An Approximation of FOCUSS Mean Squared Error

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- Several operations in science and engineering need to get back
  - a desired signal $\boldsymbol{x} \in \mathbb{R}^{N \times 1}$

$$\boldsymbol{x} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}_{1} \left.\vphantom{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}\right\} N \quad (1)$$

  - from a set of observed data or measured data $\boldsymbol{b} \in \mathbb{R}^{M \times 1}$

$$\boldsymbol{b} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}}_{1} \left.\vphantom{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}}\right\} M \quad (2)$$

- based on a modeling matrix or measurement matrix $\boldsymbol{A} \in \mathbb{R}^{M \times N}$,

$$\boldsymbol{A} = \underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{M,1} & a_{M,2} & \cdots & a_{M,N} \end{bmatrix}}_{N} \Bigg\} M \tag{3}$$

  which either
  - depends on the model or
  - can be chosen beforehand,

where

- $M \in \mathbb{N}^{1 \times 1}$ and
- $N \in \mathbb{N}^{1 \times 1}$

are the lengths of

- real-valued output data $\boldsymbol{b}$ and
- real-valued input data $\boldsymbol{x}$,

respectively.

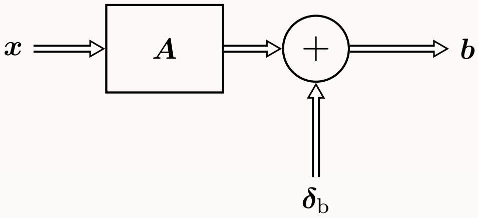

Figure 1: A linear system with noisy output.

- In Fig. 1, the linear system can be written as

$$\boldsymbol{A}\boldsymbol{x} + \boldsymbol{\delta}_{\mathrm{b}} = \boldsymbol{b}, \tag{4}$$

or, if the perturbation $\boldsymbol{\delta}_{\mathrm{b}}$ is negligible, approximately as

$$\boldsymbol{A}\boldsymbol{x} \approx \boldsymbol{b}. \tag{5}$$

- An iterative computation was introduced previously as follows.

Algorithm 1 FOCal Underdetermined System Solver (FOCUSS) [1, 2, 3]

Input: $\boldsymbol{A} \in \mathbb{R}^{M \times N}$, $\boldsymbol{b} \in \mathbb{R}^{M \times 1}$, $p \in (0, 1]$,
$N_{\max} \in \mathbb{Z}_{+}^{1 \times 1}$, $\epsilon_{\min} \in \mathbb{R}_{+}^{1 \times 1}$

Output: $\hat{\boldsymbol{x}}_p \in \mathbb{R}^{N \times 1}$

$\hat{\boldsymbol{x}}[0] \leftarrow \mathbf{1}$

$i \leftarrow 0$

$\epsilon_{\hat{x}} \leftarrow \epsilon_{\min} + 1$

while $\epsilon_{\hat{x}} > \epsilon_{\min} \wedge i \leq N_{\max}$ **do**

$i \leftarrow i + 1$

$\hat{\boldsymbol{x}}[i] \leftarrow \boldsymbol{\Phi}_p(\hat{\boldsymbol{x}}[i-1])\boldsymbol{b}$

$\epsilon_{\hat{x}} \leftarrow \frac{\|\hat{\boldsymbol{x}}[i] - \hat{\boldsymbol{x}}[i-1]\|_2}{\|\hat{\boldsymbol{x}}[i-1]\|_2}$

end while

return $\hat{\boldsymbol{x}}[i]$

- We may approximate the mean-squared error of the FOCUSS estimate as

$$\begin{aligned}\mathcal{E}_{\delta_\mathrm{b}}\{\|\hat{\boldsymbol{x}}_p - \boldsymbol{x}_0\|_2^2\} \\ \approx \sigma_{\delta_\mathrm{b}}^2 \operatorname{tr}\left(\boldsymbol{\Phi}_p^\mathsf{T}(\boldsymbol{x}_0)\boldsymbol{\Phi}_p(\boldsymbol{x}_0)\right) - 2\boldsymbol{x}_0^\mathsf{T}\boldsymbol{\Phi}_p(\boldsymbol{x}_0)\boldsymbol{A}\boldsymbol{x}_0 \\ + \boldsymbol{x}_0^\mathsf{T}\boldsymbol{A}^\mathsf{T}\boldsymbol{\Phi}_p^\mathsf{T}(\boldsymbol{x}_0)\boldsymbol{\Phi}_p(\boldsymbol{x}_0)\boldsymbol{A}\boldsymbol{x}_0 + \boldsymbol{x}_0^\mathsf{T}\boldsymbol{x}_0,\end{aligned} \tag{6}$$

  where
  - $\|\cdot\|_p$ is the ℓ_p norm,
  - $\sigma_{\delta_\mathrm{b}}^2$ is the variance of $\boldsymbol{\delta}_\mathrm{b}$,
  - $\boldsymbol{\Phi}_p(\boldsymbol{x}) \in \mathbb{R}^{N\times M}$ is given by

$$\boldsymbol{\Phi}_p(\boldsymbol{x}) = \boldsymbol{D}(|\boldsymbol{x}|^{2-p})\boldsymbol{A}^\mathsf{T}\left(\boldsymbol{A}\boldsymbol{D}(|\boldsymbol{x}|^{2-p})\boldsymbol{A}^\mathsf{T}\right)^{-1}, \tag{7}$$

  - $\cdot^\mathsf{T}$ being the transpose of a vector or a matrix $\cdot$, and
  - $\boldsymbol{x}_0$ is the true value of $\boldsymbol{x}$.

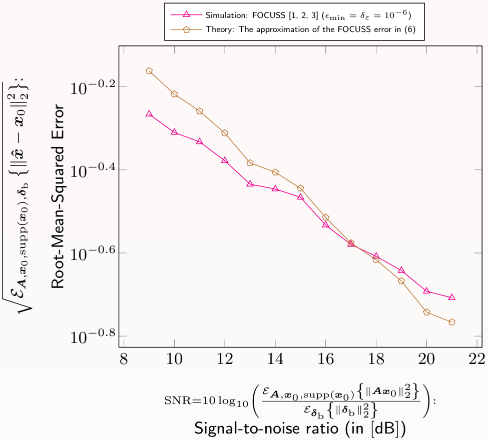

Figure 2: RMSE as a function of SNR from $N_{\boldsymbol{A}} = N_{\boldsymbol{x}_0} = 334$, $N_{\mathrm{supp}(\boldsymbol{x}_0)} = 100$, $N_{\boldsymbol{\delta}_\mathrm{b}} = 100$, and $N_\mathrm{R} = N_{\boldsymbol{A}} N_{\mathrm{supp}(\boldsymbol{x}_0)} N_{\boldsymbol{\delta}_\mathrm{b}} = 3{,}340{,}000$ independent runs for $K = 3$, $M = 19$, $N = 64$, $N_{\max} = 100$, $p = 0.9$, and $\epsilon_{\min} = \delta_\varepsilon = 10^{-6}$.

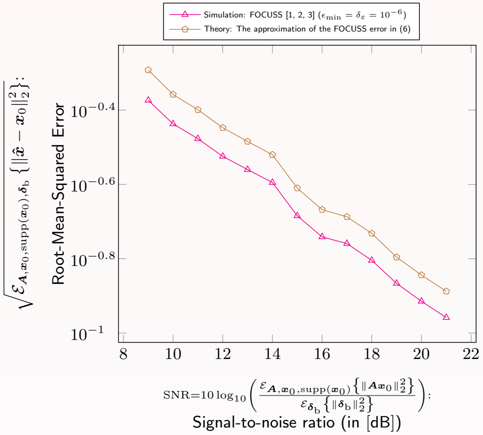

Figure 3: RMSE as a function of SNR from $N_{\boldsymbol{A}} = N_{\boldsymbol{x}_0} = 500$, $N_{\mathrm{supp}(\boldsymbol{x}_0)} = 100$, $N_{\boldsymbol{\delta}_\mathrm{b}} = 100$, and $N_\mathrm{R} = N_{\boldsymbol{A}} N_{\mathrm{supp}(\boldsymbol{x}_0)} N_{\boldsymbol{\delta}_\mathrm{b}} = 5{,}000{,}000$ independent runs for $K = 2$, $M = 51$, $N = 128$, $N_{\max} = 100$, $p = 0.9$, and $\epsilon_{\min} = \delta_\varepsilon = 10^{-6}$.

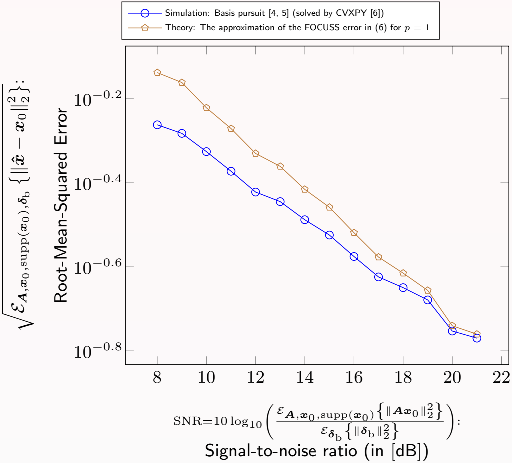

Figure 4: RMSE as a function of SNR from $N_{\boldsymbol{A}} = N_{\boldsymbol{x}_0} = 334$, $N_{\mathrm{supp}(\boldsymbol{x}_0)} = 100$, $N_{\boldsymbol{\delta}_\mathrm{b}} = 100$, and $N_\mathrm{R} = N_{\boldsymbol{A}} N_{\mathrm{supp}(\boldsymbol{x}_0)} N_{\boldsymbol{\delta}_\mathrm{b}} = 3{,}340{,}000$ independent runs for $K = 3$, $M = 19$, and $N = 64$.

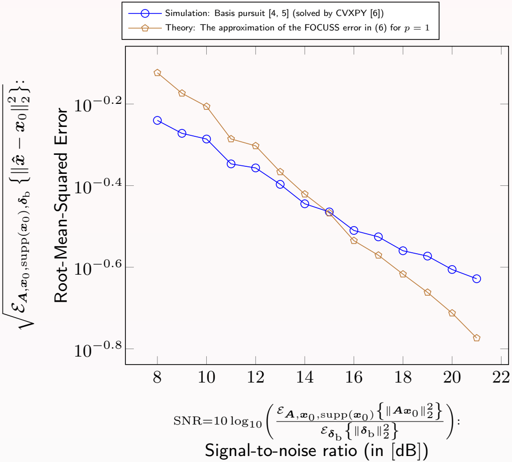

Figure 5: RMSE as a function of SNR from $N_{\boldsymbol{A}} = N_{\boldsymbol{x}_0} = 334$, $N_{\mathrm{supp}(\boldsymbol{x}_0)} = 100$, $N_{\boldsymbol{\delta}_\mathrm{b}} = 100$, and $N_\mathrm{R} = N_{\boldsymbol{A}} N_{\mathrm{supp}(\boldsymbol{x}_0)} N_{\boldsymbol{\delta}_\mathrm{b}} = 3{,}340{,}000$ independent runs for $K = 3$, $M = 20$, and $N = 128$.

- It is shown that the FOCUSS is approximately a biased estimator.
- A closed-form expression is provided for approximating the MSE from the FOCUSS.
- The derived MSE is quite close to the actual estimation error.
- For $p = 1$, our expression still can capture the error performance by the basis pursuit.

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Thank you for your attention