2c^2\mu_0} = \frac{\varepsilon_0 E^2}{2} = u_E\,. \tag{13.6.10}$$

The time-averaged energy density of the wave is then

$$\begin{aligned}\left\langle u\right\rangle = \left\langle u_E + u_B\right\rangle = \varepsilon_0\left\langle E^2\right\rangle &= \frac{\varepsilon_0}{2}E_0^2\\ = \frac{1}{\mu_0}\left\langle B^2\right\rangle &= \frac{B_0^2}{2\mu_0}.\end{aligned} \tag{13.6.11}$$

Thus, comparing Eqs. (13.6.8) and (13.6.11), we can conclude that the intensity is related to the average energy density by

$$I = \left\langle S\right\rangle = c\left\langle u\right\rangle\,. \tag{13.6.12}$$

Example 13.1: Solar Constant

At the upper surface of the Earth's atmosphere, the time-averaged magnitude of the Poynting vector, $\left\langle S\right\rangle = 1.35\times10^3\ \text{W/m}^2$, is referred to as the *solar constant*.

(a) Assuming that the Sun's electromagnetic radiation is a plane sinusoidal wave, what are the magnitudes of the electric and magnetic fields?

(b) What is the total time-averaged power radiated by the Sun? The mean Sun-Earth distance is $R = 1.50\times10^{11}\,\text{m}$.

Solution:

(a) The time-averaged Poynting vector is related to the amplitude of the electric field by

$$\left\langle S\right\rangle = \frac{c}{2}\varepsilon_0 E_0^2\,.$$

Thus, the amplitude of the electric field is

$$E_0 = \sqrt{\frac{2\langle S\rangle}{c\varepsilon_0}} = \sqrt{\frac{2(1.35\times 10^3\ \mathrm{W/m^2})}{(3.0\times 10^8\ \mathrm{m/s})(8.85\times 10^{-12}\ \mathrm{C^2/N\cdot m^2})}} = 1.01\times 10^3\ \mathrm{V/m}.$$

The corresponding amplitude of the magnetic field is

$$B_0 = \frac{E_0}{c} = \frac{1.01\times 10^3\ \mathrm{V/m}}{3.0\times 10^8\ \mathrm{m/s}} = 3.4\times 10^{-6}\ \mathrm{T}.$$

The associated magnetic field is less than one-tenth the Earth's magnetic field.

(b) The total time averaged power radiated by the Sun at the distance R is

$$\langle P\rangle = \langle S\rangle A = \langle S\rangle 4\pi R^2 = (1.35\times 10^3\ \mathrm{W/m^2})4\pi(1.50\times 10^{11}\mathrm{m})^2 = 3.8\times 10^{26}\ \mathrm{W}.$$

The type of wave discussed in the Example 13.1 is a spherical wave (Figure 13.6.3a), which originates from a "point-like" source. The intensity at a distance r from the source is

$$I = \langle S\rangle = \frac{\langle P\rangle}{4\pi r^2}, \tag{13.6.13}$$

which decreases as $1/r^2$. The intensity of a plane wave (Figure 13.6.3b) remains constant and there is no spreading of its energy.

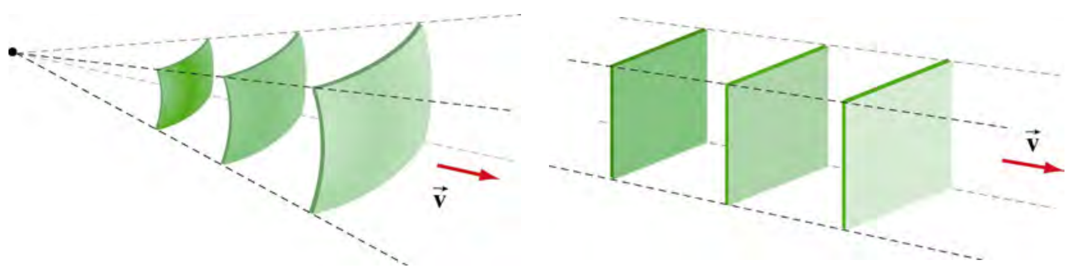


Figure 13.6.3 (a) a spherical wave, and (b) plane wave.

Example 13.2: Intensity of a Standing Wave

Compute the intensity of the standing electromagnetic wave given by Eqs. (13.5.6) and (13.5.7)

$$E_y(x,t) = 2E_0\sin(kx)\cos(\omega t),\quad B_z(x,t) = 2B_0\cos(kx)\sin(\omega t).$$

Solution: The Poynting vector for the standing wave is

$$\begin{aligned}\vec{\mathbf{S}} = \frac{\vec{\mathbf{E}}\times\vec{\mathbf{B}}}{\mu_0} &= \frac{1}{\mu_0}(2E_0\sin(kx)\cos(\omega t)\,\hat{\mathbf{j}})\times(2B_0\cos(kx)\sin(\omega t)\,\hat{\mathbf{k}}) \\ &= \frac{4E_0B_0}{\mu_0}(\sin(kx)\cos(kx)\sin(\omega t)\cos(\omega t))\hat{\mathbf{i}} \\ &= \frac{E_0B_0}{\mu_0}(\sin(2kx)\sin(2\omega t))\hat{\mathbf{i}}.\end{aligned} \tag{13.6.14}$$

The time average of S is

$$\langle S \rangle = \frac{E_0B_0}{\mu_0}\sin(2kx)\langle \sin(2\omega t) \rangle = 0 \tag{13.6.15}$$

because $\langle \sin(2\omega t) \rangle = 0$. The result is to be expected since the standing wave does not propagate. Alternatively, we may say that the energy carried by the two waves traveling in the opposite directions that form the standing wave exactly cancels each other, resulting in no energy flow.

13.6.1 Energy Transport

The Poynting vector $\vec{\mathbf{S}}$ represents the rate of the energy flow per unit area, therefore the rate of change of energy in a closed volume can be written as a surface integral on the surface enclosing the volume

$$\boxed{\frac{dU}{dt} = -\oint\!\!\oint \vec{\mathbf{S}}\cdot d\vec{\mathbf{A}}}, \tag{13.6.16}$$

where $d\vec{\mathbf{A}} = dA\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector in the *outward* normal direction. The above expression allows us to interpret $\vec{\mathbf{S}}$ as the *power density*, in analogy to the current density $\vec{\mathbf{J}}$ in the expression for the flow of charge through a surface

$$I = \frac{dQ}{dt} = \iint \vec{\mathbf{J}}\cdot d\vec{\mathbf{A}}\,. \tag{13.6.17}$$

If energy flows out of the volume, then $\vec{\mathbf{S}} = S\hat{\mathbf{n}}$ and $dU/dt < 0$, showing an overall decrease of energy in the volume. On the other hand, if energy flows into the volume, then $\vec{\mathbf{S}} = S(-\hat{\mathbf{n}})$ and $dU/dt > 0$, indicating an overall increase of energy in the volume.

13.6.2 Power flow in a Solenoid with Changing Current

As an example to elucidate the physical meaning of the above equation, let's consider an inductor made up of a section of a very long air-core solenoid of length l, radius r and n turns per unit length. Suppose at some instant the current is changing at a rate $dI/dt > 0$. Using Ampere's law, the magnetic field in the solenoid is

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = Bl = \mu_0 (NI) .$$

Therefore the magnetic field is

$$\vec{\mathbf{B}} = \mu_0 n I \,\hat{\mathbf{k}} . \qquad (13.6.18)$$

Thus, the rate of increase of the magnitude of the magnetic field is

$$\frac{dB}{dt} = \mu_0 n \frac{dI}{dt} . \qquad (13.6.19)$$

According to Faraday's law, changing magnetic flux results in an induced electric field

$$\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt} . \qquad (13.6.20)$$

Calculating and equating the two sides of Faraday's Law yields

$$E 2\pi r = -\mu_0 n \frac{dI}{dt} \pi r^2 .$$

The electric field is therefore

$$\vec{\mathbf{E}} = -\frac{\mu_0 n r}{2} \frac{dI}{dt} \hat{\boldsymbol{\theta}} . \qquad (13.6.21)$$

The direction of $\vec{\mathbf{E}}$ is clockwise the same as the induced current, ($\hat{\boldsymbol{\theta}}$ is a unit vector tangent to the coils with direction as shown in Figure 13.6.4.)

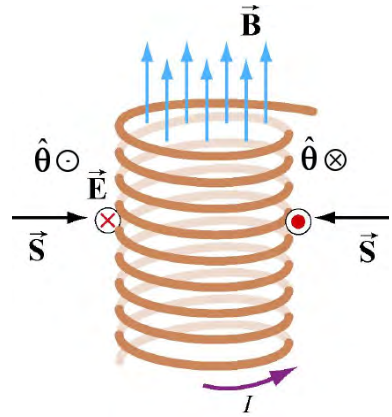


Figure 13.6.4 Poynting vector for a solenoid with $dI/dt > 0$

The corresponding Poynting vector can then be obtained as

$$\vec{\mathbf{S}} = \frac{\vec{\mathbf{E}} \times \vec{\mathbf{B}}}{\mu_0} = \frac{1}{\mu_0}\left(-\frac{\mu_0 n r}{2}\frac{dI}{dt}\hat{\boldsymbol{\theta}}\right) \times \mu_0 n I\,\hat{\mathbf{k}} = -\frac{\mu_0 n^2 r I}{2}\frac{dI}{dt}\hat{\mathbf{r}}, \qquad (13.6.22)$$

which points radially inward, in the negative $\hat{\mathbf{r}}$-direction. The directions of the fields and the Poynting vector are shown in Figure 13.6.4. The magnetic energy stored in the inductor is

$$U_B = \frac{B^2}{2\mu_0}\pi r^2 l = \frac{1}{2}\mu_0 \pi n^2 I^2 r^2 l\,. \qquad (13.6.23)$$

Therefore the rate of change of U_B is given by

$$\frac{dU_B}{dt} = \mu_0 \pi n^2 I r^2 l \frac{dI}{dt} = I\,|\varepsilon| \qquad (13.6.24)$$

where

$$\varepsilon = -N\frac{d\Phi_B}{dt} = -nl\frac{dB}{dt}\pi r^2 = -\mu_0 n^2 l \pi r^2 \frac{dI}{dt} \qquad (13.6.25)$$

is the induced emf. One may readily verify that this is the same as

$$-\oint\!\!\oint \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}} = \frac{\mu_0 n^2 r I}{2}\frac{dI}{dt} 2\pi r l = \mu_0 \pi n^2 I r^2 l \frac{dI}{dt}\,. \qquad (13.6.26)$$

Thus, we have

$$\frac{dU_B}{dt} = -\oint\!\!\oint \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}} > 0\,. \qquad (13.6.27)$$

When $dI/dt > 0$, the energy inside the solenoid increases, as expected. When $dI/dt < 0$, the energy inside the solenoid decreases, with $dU_B/dt < 0$.

13.7 Momentum and Radiation Pressure

An electromagnetic wave transports not only energy but also momentum, and hence can exert a *radiation pressure* on a surface due to the absorption and reflection of the momentum. When a plane electromagnetic wave is completely absorbed by a surface, the momentum transferred is related to the energy absorbed by

$$\Delta p = \frac{\Delta U}{c} \quad \text{(complete absorption)}. \tag{13.7.1}$$

(We shall not prove this result as it involves a more complicated description of energy and momentum stored in electromagnetic fields.)

If the electromagnetic wave is completely reflected by a surface such as a mirror, the result becomes

$$\Delta p = \frac{2\Delta U}{c} \quad \text{(complete reflection)}. \tag{13.7.2}$$

For a wave that is completely absorbed, the time-averaged radiation pressure (force per unit area) is given by

$$P = \frac{\langle F \rangle}{A} = \frac{1}{A}\left\langle \frac{dp}{dt} \right\rangle = \frac{1}{Ac}\left\langle \frac{dU}{dt} \right\rangle. \tag{13.7.3}$$

Because the time-averaged rate that energy delivered to the surface is

$$\left\langle \frac{dU}{dt} \right\rangle = \langle S \rangle A, \tag{13.7.4}$$

Substitute Eq. (13.7.4) into Eq. (13.7.3) yielding

$$P = \frac{\langle S \rangle}{c} \quad \text{(complete absorption)}. \tag{13.7.5}$$

Similarly, if the radiation is completely reflected, the radiation pressure is twice as great as the case of complete absorption:

$$P = \frac{2\langle S \rangle}{c} \quad \text{(complete reflection)}. \tag{13.7.6}$$

13.8 Production of Electromagnetic Waves

Electromagnetic waves are produced when electric charges are accelerated. Stationary charges or steady currents on not radiate electromagnetic waves. Figure 13.8.1 depicts the electric field lines produced by an oscillating charge at some instant.

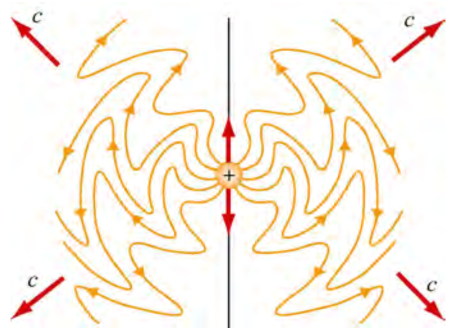


Figure 13.8.1 Electric field lines of an oscillating point charge

A common way of producing electromagnetic waves is to apply a sinusoidal voltage source to an antenna, causing the charges to accumulate near the tips of the antenna. The effect is to produce an oscillating electric dipole. The production of electric-dipole radiation is depicted in Figure 13.8.2.

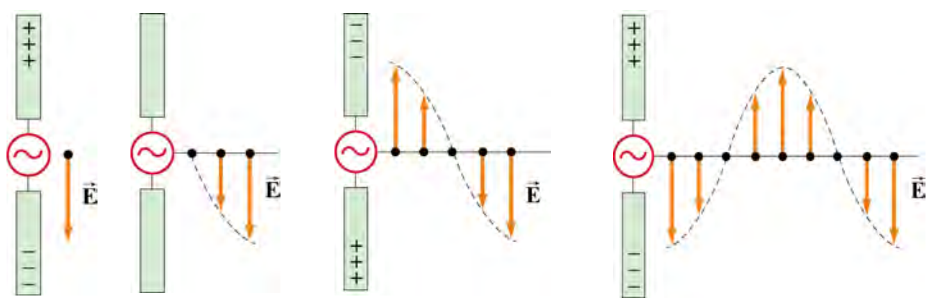


Figure 13.8.2 Electric fields produced by an electric-dipole antenna.

At time $t = 0$ the ends of the rods are charged so that the upper rod has a maximum positive charge and the lower rod has an equal amount of negative charge. At this instant the electric field near the antenna points downward. The charges then begin to decrease. After one-fourth period, $t = T/4$, the charges vanish momentarily and the electric field strength is zero. Subsequently, the polarities of the rods are reversed with negative charges continuing to accumulate on the upper rod and positive charges on the lower until $t = T/2$, when the electric field strength is maximal. At this moment, the electric field near the rod points upward. As the charges continue to oscillate between the rods, electric fields are produced and move away with speed of light. The motion of the charges also produces a current that in turn sets up a magnetic field encircling the rods. However, the behavior of the fields near the antenna is expected to be very different from the behavior far away from the antenna.

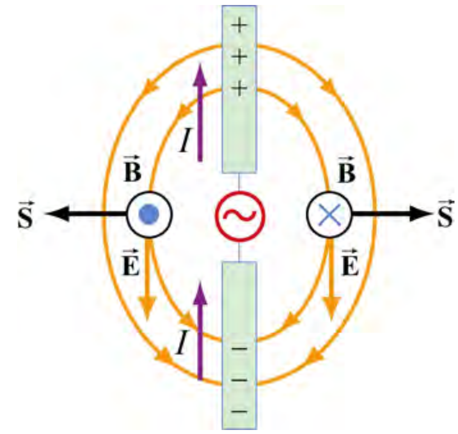


Figure 13.8.3 Electric and magnetic field lines produced by an electric-dipole antenna.

Let us consider a half-wavelength antenna, in which the length of each rod is equal to one quarter of the wavelength of the emitted radiation. Since charges are driven to oscillate back and forth between the rods by the alternating voltage, the antenna may be approximated as an oscillating electric dipole. Figure 13.8.3 depicts the electric and the magnetic field lines at the instant the current is upward. The Poynting vector at the positions shown in figure 13.8.3 are directed outward.

In general, the radiation pattern produced is very complex. However, at distances much greater than the dimensions of the system and the wavelength of the radiation, the fields exhibit a very different behavior. In this *far region*, the radiation is caused by the continuous induction of a magnetic field due to a time-varying electric field and vice versa. Both fields oscillate in phase and vary in amplitude as $1/r$.

The intensity of the variation can be shown to vary as $\sin^2\theta/r^2$, where θ is the angle measured from the axis of the antenna. The angular dependence of the intensity $I(\theta)$ is shown in Figure 13.8.4. The intensity is at a maximum in a plane that passes through the midpoint of the antenna and perpendicular to it.

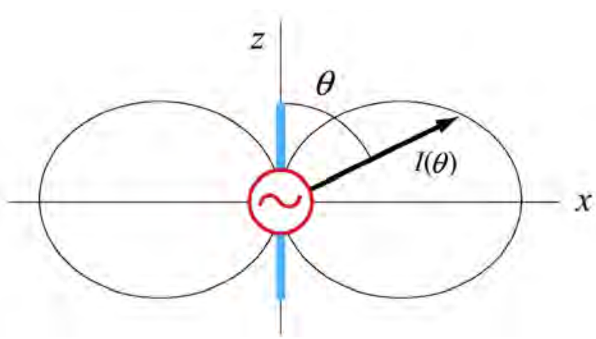


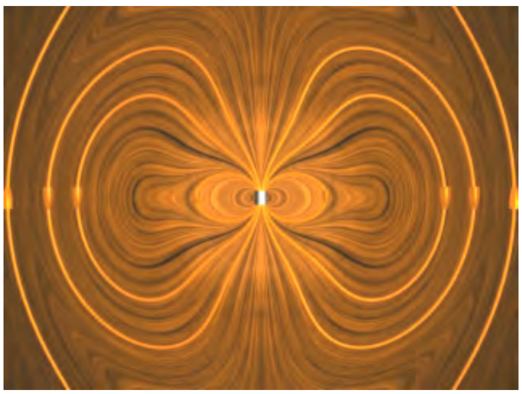
Figure 13.8.4 Angular dependence of the radiation intensity.

13.8.1 Electric Dipole Radiation Movie

Consider an electric dipole whose dipole moment varies in time according to

$$\vec{\mathbf{p}}(t) = p_0 \left[1 + \frac{1}{10}\cos\left(\frac{2\pi t}{T}\right)\right]\hat{\mathbf{k}} \,. \tag{13.8.1}$$

Figure 13.8.5 shows one frame of an animation of these fields. Close to the dipole, the field line motion and thus the Poynting vector is first outward and then inward, corresponding to energy flow outward as the quasi-static dipolar electric field energy is being built up, and energy flow inward as the quasi-static dipole electric field energy is being destroyed.



Link to movie

Figure 13.8.5 Radiation from an electric dipole whose dipole moment varies by 10%.

Even though the energy flow direction changes sign in these regions, there is still a small time-averaged energy flow outward. This small energy flow outward represents the small amount of energy radiated away to infinity. Outside of the point at which the outer field lines detach from the dipole and move off to infinity, the velocity of the field lines, and thus the direction of the electromagnetic energy flow, is always outward. This is the region dominated by radiation fields, which consistently carry energy outward to infinity.

13.8.2 Another Electric Dipole Radiation Movie

Figure 13.8.6 shows one frame of an animation of an electric dipole characterized by

$$\vec{\mathbf{p}}(t) = p_0 \cos\left(\frac{2\pi t}{T}\right)\hat{\mathbf{k}} \,. \tag{13.8.2}$$

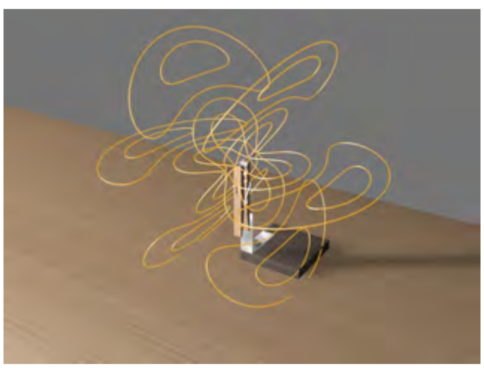
The equation shows that the direction of the dipole moment varies between $+\hat{\mathbf{k}}$ and $-\hat{\mathbf{k}}$.

 Link to Movie

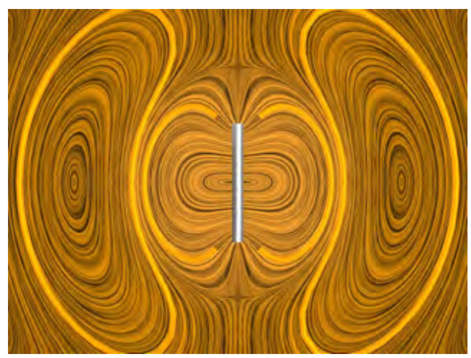
Figure 13.8.6 Radiation from an electric dipole whose dipole moment completely reverses with time.

13.8.3 Radiation From a Quarter-Wave Antenna Animation

Figure 13.8.7(a) shows the radiation pattern at one instant of time from a quarter-wave antenna. Figure 13.8.7(b) shows this radiation pattern in a plane over the full period of the radiation. A quarter-wave antenna produces radiation whose wavelength is twice the tip-to-tip length of the antenna. This is evident in the animation of Figure 13.8.7(b).



(a) Link to Movie



(b) Link to Movie

Figure 13.8.7 Radiation pattern from a quarter-wave antenna: (a) The azimuthal pattern at one instant of time, and (b) the radiation pattern in one plane over the full period.

13.8.4 Plane Waves

There is an interactive simulation for generating plane waves in Section 11.15. You should both read the development below and generate your own plane waves using the simulation.

We have seen that electromagnetic plane waves propagate in empty space at the speed of light. Below we demonstrate how one would create such waves in a particularly simple planar geometry. Although physically this is not particularly applicable to the real world, it is reasonably easy to treat, and we can see directly how electromagnetic plane waves are generated, *why it takes work to make them*, and how much energy they carry away with them.

To make an electromagnetic plane wave, we do much the same thing we do when we make waves on a string. We grab the string somewhere and shake it, and thereby generate a wave on the string. We do work against the tension in the string when we shake it, and that work is carried off as an energy flux in the wave. Electromagnetic waves are much the same proposition. The electric field line serves as the "string." As we will see below, there is a tension associated with an electric field line, in that when we shake it (try to displace it from its initial position), there is a restoring force that resists the shake, and a wave propagates along the field line as a result of the shake. To understand in detail what happens in this process will involve using most of the electromagnetism we have learned thus far, from Gauss's law to Ampere's law plus the reasonable assumption that electromagnetic information propagates at speed c in a vacuum.

How do we shake an electric field line, and what do we grab on to? What we do is shake the electric charges that the field lines are attached to. After all, it is these charges that produce the electric field, and in a very real sense the electric field is "rooted" in the electric charges that produce them. Knowing this, and assuming that in a vacuum, electromagnetic signals propagate at the speed of light, we can pretty much puzzle out how to make a plane electromagnetic wave by shaking charges. Let's first figure out how to make a *kink* in an electric field line, and then we'll go on to make sinusoidal waves.

Suppose we have an infinite sheet of charge located in the yz-plane, initially at rest, with surface charge density σ, as shown in Figure 13.8.8.

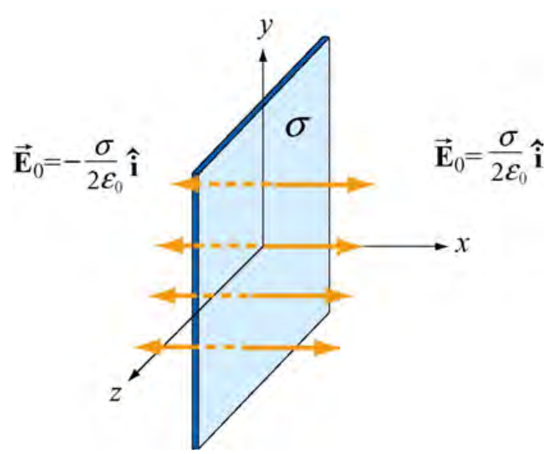


Figure 13.8.8 Electric field due to an infinite sheet with charge density σ.

From Gauss's law discussed in Chapter 3, we know that this surface charge will give rise to a static electric field ,

$$\vec{\mathbf{E}}_0 = \begin{cases} +\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{i}}, & x > 0 \\ -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{i}}, & x < 0. \end{cases} \tag{13.8.3}$$

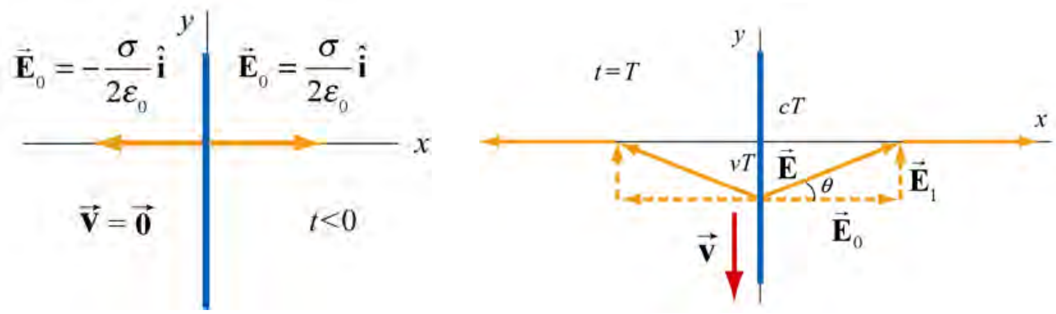


Figure 13.8.9 Electric field lines (a) through $y=0$ at $t<0$, and (b) at $t=T$

Now, at $t=0$, we grab the sheet of charge and start pulling it *downward* with *constant* velocity $\vec{\mathbf{v}}=-v\,\hat{\mathbf{j}}$. Let's examine how things will then appear at a later time $t=T$. In particular, before the sheet starts moving, let's look at the field line that goes through $y=0$ for $t<0$, as shown in Figure 13.8.9(a).

The "foot" of this electric field line, that is, where it is anchored, is rooted in the electric charge that generates it, and that "foot" must move downward with the sheet of charge, at the same speed as the charges move downward. Thus the "foot" of our electric field line, which was initially located at $y=0$ at $t=0$, will have moved a distance $y=-vT$ down the *y*-axis at time $t=T$.

We have assumed that the information that this field line is being dragged downward will propagate outward from $x=0$ at the speed of light c. Thus the portion of our field line located a distance $x>cT$ along the *x*-axis from the origin doesn't know the charges are moving, and thus has not yet begun to move downward. Our field line therefore must appear at time $t=T$ as shown in Figure 13.8.9(b). Nothing has happened outside of $|x|>cT$; the foot of the field line at $x=0$ is a distance $y=-vT$ down the *y*-axis, and we have guessed about what the field line must look like for $0<|x|<cT$ by simply connecting the two positions on the field line that we know about at time T ($x=0$ and $|x|=cT$) by a straight line. This is exactly the guess we would make if we were dealing with a string instead of an electric field. This is a reasonable thing to do, and it turns out to be the right guess.

What we have done by pulling down on the charged sheet is to generate a perturbation in the electric field, $\vec{\mathbf{E}}_1$ in addition to the static field $\vec{\mathbf{E}}_0$. Thus, the field $\vec{\mathbf{E}}$ for $0 < |x| < cT$ is the superposition

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_1 . \tag{13.8.4}$$

As shown in Figure 13.8.9(b), the field vector $\vec{\mathbf{E}}$ must be parallel to the line connecting the foot of the field line and the position of the field line at $|x| = cT$. This implies

$$\tan\theta = \frac{E_1}{E_0} = \frac{vT}{cT} = \frac{v}{c}, \tag{13.8.5}$$

where $E_1 = |\vec{\mathbf{E}}_1|$ and $E_0 = |\vec{\mathbf{E}}_0|$ are the magnitudes of the fields, and θ is the angle with the x- axis. Using Eq. (13.8.5), the perturbation field can be written as

$$\vec{\mathbf{E}}_1 = \left(\frac{v}{c}E_0\right)\hat{\mathbf{j}} = \left(\frac{v\sigma}{2\varepsilon_0 c}\right)\hat{\mathbf{j}}, \tag{13.8.6}$$

where we have used $E_0 = \sigma/2\varepsilon_0$. We have generated an electric field perturbation, and this expression tells us how large the perturbation field $\vec{\mathbf{E}}_1$ is for a given speed, v, of the charged sheet.

This explains why the electric field line has a *tension* associated with it, just as a string does. The direction of $\vec{\mathbf{E}}_1$ is such that the forces it exerts on the charges in the sheet *resist* the motion of the sheet. That is, there is an *upward* electric force on the sheet when we try to move it *downward*. For an infinitesimal area dA of the sheet containing charge $dq = \sigma\, dA$, the upward "tension" associated with the electric field is

$$d\vec{\mathbf{F}}_e = dq\vec{\mathbf{E}}_1 = (\sigma dA)\left(\frac{v\sigma}{2\varepsilon_0 c}\right)\hat{\mathbf{j}} = \left(\frac{v\sigma^2 dA}{2\varepsilon_0 c}\right)\hat{\mathbf{j}}. \tag{13.8.7}$$

Therefore, to overcome the tension, the external agent must apply an equal but opposite (*downward*) force

$$d\vec{\mathbf{F}}_{\text{ext}} = -d\vec{\mathbf{F}}_e = -\left(\frac{v\sigma^2 dA}{2\varepsilon_0 c}\right)\hat{\mathbf{j}}. \tag{13.8.8}$$

Because the amount of work done is $dW_{\text{ext}} = \vec{\mathbf{F}}_{\text{ext}} \cdot d\,\vec{\mathbf{s}}$, the work done per unit time per unit area by the external agent is

$$\frac{1}{dA}\frac{dW_{\text{ext}}}{dt} = \frac{1}{dA}d\vec{\mathbf{F}}_{\text{ext}} \cdot \frac{d\vec{\mathbf{s}}}{dt} = \left(-\frac{v\sigma^2}{2\varepsilon_0 c}\hat{\mathbf{j}}\right)\cdot(-v\,\hat{\mathbf{j}}) = \frac{v^2\sigma^2}{2\varepsilon_0 c}. \tag{13.8.9}$$

What else has happened in this process of moving the charged sheet down? Well, once the charged sheet is in motion, we have created a sheet of current with surface current density (current per unit length) $\vec{\mathbf{K}} = -\sigma v\,\hat{\mathbf{j}}$.

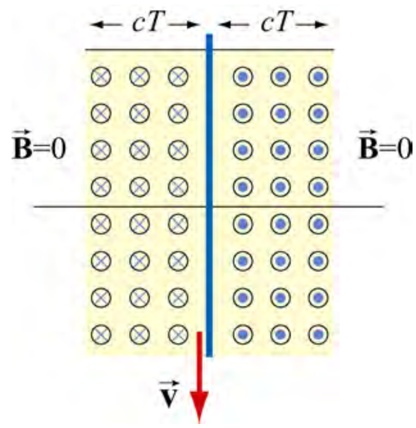


Figure 13.8.10 Magnetic field at $t = T$

From Ampere's law, we know that a *magnetic field* has been created, in addition to $\vec{\mathbf{E}}_1$. The current sheet will produce a magnetic field (see Example 9.4)

$$\vec{\mathbf{B}}_1 = \begin{cases} +(\mu_0 \sigma v/2)\hat{\mathbf{k}}, & x > 0 \\ -(\mu_0 \sigma v/2)\hat{\mathbf{k}}, & x < 0. \end{cases} \tag{13.8.10}$$

This magnetic field changes direction as we move from negative to positive values of x, (across the current sheet). The configuration of the field due to a downward current is shown in Figure 13.8.10 for $|x| < cT$. Again, the information that the charged sheet has started moving, producing a current sheet and associated magnetic field, can only propagate outward from $x = 0$ at the speed of light c. Therefore the magnetic field is still zero, $\vec{\mathbf{B}} = \vec{\mathbf{0}}$ for $|x| > cT$. The magnetic field $\vec{\mathbf{B}}_1$ generated by the current sheet is perpendicular to $\vec{\mathbf{E}}_1$ with a magnitude $B_1 = E_1 / c$, as expected for a transverse electromagnetic wave. As we expect, the ratio of the magnitudes of the fields is equal to the speed of light

$$\frac{E_1}{B_1} = \frac{v\sigma / 2\varepsilon_0 c}{\mu_0 \sigma v / 2} = \frac{1}{c\mu_0 \varepsilon_0} = c\,. \tag{13.8.11}$$

Now, let's discuss the energy carried away by these perturbation fields. The Poynting vector, $\vec{\mathbf{S}}$, gives the energy flux associated with an electromagnetic field. For $x > 0$, the energy flowing to the *right* is

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}}_1 \times \vec{\mathbf{B}}_1 = \frac{1}{\mu_0}\left(\frac{v\sigma}{2\varepsilon_0 c}\,\hat{\mathbf{j}}\right)\times\left(\frac{\mu_0 \sigma v}{2}\,\hat{\mathbf{k}}\right) = \left(\frac{v^2\sigma^2}{4\varepsilon_0 c}\right)\hat{\mathbf{i}}\,. \qquad (13.8.12)$$

This is only half of the work we do per unit time per unit area to pull the sheet down, as given by Eq. (13.8.9). Because the fields on the left carry exactly the same amount of energy flux *to the left*, (the magnetic field $\vec{\mathbf{B}}_1$ changes direction across the plane $x = 0$ whereas the electric field $\vec{\mathbf{E}}_1$ does not, so the Poynting flux also changes across $x = 0$). Therefore energy flux carried off by the perturbation electric and magnetic fields we have generated is *exactly equal* to the rate of work per unit area to pull the charged sheet down against the tension in the electric field. Thus we have generated perturbation electromagnetic fields that carry off energy at exactly the rate that it takes to create them.

Where does the energy carried off by the electromagnetic wave come from? The external agent who originally "shook" the charge to produce the wave had to do work against the perturbation electric field the shaking produces, and that agent is the ultimate source of the energy carried by the wave. An exactly analogous situation exists when one asks where the energy carried by a wave on a string comes from. The agent who originally shook the string to produce the wave had to do work to shake it against the restoring tension in the string, and that agent is the ultimate source of energy carried by a wave on a string.

13.8.5 Sinusoidal Electromagnetic Wave

How about generating a sinusoidal wave with angular frequency ω? To do this, instead of pulling the charge sheet down at constant speed, we just shake it up and down with a velocity $\vec{\mathbf{v}}(t) = -v_0 \cos\omega t\,\hat{\mathbf{j}}$. The oscillating sheet of charge will generate fields that are for $x > 0$ are given by

$$\vec{\mathbf{E}}_1 = \frac{c\mu_0\sigma v_0}{2}\cos\omega\left(t - \frac{x}{c}\right)\hat{\mathbf{j}}, \qquad \vec{\mathbf{B}}_1 = \frac{\mu_0\sigma v_0}{2}\cos\omega\left(t - \frac{x}{c}\right)\hat{\mathbf{k}}\,, \qquad (13.8.13)$$

and for $x < 0$ are given by

$$\vec{\mathbf{E}}_1 = \frac{c\mu_0\sigma v_0}{2}\cos\omega\left(t + \frac{x}{c}\right)\hat{\mathbf{j}}, \qquad \vec{\mathbf{B}}_1 = -\frac{\mu_0\sigma v_0}{2}\cos\omega\left(t + \frac{x}{c}\right)\hat{\mathbf{k}}\,. \qquad (13.8.14)$$

In Eqs. (13.8.13) and (13.8.14) we have chosen the *amplitudes* of these terms to be the amplitudes of the kink generated (Eq. (13.8.6)) in our analysis of the sheet moving at constant speed, with $E_1 / B_1 = c$, but now we allow that the speed is varying sinusoidally in time with angular frequency ω. But why have we put the $(t - x/c)$ and $(t + x/c)$ in the arguments for the cosine function in Eqs. (13.8.13) and (13.8.14)?

Consider first $x > 0$. If we are sitting at some $x > 0$ at time t, and are measuring an electric field there, the field we are observing should not depend on what the current sheet is doing *at that* observation time t. Information about what the current sheet is doing takes a time x/c to propagate out to the observer at $x > 0$. Thus what the observer at $x > 0$ sees at time t depends on what the current sheet was doing *at an earlier time*, namely $t - x/c$. The electric field as a function of time should reflect that time delay due to the finite speed of propagation from the origin to some $x > 0$, and this is the reason the $(t - x/c)$ appears in Eq. (13.8.13) and not t itself. For $x < 0$, the argument is exactly the same, except if $x < 0$, $t + x/c$ is the expression for the earlier time, and not $t - x/c$. This is exactly the time-delay effect one gets when one measures waves on a string. If we are measuring wave amplitudes on a string some distance away from the agent who is shaking the string to generate the waves, what we measure at time t depends on what the agent was doing at an earlier time, allowing for the wave to propagate from the agent to the observer.

Because $\cos(\omega(t - x/c)) = \cos(\omega t - kx)$ where $k = \omega / c$ is the (angular) wave number, we see that Eqs. (13.8.13) and (13.8.14) are precisely the kinds of plane electromagnetic waves we have studied. We can also easily arrange to get rid of our static field $\vec{\mathbf{E}}_0$ by putting a stationary charged sheet with charge per unit area $-\sigma$ at $x = 0$. That charged sheet will cancel out the static field due to the positive sheet of charge, but will not affect the perturbation field we have calculated, since the negatively charged sheet is not moving. In reality, that is how electromagnetic waves are generated; an overall neutral medium where charges of one sign (usually the electrons) are accelerated while an equal number of charges of the opposite sign essentially remain at rest. Thus an observer only sees the wave fields, and not the static fields. In the following, we will assume that we have set $\vec{\mathbf{E}}_0$ to zero in this way.

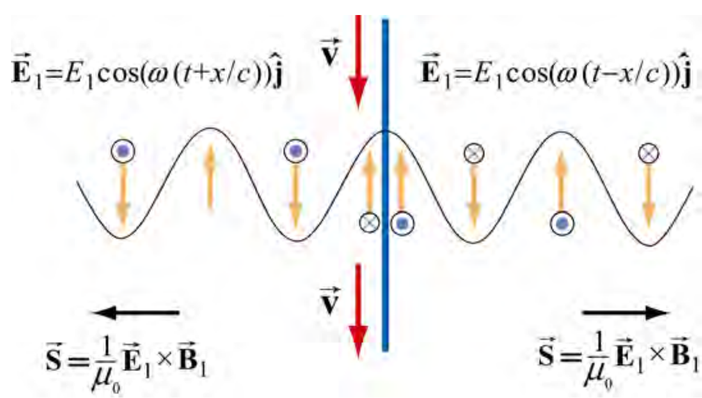


Figure 13.8.11 Electric field generated by the oscillation of a current sheet

The electric field generated by the oscillation of the current sheet is shown in Figure 13.8.11, for the instant when the sheet is moving *down* and the perturbation electric field is *up*. The magnetic fields that point both into and out of the plane of the figure at different locations are also shown.

What we have accomplished in the construction here, which really only assumes that the feet of the electric field lines move with the charges, and that information propagates at c is to show we can generate such a wave by shaking a plane of charge sinusoidally. The wave we generate has electric and magnetic fields perpendicular to one another, and transverse to the direction of propagation, with the ratio of the magnitudes of the electric field to the magnetic field equal to the speed of light. Moreover, we see directly where the energy flux $\vec{\mathbf{S}} = \vec{\mathbf{E}} \times \vec{\mathbf{B}} / \mu_0$ carried off by the wave comes from. The agent who shakes the charges, and thereby generates the electromagnetic wave puts the energy in. If we go to more complicated geometries, these statements become much more complicated in detail, but the overall picture remains as we have presented it.

Let us rewrite slightly the expressions given in Eqs. (13.8.13) and (13.8.14) for the fields generated by our oscillating charged sheet, in terms of the current per unit length in the sheet, $\vec{\mathbf{K}}(t) = \sigma v(t)\,\hat{\mathbf{j}}$. Because $\vec{\mathbf{v}}(t) = -v_0 \cos\omega t\,\hat{\mathbf{j}}$, it follows that $\vec{\mathbf{K}}(t) = -\sigma v_0 \cos\omega t\,\hat{\mathbf{j}}$. Thus, for $x > 0$

$$\vec{\mathbf{E}}_1(x,t) = -\frac{c\mu_0}{2}\vec{\mathbf{K}}(t - x/c), \quad \vec{\mathbf{B}}_1(x,t) = \hat{\mathbf{i}} \times \frac{\vec{\mathbf{E}}_1(x,t)}{c}, \tag{13.8.15}$$

and for $x < 0$

$$\vec{\mathbf{E}}_1(x,t) = -\frac{c\mu_0}{2}\vec{\mathbf{K}}(t + x/c), \quad \vec{\mathbf{B}}_1(x,t) = -\hat{\mathbf{i}} \times \frac{\vec{\mathbf{E}}_1(x,t)}{c}. \tag{13.8.16}$$

The magnetic field, $\vec{\mathbf{B}}_1(x,t)$, reverses direction across the current sheet, with a jump of $\mu_0 \left|\vec{\mathbf{K}}(t)\right|$ at the sheet, as it must from Ampere's law. *Any* oscillating sheet of current *must* generate the plane electromagnetic waves described by these equations, just as *any* stationary electric charge *must* generate a Coulomb electric field.

Note: To avoid possible future confusion, we point out that in a more advanced electromagnetism course, you will study the radiation fields generated by a *single* oscillating charge, and find that they are proportional to the *acceleration* of the charge. This is very different from the case here, where the radiation fields of our oscillating sheet of charge are proportional to the *velocity* of the charges. However, there is no contradiction, because when you add up the radiation fields due to all the single charges making up our sheet, you recover the same result we give in Eqs. (13.8.15) and (13.8.16) (see Chapter 30, Section 7, of Feynman, Leighton, and Sands, *The Feynman Lectures on Physics, Vol.* 1, Addison-Wesley, 1963).

13.9 Summary

- The **Ampere-Maxwell law** reads

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 (I + I_d)\,,$$

where

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt}$$

is called the **displacement current**. The equation describes how changing electric flux is associated with a magnetic field.

- Gauss's law for magnetism is

$$\Phi_B = \oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0\,.$$

The law states that the magnetic flux through a closed surface must be zero, and implies the absence of magnetic monopoles.

- Electromagnetic phenomena are described by the **Maxwell's equations**:

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{\varepsilon_0} \qquad \oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}$$

$$\oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0 \qquad \oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}\,,$$

- In free space, the electric and magnetic components of the electromagnetic wave obey a wave equation

$$\frac{\partial^2 E_y(x,t)}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y(x,t)}{\partial t^2}$$
$$\frac{\partial^2 B_z(x,t)}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B_z(x,t)}{\partial t^2}\,.$$

- The magnitudes and the amplitudes of the electric and magnetic fields in an electromagnetic wave are related by

$$\frac{E}{B} = \frac{E_0}{B_0} = \frac{\omega}{k} = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2997092458\ \mathrm{m \cdot s^{-1}}\,.$$

- A **standing electromagnetic wave** does not propagate, but instead the electric and magnetic fields execute simple harmonic motion perpendicular to the would-be direction of propagation. An example of a standing wave is

$$E_y(x,t) = 2E_0 \sin(kx)\cos(\omega t), \quad B_z(x,t) = 2B_0 \cos(kx)\sin(\omega t).$$

- The energy flow rate of an electromagnetic wave through a closed surface is given by

$$\frac{dU}{dt} = -\oint\!\!\oint \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}},$$

  where

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}$$

  is the **Poynting vector**, and $\vec{\mathbf{S}}$ points in the direction the wave propagates.

- The **intensity** of an electromagnetic wave is related to the average energy density by

$$I = \langle S \rangle = c\langle u \rangle.$$

- The momentum transferred is related to the energy absorbed by

$$\Delta p = \begin{cases} \dfrac{\Delta U}{c} & \text{(complete absorption)} \\ 2\dfrac{\Delta U}{c} & \text{(complete reflection)} \end{cases}$$

- The average **radiation pressure** on a surface by a normally incident electromagnetic wave is

$$P = \begin{cases} \dfrac{\langle S \rangle}{c} & \text{(complete absorption)} \\ \dfrac{2\langle S \rangle}{c} & \text{(complete reflection).} \end{cases}$$

13.10 Appendix: Reflection of Electromagnetic Waves at Conducting Surfaces

How does a very good conductor reflect an electromagnetic wave falling on it? The time-varying electric field of the incoming wave drives an oscillating current on the surface of the conductor, following Ohm's law. That oscillating current sheet, of necessity, must generate waves propagating in both directions from the sheet. One of these waves is the reflected wave. The other wave cancels out the incoming wave inside the conductor. Let us make this qualitative description quantitative.

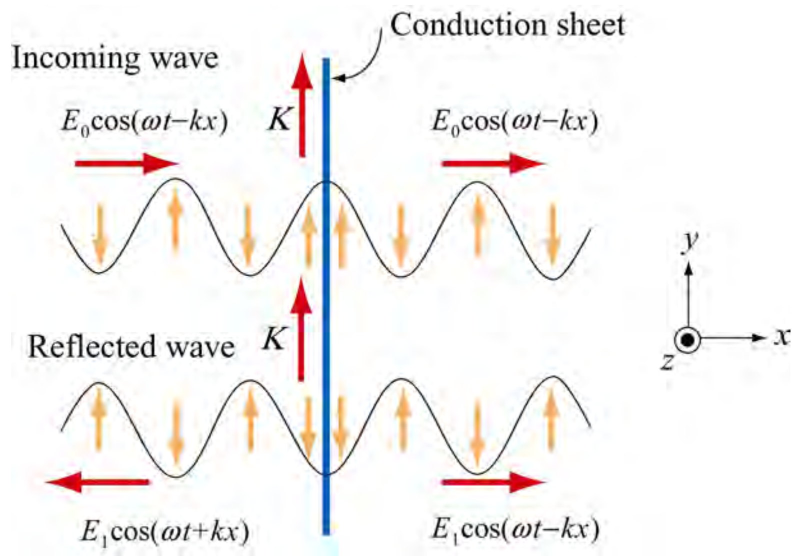


Figure 13.10.1 Reflection of electromagnetic waves at conducting surface

Suppose we have an infinite plane wave propagating in the positive x-direction, with

$$\vec{\mathbf{E}}_0 = E_0 \cos(\omega t - kx)\hat{\mathbf{j}}, \qquad \vec{\mathbf{B}}_0 = B_0 \cos(\omega t - kx)\hat{\mathbf{k}}, \tag{13.10.1}$$

as shown in the top portion of Figure 13.10.1. We put at the origin ($x = 0$) a conducting sheet with width D, which is much smaller than the wavelength of the incoming wave.

This conducting sheet will *reflect* our incoming wave. How? The electric field of the incoming wave will cause a current $\vec{\mathbf{J}} = \vec{\mathbf{E}}/\rho$ to flow in the sheet, where ρ is the resistivity (not to be confused with charge per unit volume), and is equal to the reciprocal of conductivity σ (not to be confused with charge per unit area). Moreover, the direction of $\vec{\mathbf{J}}$ *will be in the same direction as the electric field of the incoming wave*, as shown in the sketch. Thus our incoming wave sets up an oscillating sheet of current with current per unit length $\vec{\mathbf{K}} = \vec{\mathbf{J}}D$. As in our discussion of the generation of plane electromagnetic waves above, this current sheet *will also generate electromagnetic waves*, moving both to the right and to the left (see lower portion of Figure 13.10.1) away from the oscillating sheet of charge. Using Eq. (13.8.15) for $x > 0$ the wave generated by the current will be

$$\vec{\mathbf{E}}_1(x,t) = -\frac{c\mu_0 JD}{2}\cos(\omega t - kx)\hat{\mathbf{j}}, \tag{13.10.2}$$

where $J = |\vec{\mathbf{J}}|$. For $x < 0$, we will have a similar expression, except that the argument will be $(\omega t + kx)$ (see Figure 13.10.1). The sign of this electric field $\vec{\mathbf{E}}_1$ at $x = 0$ is *down* ($-\hat{\mathbf{j}}$) when the sheet of current is *up* (and $\vec{\mathbf{E}}_0$ is *up*, $+\hat{\mathbf{j}}$), and vice-versa, just as we saw before. Thus, for $x > 0$, the generated electric field $\vec{\mathbf{E}}_1$ will always be *opposite* the direction of the electric field of the incoming wave, and *it will tend to cancel out the incoming wave for* $x > 0$. For a very good conductor, the magnitude of the surface current is (see next section)

$$K = |\vec{\mathbf{K}}| = JD = \frac{2E_0}{c\mu_0}, \tag{13.10.3}$$

Thus, for $x > 0$, the electric field is

$$\vec{\mathbf{E}}_1(x,t) = -E_0 \cos(\omega t - kx)\hat{\mathbf{j}}. \tag{13.10.4}$$

For a very good conductor, the electric field of the wave generated by the current *will exactly cancel* the electric field of the incoming wave *for* $x > 0$! That's what a very good conductor does. It supports exactly the amount of current per unit length $K = 2E_0 / c\mu_0$ needed to cancel out the incoming wave for $x > 0$. For $x < 0$, *this same current* generates a "reflected" wave propagating back in the direction from which the original incoming wave came, *with the same amplitude as the original incoming wave*. This is how a very good conductor *totally* reflects electromagnetic waves. Below we shall show that K will in fact approach the value needed to accomplish this in the limit that the resistivity ρ approaches zero.

In the process of reflection, there is a force per unit area exerted on the conductor. This is just the $\vec{\mathbf{v}} \times \vec{\mathbf{B}}$ force due to the current $\vec{\mathbf{J}}$ flowing in the presence of the magnetic field of the incoming wave, or a force per unit volume of $\vec{\mathbf{J}} \times \vec{\mathbf{B}}_0$. If we calculate the total force $d\vec{\mathbf{F}}$ acting on a cylindrical volume with area dA and length D of the conductor, we find that it is in the positive x-direction, with magnitude

$$dF = D\,|\vec{\mathbf{J}} \times \vec{\mathbf{B}}_0|\,dA = DJB_0\,dA = \frac{2E_0 B_0 dA}{c\mu_0}. \tag{13.10.5}$$

The force per unit area or the radiation pressure, is therefore

$$\frac{dF}{dA} = \frac{2E_0 B_0}{c\mu_0} = \frac{2S}{c}, \tag{13.10.6}$$

twice the Poynting flux divided by the speed of light c.

We shall show that a perfect conductor will perfectly reflect an incident wave. To approach the limit of a perfect conductor, we first consider the finite resistivity case, and then let the resistivity go to zero.

For simplicity, we assume that the sheet is thin compared to a wavelength, so that the entire sheet sees essentially the same electric field. This implies that the current density

$\vec{\mathbf{J}}$ will be uniform across the thickness of the sheet, and outside of the sheet we will see fields appropriate to an equivalent surface current $\vec{\mathbf{K}}(t) = D\vec{\mathbf{J}}(t)$. This current sheet will generate additional electromagnetic waves, moving both to the right and to the left, away from the oscillating sheet of charge. The total electric field, $\vec{\mathbf{E}}(x,t)$, will be the sum of the incident electric field and the electric field generated by the current sheet. Using Eqs. (13.8.15) and (13.8.16) above, we obtain the following expressions for the total electric field

$$\vec{\mathbf{E}}(x,t) = \vec{\mathbf{E}}_0(x,t) + \vec{\mathbf{E}}_1(x,t) = \begin{cases} \vec{\mathbf{E}}_0(x,t) - \dfrac{c\mu_0}{2}\vec{\mathbf{K}}(t - x/c), & x > 0 \\ \vec{\mathbf{E}}_0(x,t) - \dfrac{c\mu_0}{2}\vec{\mathbf{K}}(t + x/c), & x < 0. \end{cases} \tag{13.10.7}$$

We also have a relation between the current density $\vec{\mathbf{J}}$ and $\vec{\mathbf{E}}$ from the microscopic form of Ohm's law: $\vec{\mathbf{J}}(t) = \vec{\mathbf{E}}(0,t)/\rho$, where $\vec{\mathbf{E}}(0,t)$ is the total electric field at the position of the conducting sheet. It is appropriate to use the total electric field in Ohm's law; the currents arise from the total electric field, irrespective of the origin of that field. Thus, we have

$$\vec{\mathbf{K}}(t) = D\vec{\mathbf{J}}(t) = \frac{D\vec{\mathbf{E}}(0,t)}{\rho}. \tag{13.10.8}$$

At $x = 0$, either expression in Eq. (13.10.7) gives

$$\begin{aligned} \vec{\mathbf{E}}(0,t) &= \vec{\mathbf{E}}_0(0,t) + \vec{\mathbf{E}}_1(0,t) = \vec{\mathbf{E}}_0(0,t) - \frac{c\mu_0}{2}\vec{\mathbf{K}}(t) \\ &= \vec{\mathbf{E}}_0(0,t) - \frac{Dc\mu_0\vec{\mathbf{E}}(0,t)}{2\rho}, \end{aligned} \tag{13.10.9}$$

where we have used Eq. (13.10.9) for the last step. Solving for $\vec{\mathbf{E}}(0,t)$, we obtain

$$\vec{\mathbf{E}}(0,t) = \frac{\vec{\mathbf{E}}_0(0,t)}{1 + Dc\mu_0/2\rho}. \tag{13.10.10}$$

Using Eq. (13.10.10), the surface current density in Eq. (13.10.8) can be rewritten as

$$\vec{\mathbf{K}}(t) = D\vec{\mathbf{J}}(t) = \frac{D\vec{\mathbf{E}}_0(0,t)}{\rho + Dc\mu_0/2}. \tag{13.10.11}$$

In the limit where $\rho \simeq 0$ (no resistance, a perfect conductor), $\vec{\mathbf{E}}(0,t) = \vec{\mathbf{0}}$, as can be seen from Eq. (13.10.8) and the surface current becomes

$$\vec{\mathbf{K}}(t) = \frac{2\vec{\mathbf{E}}_0(0,t)}{c\mu_0} = \frac{2E_0}{c\mu_0}\cos\omega t\,\hat{\mathbf{j}} = \frac{2B_0}{\mu_0}\cos\omega t\,\hat{\mathbf{j}}. \tag{13.10.12}$$

In this same limit, the total electric fields can be written as

$$\vec{\mathbf{E}}(x,t) = \begin{cases} (E_0 - E_0)\cos(\omega t - kx)\hat{\mathbf{j}} = \vec{\mathbf{0}}, & x > 0 \\ E_0[\cos(\omega t - kx) - \cos(\omega t + kx)]\hat{\mathbf{j}} = 2E_0\sin\omega t\sin kx\,\hat{\mathbf{j}}, & x < 0. \end{cases} \tag{13.10.13}$$

Similarly, the total magnetic fields in this limit are given by for $x > 0$,

$$\begin{aligned} \vec{\mathbf{B}}(x,t) &= \vec{\mathbf{B}}_0(x,t) + \vec{\mathbf{B}}_1(x,t) = B_0\cos(\omega t - kx)\hat{\mathbf{k}} + \hat{\mathbf{i}} \times \frac{\vec{\mathbf{E}}_1(x,t)}{c} \\ &= B_0\cos(\omega t - kx)\hat{\mathbf{k}} - B_0\cos(\omega t - kx)\hat{\mathbf{k}} = \vec{\mathbf{0}}, \end{aligned} \tag{13.10.14}$$

and for $x < 0$,

$$\vec{\mathbf{B}}(x,t) = B_0[\cos(\omega t - kx) + \cos(\omega t + kx)]\hat{\mathbf{k}} = 2B_0\cos\omega t\cos kx\,\hat{\mathbf{k}}. \tag{13.10.15}$$

Thus, from Eqs. (13.10.13) - (13.10.15), we see that we get no electromagnetic wave for $x > 0$, and standing electromagnetic waves for $x < 0$. At $x = 0$, the total electric field vanishes. The current per unit length at $x = 0$ is

$$\vec{\mathbf{K}}(t) = \frac{2B_0}{\mu_0}\cos\omega t\,\hat{\mathbf{j}}. \tag{13.10.16}$$

This is just the current per length we need to bring the magnetic field down from its value at $x < 0$ to zero for $x > 0$.

You may be perturbed by the fact that in the limit of a perfect conductor, the electric field vanishes at $x = 0$, since it is the electric field at $x = 0$ that is driving the current there! In the limit of very small resistance, the electric field required to drive any finite current is very small. In the limit where $\rho = 0$, the electric field is zero, but as we approach that limit, we can still have a perfectly finite and well-determined value of $\vec{\mathbf{J}} = \vec{\mathbf{E}}/\rho$, as we found by taking this limit in Eqs. (13.10.8) and (13.10.12).

13.11 Problem-Solving Strategy: Traveling Electromagnetic Waves

This chapter explores various properties of the electromagnetic waves. The electric and magnetic fields of the wave obey the wave equation. Once the functional form of either one of the fields is given, the other can be determined from Maxwell's equations. As an example, let's consider a sinusoidal electromagnetic wave with electric field given by

$$\vec{\mathbf{E}}(z,t) = E_0 \sin(kz - \omega t)\hat{\mathbf{i}} .$$

The equation above contains the complete information about the electromagnetic wave.

1. Direction of wave propagation: The argument of the sine function in the electric field can be rewritten as $(kz - \omega t) = k(z - vt)$, which indicates that the wave is propagating in the positive z-direction.

2. Wavelength: The wavelength, λ, is related to the (angular) wave number, k, by $\lambda = 2\pi / k$.

3. Frequency: The frequency of the wave, f, is related to the angular frequency, ω, by $f = \omega / 2\pi$, and period, T, by $f = 1/T$.

4. Speed of propagation: The speed of the wave is given by

   $$v = \lambda f = \frac{2\pi}{k} \cdot \frac{\omega}{2\pi} = \frac{\omega}{k}$$

   In vacuum, the speed of the electromagnetic wave is equal to the speed of light, c.

5. Magnetic field $\vec{\mathbf{B}}$: The magnetic field $\vec{\mathbf{B}}$ is perpendicular to both $\vec{\mathbf{E}}$ and $+\hat{\mathbf{k}}$, the unit vector pointing in the positive z-direction, which is the direction of propagation, as we have found. In addition, since the wave propagates in the same direction as the cross product $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$. When $\sin(kz - \omega t) > 0$, $\vec{\mathbf{B}}$ must point in the positive y-direction (since $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$).

   Because $\vec{\mathbf{B}}$ is always in phase with $\vec{\mathbf{E}}$, the two fields have the same functional form. Thus, we may write the magnetic field as

   $$\vec{\mathbf{B}}(z,t) = B_0 \sin(kz - \omega t)\hat{\mathbf{j}},$$

   where B_0 is the amplitude. Using Maxwell's equations one may show that $B_0 = E_0(k/\omega) = E_0 / c$ in vacuum.

6. The Poynting vector: Using Eq. (13.6.5), the Poynting vector can be obtained as

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}}\times\vec{\mathbf{B}} = \frac{1}{\mu_0}\left[E_0\sin(kz-\omega t)\,\hat{\mathbf{i}}\right]\times\left[B_0\sin(kz-\omega t)\,\hat{\mathbf{j}}\right] = \frac{E_0 B_0 \sin^2(kz-\omega t)}{\mu_0}\,\hat{\mathbf{k}}\,.$$

7. Intensity: The intensity of the wave is equal to the time-average of S:

$$I = \langle S\rangle = \frac{E_0 B_0}{\mu_0}\langle\sin^2(kz-\omega t)\rangle = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2c\mu_0} = \frac{cB_0^2}{2\mu_0}$$

8. Radiation pressure: If the electromagnetic wave is normally incident on a surface and the radiation is completely *reflected*, the radiation pressure is

$$P = \frac{2I}{c} = \frac{E_0 B_0}{c\mu_0} = \frac{E_0^2}{c^2\mu_0} = \frac{B_0^2}{\mu_0}$$

13.12 Solved Problems

13.12.1 Plane Electromagnetic Wave

Suppose the electric field of a plane electromagnetic wave is given by

$$\vec{\mathbf{E}}(z,t) = E_0\cos(kz-\omega t)\,\hat{\mathbf{i}}\,. \tag{13.12.1}$$

Find the following quantities:

(a) The direction of wave propagation.

(b) The corresponding magnetic field $\vec{\mathbf{B}}$.

Solutions:

(a) By writing the argument of the cosine function as $kz-\omega t = k(z-ct)$ where $\omega = ck$, we see that the wave is traveling in the positive z-direction.

(b) The direction of propagation of the electromagnetic waves coincides with the direction of the Poynting vector, which is given by $\vec{\mathbf{S}} = \vec{\mathbf{E}}\times\vec{\mathbf{B}}/\mu_0$. In addition, $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are perpendicular to each other. Therefore, if $\vec{\mathbf{E}} = E(z,t)\,\hat{\mathbf{i}}$ and $\vec{\mathbf{S}} = S\hat{\mathbf{k}}$, then $\vec{\mathbf{B}} = B(z,t)\,\hat{\mathbf{j}}$. That is, when $\cos(kz-\omega t) > 0$, $\vec{\mathbf{B}}$ points in the positive y-direction. Because $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are in phase with each other, the magnetic field is

$$\vec{\mathbf{B}}(z,t) = B_0\cos(kz-\omega t)\,\hat{\mathbf{j}}\,. \tag{13.12.2}$$

To find the magnitude of $\vec{\mathbf{B}}$, we make use of Faraday's law:

$$\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}, \tag{13.12.3}$$

which by a similar calculation as in Section 13.4,

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}. \tag{13.12.4}$$

From the above equations, we obtain

$$-E_0 k \sin(kz - \omega t) = -B_0 \omega \sin(kz - \omega t). \tag{13.12.5}$$

Therefore

$$\frac{E_0}{B_0} = \frac{\omega}{k} = c. \tag{13.12.6}$$

Thus, the magnetic field is given by

$$\vec{\mathbf{B}}(z,t) = (E_0 / c)\cos(kz - \omega t)\,\hat{\mathbf{j}}. \tag{13.12.7}$$

13.12.2 One-Dimensional Wave Equation

Verify that, for $\omega = kc$,

$$\begin{aligned} E_y(x,t) &= E_0 \cos(kx - \omega t) \\ B_z(x,t) &= B_0 \cos(kx - \omega t) \end{aligned} \tag{13.12.8}$$

satisfy the one-dimensional wave equations

$$\begin{aligned} \frac{\partial^2 E_y(x,t)}{\partial x^2} &= \frac{1}{c^2}\frac{\partial^2 E_y(x,t)}{\partial t^2}, \\ \frac{\partial^2 B_z(x,t)}{\partial x^2} &= \frac{1}{c^2}\frac{\partial^2 B_z(x,t)}{\partial t^2}. \end{aligned} \tag{13.12.9}$$

Solution: Differentiating $E_y(x,t)$ with respect to x gives

$$\frac{\partial E_y(x,t)}{\partial x} = -kE_0 \sin(kx - \omega t), \quad \frac{\partial^2 E_y(x,t)}{\partial x^2} = -k^2 E_0 \cos(kx - \omega t) \quad . \tag{13.12.10}$$

Similarly, differentiating $E_y(x,t)$ with respect to t yields

$$\frac{\partial E_y(x,t)}{\partial t} = \omega E_0 \sin(kx-\omega t), \quad \frac{\partial^2 E_y(x,t)}{\partial t^2} = -\omega^2 E_0 \cos(kx-\omega t) . \qquad (13.12.11)$$

Thus,

$$\frac{\partial^2 E_y(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E_y(x,t)}{\partial t^2} = \left(-k^2 + \frac{\omega^2}{c^2}\right) E_0 \cos(kx-\omega t) = 0 , \qquad (13.12.12)$$

where we have made used of the relation $\omega = kc$. One may follow a similar procedure to verify that the magnetic field in Eq. (13.12.8) satisfies the appropriate one-dimensional wave equation.

13.12.3 Poynting Vector of a Charging Capacitor

A parallel-plate capacitor with circular plates of radius R and separated by a distance h is charged through a straight wire carrying current I , as shown in the Figure 13.12.1.

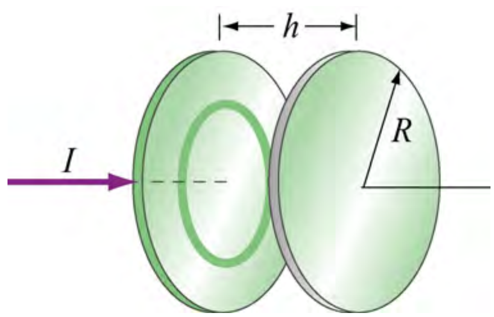


Figure 13.12.1 Parallel plate capacitor

(a) Show that as the capacitor is being charged, the Poynting vector $\vec{\mathbf{S}}$ points radially inward toward the center of the capacitor.

(b) By integrating $\vec{\mathbf{S}}$ over the cylindrical boundary, show that the rate at which energy enters the capacitor is equal to the rate at which electrostatic energy is being stored in the electric field.

Solutions:

(a) Let the axis of the circular plates be the z-axis, with current flowing in the positive z -direction. Suppose at some instant the amount of charge accumulated on the positive plate is $+Q$. The electric field is

$$\vec{\mathbf{E}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{k}} = \frac{Q}{\pi R^2 \varepsilon_0}\hat{\mathbf{k}}\,. \tag{13.12.13}$$

According to the Ampere-Maxwell's equation, there is an associated magnetic field to the changing electric flux,

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d}{dt} \iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}\,. \tag{13.12.14}$$

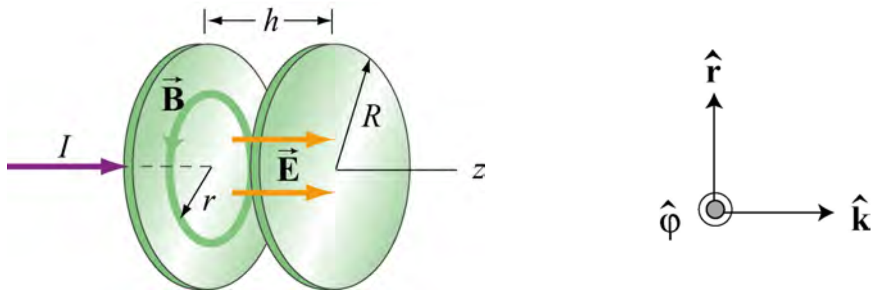


Figure 13.12.2

From the cylindrical symmetry of the system, we see that the magnetic field will be circular, centered on the z-axis, $\vec{\mathbf{B}} = B\hat{\boldsymbol{\varphi}}$ (see Figure 13.12.2.)

Consider a circular path of radius $r < R$ between the plates. Calculating both sides of Eq. (13.12.14), we obtain

$$B 2\pi r = 0 + \mu_0 \varepsilon_0 \frac{d}{dt}\left(\frac{Q}{\pi R^2 \varepsilon_0}\pi r^2\right) = \frac{\mu_0 r^2}{R^2}\frac{dQ}{dt}\,. \tag{13.12.15}$$

Therefore the magnetic field is given by

$$\vec{\mathbf{B}} = \frac{\mu_0 r}{2\pi R^2}\frac{dQ}{dt}\hat{\boldsymbol{\varphi}}\,. \tag{13.12.16}$$

The Poynting $\vec{\mathbf{S}}$ vector can then be written as

$$\begin{aligned} \vec{\mathbf{S}} &= \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}} = \frac{1}{\mu_0}\left(\frac{Q}{\pi R^2 \varepsilon_0}\hat{\mathbf{k}}\right) \times \left(\frac{\mu_0 r}{2\pi R^2}\frac{dQ}{dt}\hat{\boldsymbol{\varphi}}\right) \\ &= -\frac{Qr}{2\pi^2 R^4 \varepsilon_0}\frac{dQ}{dt}\hat{\mathbf{r}}. \end{aligned} \tag{13.12.17}$$

When $dQ/dt > 0$, $\vec{\mathbf{S}}$ points in the negative $\hat{\mathbf{r}}$-direction, or radially inward toward the center of the capacitor.

(b) The energy per unit volume stored by the electric field is $u_E = \varepsilon_0 E^2/2$. The total energy stored in the electric field then becomes

$$U_E = u_E V = \frac{\varepsilon_0}{2} E^2(\pi R^2 h) = \frac{1}{2}\varepsilon_0 \left(\frac{Q}{\pi R^2 \varepsilon_0}\right)^2 \pi R^2 h = \frac{Q^2 h}{2\pi R^2 \varepsilon_0}. \qquad (13.12.18)$$

Differentiating the above expression with respect to t, we obtain the rate at which this energy is being stored,

$$\frac{dU_E}{dt} = \frac{d}{dt}\left(\frac{Q^2 h}{2\pi R^2 \varepsilon_0}\right) = \frac{Qh}{\pi R^2 \varepsilon_0}\frac{dQ}{dt}. \qquad (13.12.19)$$

The rate at which energy flows into the capacitor through the cylinder at $r = R$ can be obtained by integrating $\vec{\mathbf{S}}$ over the surface area,

$$\oiint_S \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}} = SA_R = \frac{Qr}{2\pi^2 \varepsilon_o R^4}\frac{dQ}{dt} 2\pi R h = \frac{Qh}{\varepsilon_0 \pi R^2}\frac{dQ}{dt}, \qquad (13.12.20)$$

which is equal to the rate at which energy stored in the electric field is changing.

13.12.4 Poynting Vector of a Conductor

A cylindrical conductor of radius a and conductivity σ carries a steady current I, which is distributed uniformly over its cross-section, as shown in Figure 13.12.3.

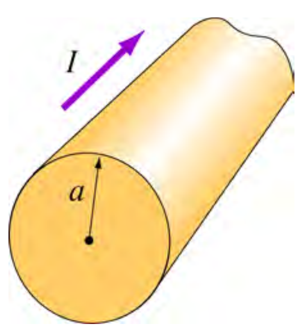


Figure 13.12.3

(a) Compute the electric field $\vec{\mathbf{E}}$ inside the conductor.

(b) Compute the magnetic field $\vec{\mathbf{B}}$ just outside the conductor.

(c) Compute the Poynting vector $\vec{\mathbf{S}}$ at the surface of the conductor. In which direction does $\vec{\mathbf{S}}$ point?

(d) By integrating $\vec{\mathbf{S}}$ over the surface area of the conductor, show that the rate at which electromagnetic energy enters the surface of the conductor is equal to the rate at which energy is dissipated.

Solutions:

(a) Let the direction of the current be along the positive z- axis. The electric field is given by

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{J}}}{\sigma} = \frac{I}{\sigma \pi a^2}\hat{\mathbf{k}} . \qquad (13.12.21)$$

where R is the resistance and l is the length of the conductor.

(b) The magnetic field can be computed using Ampere's law,

$$\oint_C \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} . \qquad (13.12.22)$$

Choosing the Amperian loop to be a circle of radius r, calculating both sides of Eq. (13.12.22) yields

$$B(2\pi r) = \mu_0 I . \qquad (13.12.23)$$

The magnetic field is then

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\varphi}} . \qquad (13.12.24)$$

(c) The Poynting vector on the surface of the wire ($r = a$) is

$$\vec{\mathbf{S}} = \frac{\vec{\mathbf{E}} \times \vec{\mathbf{B}}}{\mu_0} = \frac{1}{\mu_0}\frac{I}{\sigma \pi a^2}\hat{\mathbf{k}} \times \frac{\mu_0 I}{2\pi a}\hat{\boldsymbol{\varphi}} = -\frac{I^2}{2\pi^2 \sigma a^3}\hat{\mathbf{r}} . \qquad (13.12.25)$$

The direction of $\vec{\mathbf{S}}$ is radially inward toward the center of the conductor.

(d) The rate at which electromagnetic energy flows into the conductor is given by

$$P = \frac{dU}{dt} = \oiint_S \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}} = \frac{I^2}{2\sigma \pi^2 a^3} 2\pi a l = \frac{I^2 l}{\sigma \pi a^2} , \qquad (13.12.26)$$

The conductivity σ is related to the resistance R by

$$\sigma = \frac{1}{\rho} = \frac{l}{AR} = \frac{l}{\pi a^2 R}. \qquad (13.12.27)$$

Substitute Eq. (13.12.27) into Eq. (13.12.26) yielding

$$P = I^2 R, \qquad (13.12.28)$$

which is equal to the rate of energy dissipation in a resistor with resistance R.

13.13 Conceptual Questions

1. In the Ampere-Maxwell's equation, is it possible that both a conduction current and a displacement current are non-vanishing?

2. What causes electromagnetic radiation?

3. When you touch the indoor antenna on a TV, the reception usually improves. Why?

4. Explain why the reception for cellular phones often becomes poor when used inside a steel-framed building.

5. Compare sound waves with electromagnetic waves.

6. Can parallel electric and magnetic fields make up an electromagnetic wave in vacuum?

7. What happens to the intensity of an electromagnetic wave if the amplitude of the electric field is halved? Doubled?

13.14 Additional Problems

13.14.1 Solar Sailing

It has been proposed that a spaceship might be propelled in the solar system by radiation pressure, using a large sail made of a lightweight reflecting material. How large must the sail be if the radiation force is to be equal in magnitude to the Sun's gravitational attraction? Assume that the mass of the ship and sail is 1650 kg, that the sail is perfectly reflecting, and that the sail is oriented at right angles to the Sun's rays. Does your answer depend on where in the solar system the spaceship is located?

13.14.2 Reflections of True Love

(a) A light bulb puts out 100 W of electromagnetic radiation. What is the intensity of radiation from this light bulb at a distance of one meter from the bulb? What are the maximum values of electric and magnetic fields, E_0 and B_0, at this distance from the bulb? You may assume that the wave is a plane wave.

(b) The face of your true love is one meter from this 100 W bulb. What maximum surface current must flow on your true love's face in order to reflect the light from the bulb into your adoring eyes? Assume that your true love's face is (what else?) perfect--perfectly smooth and perfectly reflecting--and that the incident light and reflected light are normal to the surface.

13.14.3 Coaxial Cable and Power Flow

A coaxial cable consists of two concentric long hollow cylinders of zero resistance; the inner has radius a, the outer has radius b, and the length of both is l, with $l >> b$. The cable transmits DC power from a battery to a load. The battery provides an electromotive force ε between the two conductors at one end of the cable, and the load is a resistance R connected between the two conductors at the other end of the cable. A current I is directed to the right in the inner conductor and to the left in the outer one. The battery charges the inner conductor to a charge $-Q$ and the outer conductor to a charge $+Q$.

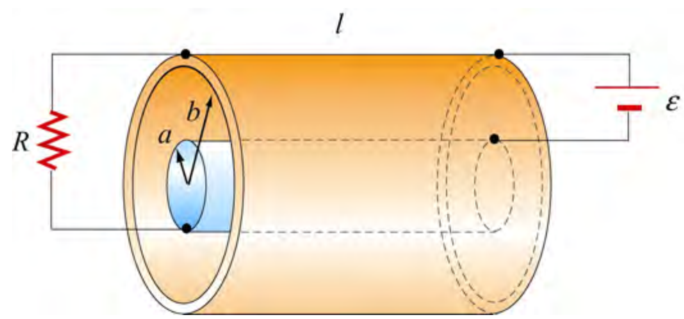


Figure 13.14.1

(a) Find the direction and magnitude of the electric field $\vec{\mathbf{E}}$ everywhere.

(b) Find the direction and magnitude of the magnetic field $\vec{\mathbf{B}}$ everywhere.

(c) Calculate the Poynting vector $\vec{\mathbf{S}}$ in the cable.

(d) By integrating $\vec{\mathbf{S}}$ over appropriate surface, find the power that flows into the coaxial cable. Which surface does it flow into?

(e) How does your result in (d) compare to the power dissipated in the resistor?

13.14.4 Superposition of Electromagnetic Waves

Electromagnetic waves are emitted from two different sources with electric fields

$$\vec{\mathbf{E}}_1(x,t) = E_{10}\cos(kx-\omega t)\,\hat{\mathbf{j}}, \quad \vec{\mathbf{E}}_2(x,t) = E_{20}\cos(kx-\omega t+\phi)\,\hat{\mathbf{j}}\,.$$

(a) Find the Poynting vector associated with the resultant electromagnetic wave.

(b) Find the intensity of the resultant electromagnetic wave

(c) Repeat the calculations above if the direction of propagation of the second electromagnetic wave is reversed so that the electric fields are now

$$\vec{\mathbf{E}}_1(x,t) = E_{10}\cos(kx-\omega t)\,\hat{\mathbf{j}}, \quad \vec{\mathbf{E}}_2(x,t) = E_{20}\cos(kx+\omega t+\phi)\,\hat{\mathbf{j}}\,.$$

13.14.5 Sinusoidal Electromagnetic Wave

The electric field of an electromagnetic wave is given by

$$\vec{\mathbf{E}}(z,t) = E_0\cos(kz-\omega t)\,(\hat{\mathbf{i}}+\hat{\mathbf{j}})\,.$$

(a) What is the maximum amplitude of the electric field?

(b) Compute the corresponding magnetic field $\vec{\mathbf{B}}$.

(c) Find the Poynting vector $\vec{\mathbf{S}}$.

(d) What is the radiation pressure if the wave is incident normally on a surface and is perfectly reflected?

13.14.6 Radiation Pressure of Electromagnetic Wave

A plane electromagnetic wave is described by

$$\vec{\mathbf{E}} = E_0\sin(kx-\omega t)\,\hat{\mathbf{j}}, \qquad \vec{\mathbf{B}} = B_0\sin(kx-\omega t)\,\hat{\mathbf{k}}\,,$$

where $E_0 = cB_0$.

(a) Show that for any point in this wave, the density of the energy stored in the electric field equals the density of the energy stored in the magnetic field. What is the time-averaged (electric plus magnetic) energy density in this wave, in terms of E_0? In terms of B_0?

(b) This wave falls on and is totally *absorbed* by an object. Assuming total absorption, show that the radiation pressure on the object is just given by the time-averaged total energy density in the wave. (Hint: the dimensions of energy density are the same as the dimensions of pressure.)

(c) Sunlight strikes the Earth, just outside its atmosphere, with an average intensity of 1350 W/m^2. What is the time-averaged total energy density of this sunlight? An object in orbit about the Earth totally absorbs sunlight. What radiation pressure does it feel?

13.14.7 Energy of Electromagnetic Waves

(a) If the electric field of an electromagnetic wave has an rms (root-mean-square) strength of $3.0\times10^{-2}\ \text{V/m}$, how much energy is transported across a 1.00-cm^2 area in one hour?

(b) The intensity of the solar radiation incident on the upper atmosphere of the Earth is approximately 1350 W/m^2. Using this information, estimate the energy contained in a 1.00-m^3 volume near the Earth's surface due to radiation from the Sun.

13.14.8 Wave Equation

Consider a plane electromagnetic wave with the electric and magnetic fields given by

$$\vec{\mathbf{E}}(x,t) = E_z(x,t)\hat{\mathbf{k}}, \quad \vec{\mathbf{B}}(x,t) = B_y(x,t)\hat{\mathbf{j}}.$$

Applying arguments similar to that presented in 13.4, show that the fields satisfy the following relationships:

$$\frac{\partial E_z}{\partial x} = \frac{\partial B_y}{\partial t}, \quad \frac{\partial B_y}{\partial x} = \mu_0\varepsilon_0\frac{\partial E_z}{\partial t}.$$

13.14.9 Electromagnetic Plane Wave

An electromagnetic plane wave, propagating in vacuum, has a magnetic field given by

$$\vec{\mathbf{B}} = B_0 f(ax+bt)\hat{\mathbf{j}} \qquad f(u) = \begin{cases} 1 & 0 < u < 1 \\ 0 & u \le 0,\ u \ge 1. \end{cases}$$

The wave encounters an infinite, dielectric sheet at $x = 0$ of such a thickness that half of the energy of the wave is reflected and the other half is transmitted and emerges on the other side of the sheet.

(a) What condition between a and b must be met in order for this wave to satisfy Maxwell's equations?

(b) What is the magnitude and direction of the $\vec{\mathbf{E}}$ field of the incoming wave?

(c) What is the magnitude and direction of the energy flux (power per unit area) carried by the incoming wave, in terms of B_0 and universal quantities?

(d) What is the pressure (force per unit area) that this wave exerts on the sheet while it is impinging on it?

13.14.10 Sinusoidal Electromagnetic Wave

An electromagnetic plane wave has an electric field given by

$$\vec{\mathbf{E}} = (300\,\mathrm{V/m})\cos\left(\frac{2\pi}{3}x - 2\pi\times10^{6}\,t\right)\hat{\mathbf{k}}\,.$$

where x and t are in SI units. The wave is propagating through ferrite, a ferromagnetic insulator, which has a relative magnetic permeability $\kappa_m = 1000$ and dielectric constant $\kappa = 10$.

(a) What direction does this wave travel?

(b) What is the wavelength of the wave (in meters)?

(c) What is the frequency of the wave (in Hz)?

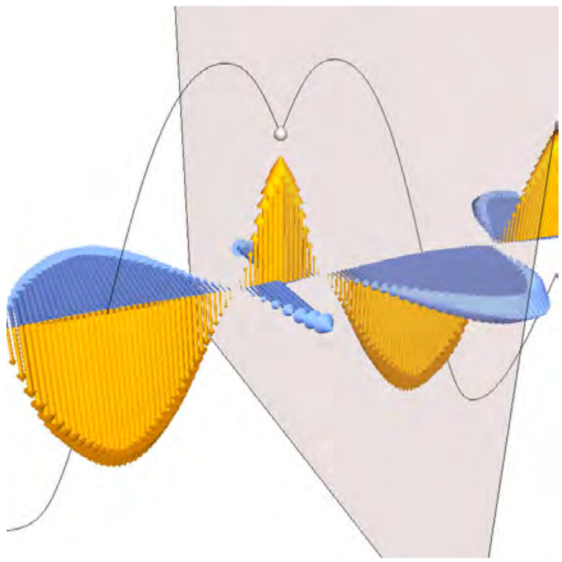
(d) What is the speed of the wave (in m/s)?

(e) Write an expression for the associated magnetic field required by Maxwell's equations. Indicate the vector direction of $\vec{\mathbf{B}}$ with a unit vector and a + or − sign, and you should give a numerical value for the amplitude in units of tesla.

(g) The wave emerges from the medium through which it has been propagating and continues in vacuum. What is the new wavelength of the wave (in meters)?

13.15 Generating Plane Electromagnetic Waves Simulation

In this section we give you access to an interactive 3D simulation that allows you to generate a plane wave in the manner described quantitatively in Sections 13.8.4 and 13.8.5 above.



Simulation

Figure 13.15.1 Screen Shot of Generating Plane Wave Simulation

The simulation allows you to generate plane electromagnetic waves by shaking a positive sheet of charge (the pink transparent sheet in Figure 13.15.1). If the "Motion on" box is checked (as it is when you first enter the simulation), the sheet of charge oscillates up and down automatically sinusoidally, generating the plane waves we expect to see from the explanation in Sections 13.8.4 and 13.8.5 above. If the "Motion on" box is unchecked, this motion ceases, and you can generate your own plane wave by left clicking and dragging on the silver ball embedded in the pink sheet. You can generate a "burst" of radiation or you can try to generate a sinusoidal wave, as you wish.

In this simulation, the black line perpendicular to the pink sheet represents a field line of the total electric field, including the static field and the radiation field. The orange arrows represent the radiation field only, with no static field. The blue arrows represent the radiation magnetic field.