Design and Stability Guarantee of Variable Center-Frequency Bandpass Filters

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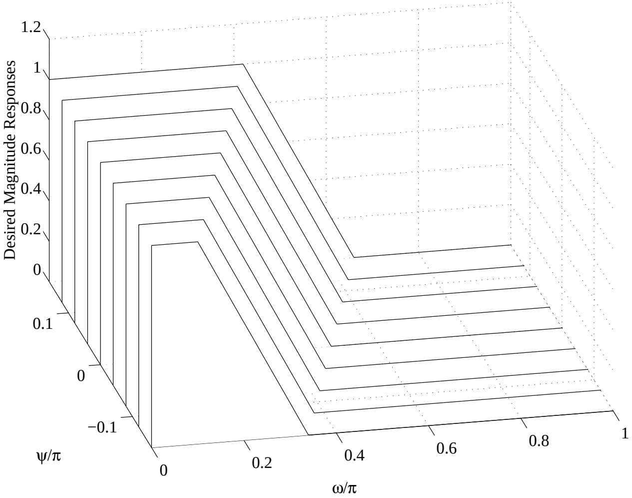
Variable Digital Filter ?

Digital filter with variable frequency response

Example: magnitude response with variable cut-off frequency

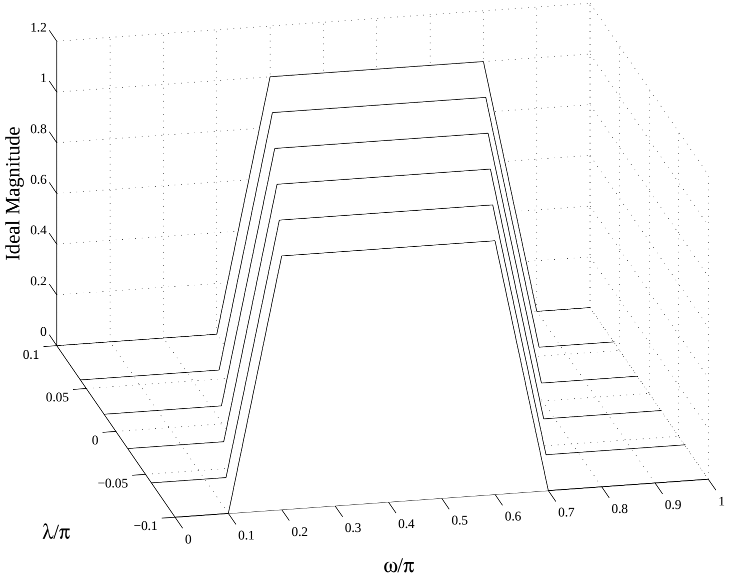


Objectives

This presentation aims to

1. propose a parameter-transformation method for guaranteeing the stability of a bandpass filter with variable center-frequency (VCF) and fixed passband width
2. present a two-step method for designing VCF bandpass filter
3. minimize the p-norm magnitude-response error using non-linear programming
4. use an example to show the achieved stability and accuracy

Bandpass Magnitude with Variable Center-Frequency (VCF)



The variable bandpass magnitude (specification) is defined by

$$F_d(\omega,\lambda) = \begin{cases} 0, & \omega \in [0, \omega_{s1}] \longrightarrow \text{stopband} \\ \dfrac{\omega - \omega_{s1}}{\omega_{p1} - \omega_{s1}}, & \omega \in [\omega_{s1}, \omega_{p1}] \longrightarrow \text{transition band} \\ 1, & \omega \in [\omega_{p1}, \omega_{p2}] \longrightarrow \text{passband} \\ \dfrac{\omega_{s2} - \omega}{\omega_{s2} - \omega_{p2}}, & \omega \in [\omega_{p2}, \omega_{s2}] \longrightarrow \text{transition band} \\ 0, & \omega \in [\omega_{s2}, \pi] \longrightarrow \text{stopband} \end{cases} \tag{1}$$

where $\omega \in [0, \pi]$ denotes the normalized angular frequency,

two passband edge frequencies

$$\begin{cases} \omega_{p1} = \omega_c - 0.2\pi = 0.3\pi + \lambda \\ \omega_{p2} = \omega_c + 0.2\pi = 0.7\pi + \lambda \end{cases}$$

two stopband edge frequencies

$$\begin{cases} \omega_{s1} = \omega_{p1} - 0.1\pi = 0.2\pi + \lambda \\ \omega_{s2} = \omega_{p2} + 0.1\pi = 0.8\pi + \lambda \end{cases}$$

The passband center-frequency

$$\omega_c = \frac{\omega_{p1} + \omega_{p2}}{2} = 0.5\pi + \lambda$$

is varied by using the parameter

$$\lambda \in [-0.10\pi, 0.10\pi]$$

Transfer Function with Variable Coefficients

We utilize the variable filter

$$F(z,\lambda) = \frac{\sum_{i=0}^{P_1} b_i(\lambda) z^{-i}}{\prod_{i=1}^{P_2} \left(1 + a_{i,1}(\lambda) z^{-1} + a_{i,2}(\lambda) z^{-2}\right)} \quad (2)$$

to approximate the magnitude-specification $M_d(\omega,\lambda)$ by minimizing the p-norm error. Here, all the coefficients are the functions of the center-frequency parameter λ.

How to get the variable coefficients ?

variable coefficients

$$b_i(\lambda), a_{i,1}(\lambda), a_{i,2}(\lambda)$$

we adopt a 2-step procedure:

- **Step-1**: design constant bandpass filters
- **Step-2**: use polynomials to fit coefficient values from Step-1

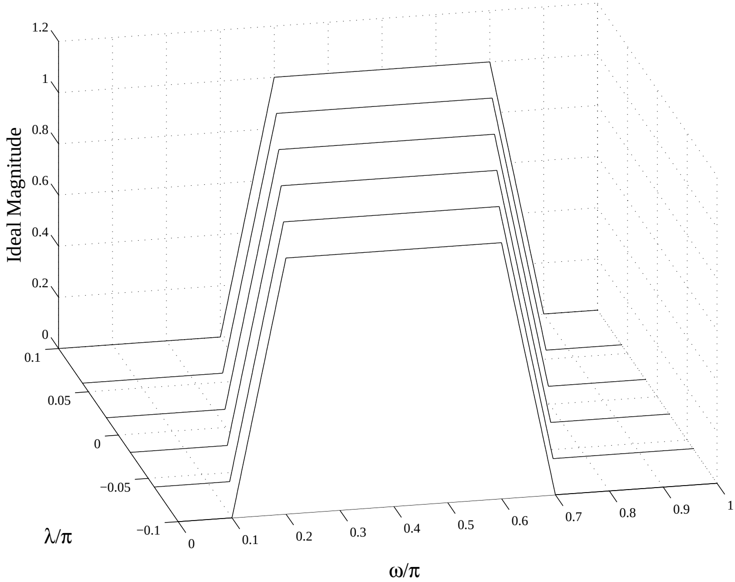
Two-Step Procedure

Step-1: Designing Constant Bandpass Filters

By discretizing $\lambda \in [-0.10\pi, 0.10\pi]$ with step size $0.20\pi/(L-1)$, and $L = 6$, we get L (6) uniformly discretized specifications

$$\begin{cases} F_d(\omega, \lambda_1) \\ F_d(\omega, \lambda_2) \\ \vdots \\ F_d(\omega, \lambda_6) \end{cases} \tag{3}$$

Discretized VCF Bandpass Specifications ($L = 6$)



Designing constant bandpass filters

$$F_l(z) = \frac{\sum_{i=0}^{P_1} b_i z^{-i}}{\prod_{i=1}^{P_2} (1 + a_{i,1} z^{-1} + a_{i,2} z^{-2})} \tag{4}$$

for approximating $F_d(\omega, \lambda_l)$ through minimizing the p-norm error of the magnitude response (non-linear minimization).

$$\begin{aligned} F_1(z) &\longmapsto F_d(\omega, \lambda_1) \\ F_2(z) &\longmapsto F_d(\omega, \lambda_2) \\ F_3(z) &\longmapsto F_d(\omega, \lambda_3) \\ F_4(z) &\longmapsto F_d(\omega, \lambda_4) \\ F_5(z) &\longmapsto F_d(\omega, \lambda_5) \\ F_6(z) &\longmapsto F_d(\omega, \lambda_6) \end{aligned} \tag{5}$$

Stability Issue

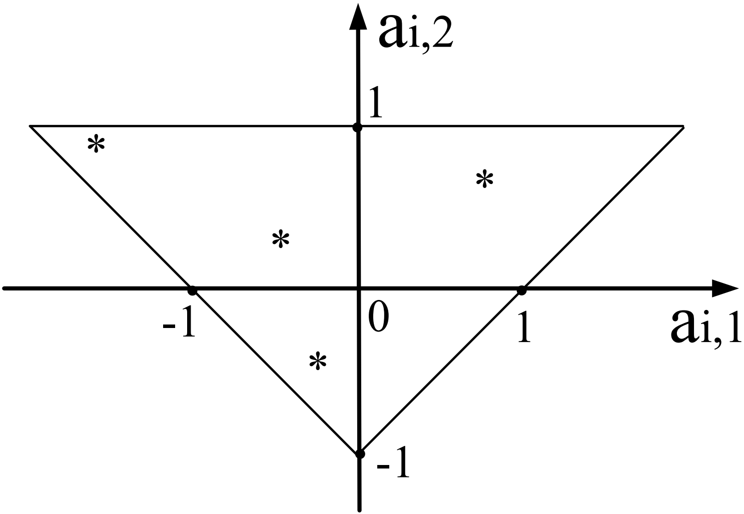
Consider the constant bandpass filter

$$F_l(z) = \frac{\sum_{i=0}^{P_1} b_i z^{-i}}{\prod_{i=1}^{P_2} (1 + a_{i,1} z^{-1} + a_{i,2} z^{-2})}$$

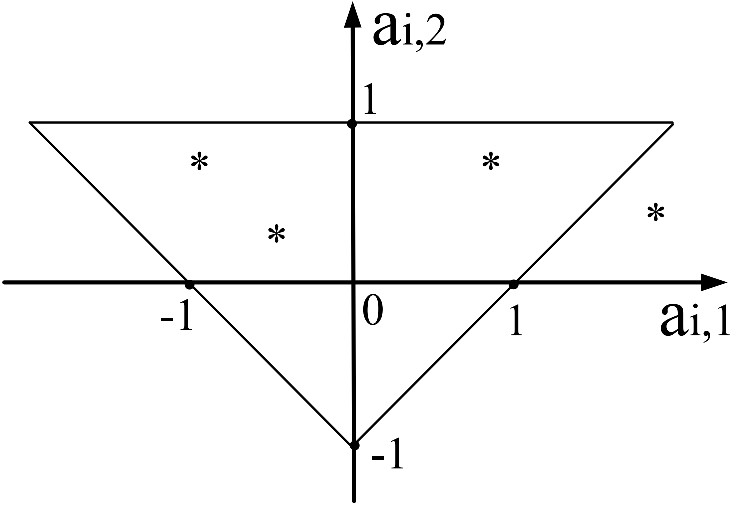
Stability Condition

$$\begin{cases} |a_{i,2}| < 1 \\ |a_{i,1}| < 1 + a_{i,2} \end{cases} \quad (6)$$

Stability Triangle (Stable Case)



Stability Triangle (Unstable Case)



Stability-Guarantee Strategy: Parameter Transformations

$$\begin{cases} a_{i,2} = \zeta \cdot I(x_{i,2}) \\ a_{i,1} = \zeta \cdot I(x_{i,1})(1 + a_{i,2}) \end{cases} \tag{7}$$

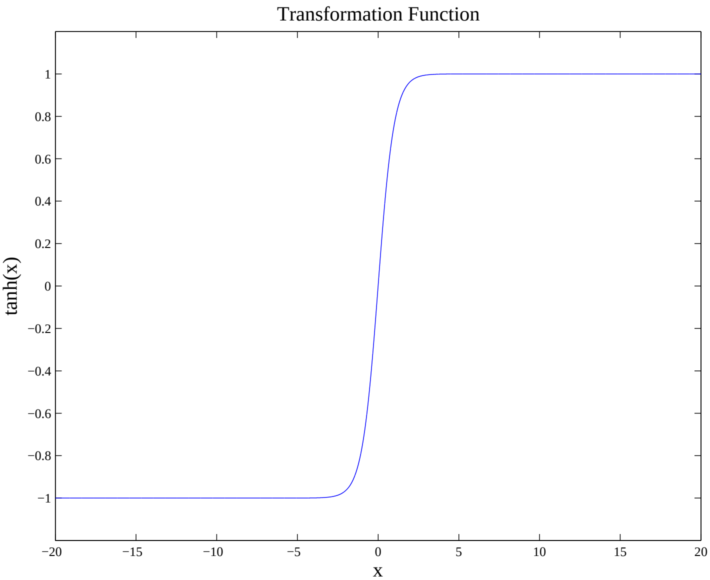
with

$$\begin{aligned} & I(x) \in [-1, 1] \\ & 0 < \zeta < 1 \end{aligned} \tag{8}$$

Here, we use

$$I(x) = \tanh(x)$$

$\tanh(x)$: Transformation Function



The unknowns to be found after parameter transformations

$$\boldsymbol{y} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{P_1} \\ x_{1,2} \\ x_{2,2} \\ \vdots \\ x_{P_2,2} \\ x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{P_2,1} \end{bmatrix}$$

Minimizing p-Norm Error

$$E_p = \left\{ \sum_{j=1}^{J} W(\omega_j)|e(\omega_j)|^p \right\}^{1/p} \tag{9}$$

where ω_j is the j-th frequency sample in $[0, \pi]$, and

$$e(\omega_j) = F_d(\omega_j, \lambda_l) - M(\omega_j)$$

is the error at ω_j, $M(\omega_j)$ is the magnitude response of $F_l(z)$, and

$$W(\omega_j) = \begin{cases} 1, & \omega_j \in \{\text{passband and stopband}\} \\ 0, & \omega_j \in \{\text{transition band}\} \end{cases} \tag{10}$$

is a weighting function.

This step produces $F_l(z)$ with different coefficients

$$
\begin{aligned}
F_1(z) &\longleftrightarrow \{b_i, x_{i,2}, x_{i,1}\} \\
F_2(z) &\longleftrightarrow \{b_i, x_{i,2}, x_{i,1}\} \\
&\vdots \\
F_L(z) &\longleftrightarrow \{b_i, x_{i,2}, x_{i,1}\}
\end{aligned} \tag{11}
$$

unknown vector: $\{b_i, x_{i,2}, x_{i,1}\}$

$$
\left[b_0 \ b_1 \ \cdots \ b_{P_1} \quad x_{1,2} \ x_{2,2} \ x_{3,2} \ \cdots \ x_{P_2,2} \quad x_{1,1} \ x_{2,1} \ x_{3,1} \ \cdots \ x_{P_2,1} \right]
$$

Step-2: Finding Polynomials as Variable Coefficients

Fitting a polynomial to the resulting values of each coefficient

$$\left[\begin{array}{c|c|c} b_0 \ b_1 \ \cdots \ b_{P_1} & x_{1,2} \ x_{2,2} \ x_{3,2} \ \cdots \ x_{P_2,2} & x_{1,1} \ x_{2,1} \ x_{3,1} \ \cdots \ x_{P_2,1} \end{array}\right]$$

produces the polynomials

$$b_i(\lambda), x_{i,2}(\lambda), x_{i,1}(\lambda)$$

Finally, we get

Variable Coefficients

$$\begin{cases} a_{i,2}(\lambda) = \zeta \cdot I(x_{i,2}(\lambda)) \\ a_{i,1}(\lambda) = \zeta \cdot I(x_{i,1}(\lambda))[1 + a_{i,2}(\lambda)] \end{cases}$$

Computer Simulations

Design Parameters

$$\begin{cases} P_1 = 8 \\ P_2 = 4 \\ p = 18 \\ \zeta = 0.999999 \\ J = 501 \longrightarrow \text{ number of frequency samples } \omega_j \\ L = 6 \longrightarrow \text{ number of samples } \lambda_l \end{cases}$$

Initial Coefficient Values

The constant bandpass filters $F_l(z)$ are designed one by one. $F_1(z)$ (the first one) is designed by using the initial values

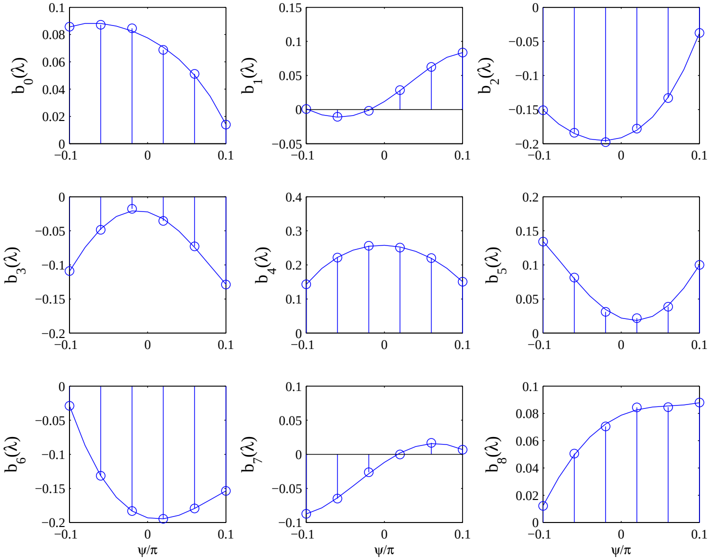
$$\begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_{1,2} & x_{2,2} & x_{3,2} & x_{4,2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

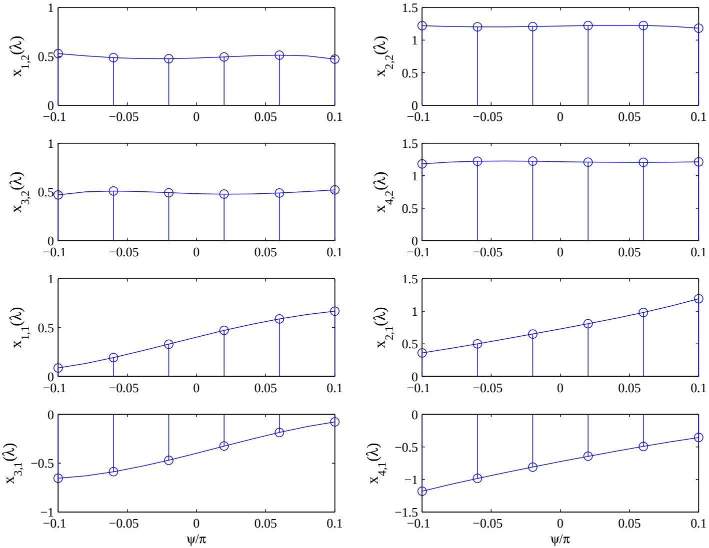
$$\begin{bmatrix} x_{1,1} & x_{2,1} & x_{3,1} & x_{4,1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

After $F_1(z)$ is designed, its coefficient values are used as the initial values for designing $F_2(z)$. This process is repeated until the last $F_L(z)$ is designed.

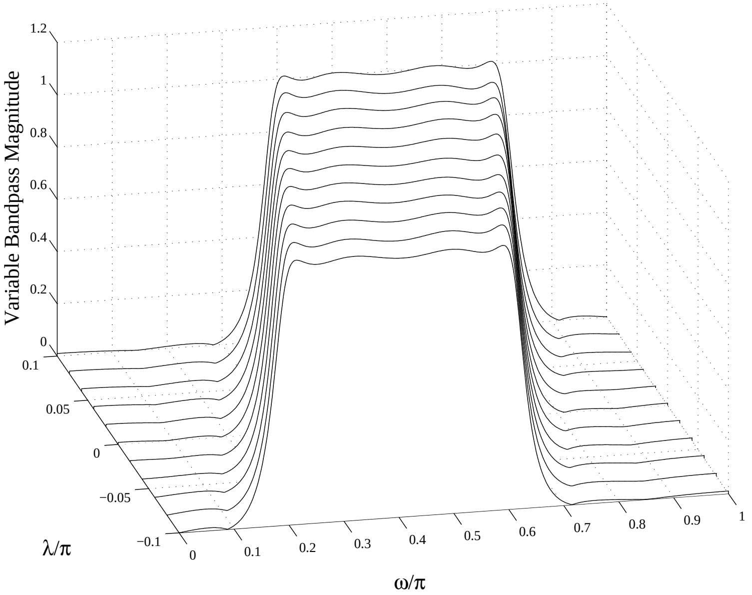
Fourth-Order Polynomials $b_i(\lambda)$



Fourth-Order Polynomials $x_{i,2}(\lambda), x_{i,1}(\lambda)$



Variable Magnitude Responses ($LL = 11$)



Error Criterion

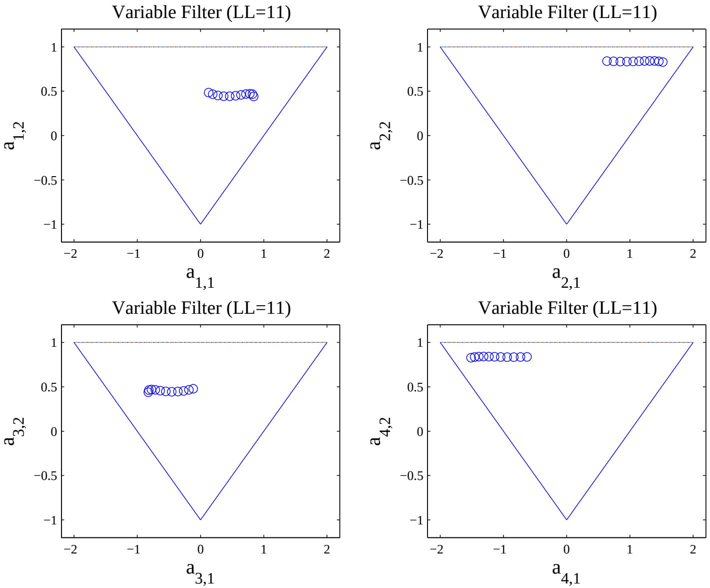
♣ p-Norm Error

$$E_p = \left\{ \sum_{j=1}^{J} |e(\omega_j)|^p \right\}^{1/p}$$

p-Norm Error (Mean Value)

$$\overline{E}_p = 0.016608$$

Stability Triangles



Conclusions

1. This paper has shown how to guarantee the stability of a bandpass filter with variable center-frequency (VCF)
2. The transformation-based stability-guarantee strategy has been incorporated into the 2-step procedure for getting a stable VCF bandpass filter
3. Computer simulations have verified the guaranteed stability and high accuracy