Robust Process Capability Indices for Multivariate Linear Profiles

Golnoosh Toosi*, Mohammad Mahdi Ahmadi**

Abstract— The quality of a product or process is an important issue for both customers and producers. The quality could be defined as a linear relationship between the response variable (s) and explanatory variable (s), which is called a linear profile. Another essential concept in quality control is the adaptation of quality specifications with customers' standards. The proper tool to measure customers' specifications is process capability indices (PCIs). To find the PCIs for profiles, the profile parameters should be estimated. These parameters can be estimated using classic estimators. However, in the presence of outliers, the classic estimators do not estimate the parameters accurately. Therefore, the performance of the classic indices using classic estimators is appropriate only in the absence of contamination. In this research, robust estimate methods such as M-estimator and LR-weighted MCD estimators are used to propose robust PCIs for multivariate linear profiles. The proposed robust indices include C_{pm} and MC_{pc} for a multivariate linear model. The performance of the proposed robust PCIs is compared with the classic PCIs in the absence and presence of contamination. The result of simulation studies shows that robust PCIs perform better than classic PCIs in the presence of outliers. In the absence of contamination, the robust PCIs perform as accurately as classic PCIs. The proposed PCIs using LR-weighted MCD outperform the M-estimator method in all considered contamination scenarios.

I. INTRODUCTION

The quality of a product or process defines by a linear relationship between the explanatory variable(s) and the response variable(s). This relationship is called a "Profile". Kang and Albin introduced the profile concept in the study in which they proposed two methods for monitoring simple linear profiles in phase II [1]. The process capability index is another quality control concept that quantifies product or process performance and measures customers' specifications. Many PCIs are investigated to evaluate this adaptation. PCIs help to specify whether a process is in statistical control or not. Many univariate process capability indices are proposed. Multivariate process capability indices are investigated for processes with multiple correlated quality specifications. The first step in calculating PCIs is to estimate the unknown parameters of the model. However, the classic estimators do not perform accurately in the presence of outliers. Outliers are those observations that do not have the same distribution in comparison to other observations normal and can cause contamination in the data. To overcome this problem, robust estimators are introduced which are not affected by the outliers and are robust in the presence of contaminations. The parameters of the profile model are usually estimated using classic estimators. The classic estimators are not robust when there are outliers in the data. To overcome this problem, robust estimators are used which are not affected by the outliers and can estimate the parameters accurately. Ebadi and Shahriari proposed a robust method to estimate the parameters in a simple linear profile and compared the performance of robust estimators with classic methods in presence of outliers [2]. Kordestani et al used three robust methods to estimate the parameters of multivariate simple linear profiles in two stages [3]. Hassanvand et al used two robust methods, M-estimator including Huber's and Bisquare to estimate the process parameters with different autocorrelation rates across the stages of two-stage process in simple linear profiles [4]. Mahpouya et al designed a multivariate robust T^2 control chart based on a bootstrap re-sampling technique and robust estimators in the presence of outliers [5]. Kang et al, Rabbani et al, and Sedaghat et al in separate studies, consider novel optimization schemes which can be used in minimizing the loss function in estimating the profile parameters [6],[7],[8]. Ahmadi and Shahriari introduced a robust control chart for simple linear profiles [9]. Based on the robust estimates they got in phase I monitoring of the simple linear profile, they constructed a control chart that can detect the shifts much quicker than the classic control charts. Khatibi and Dezyani used a regression-based model in machine learning and CNN to analyze the medical video frames [10]. Navazi et al used regression models to investigate the effect of stay-home orders during the Covid-19 pandemic [11]. Saraei et al applied the statistical analysis method in the system parameters to find the most optimum factor of an experiment [12].

The process capability indices are obtained using parameter estimates based on different estimation methods. Wang and Chen used principal components analysis (PCA) to derive the process capability index based on the multivariate normal processes data [13]. Sarrafian and Shahriari proposed a process capability index for the simple linear profile [14]. Niavarani et al presented three PCIs for the multivariate environment [15]. Two new PCIs is proposed that estimate process performance for simple linear profiles [16]. Parameters in this study are estimated by the least squares (LS) estimator which is sensitive to the presence of outliers. Ebadi and Amiri proposed three PCIs for a Multivariate simple linear profile [17]. They also employed principal component analysis to propose their new PCI. Their proposed PCIs for a multivariate simple linear profile failed to estimate the capability of the process in presence of outliers as parameters are all estimated by

*Golnoosh Toosi was with Department of Industrial Engineering, K. N. Toosi University of Technology, Tehran, Iran (email: golnoosh2c@gmail.com)

**Mohammad Mahdi Ahmadi is with Department of Systems and Industrial Engineering, The University of Arizona, Tucson, Arizona, USA (email: ahmadi@arizona.edu)

classic methods. Karimi Ghartemani et al utilized a multivariate PCI vector to estimate process capability indices in simple linear profiles [18]. Mehri et al introduced two robust capability indices for multiple linear profiles [19]. Their proposed robust PCIs are calculated using the M-estimator and Fast-τ estimator to estimate parameters in multiple linear profiles in presence of outliers. The performance of the proposed robust indices is much better than classic indices.

In this research, the robust process capability indices for multivariate simple linear profiles are introduced. To the best of our knowledge, no research has been done to determine the process capability performance in multivariate linear profiles in presence of outliers. Therefore, in this study, we investigate the parameters of multivariate linear profiles using robust estimators, M-estimator and LR-Weighted MCD, and defined robust PCIs for these types of profiles.

The rest of the paper is structured as follows: In Section 2, the robust estimation methods are introduced briefly. In Section 3, three robust principal components are proposed. In Section 4, simulation studies are analyzed. Conclusions and future works are discussed in Section 5.

II. Robust Estimation of multivariate Simple linear profiles

A multivariate simple linear profile characterizes the quality of a process by a multivariate regression model consisting of an explanatory variable and some correlated response variables. M samples of size n fixed-value explanatory variables are available [20]. For each value of the explanatory variable, p corresponding response values are recorded. The multivariate linear profile can be defined as (1):

$$\mathbf{Y}_k=\boldsymbol{X_k}\mathbf{B}_k+E_k \quad K=1,2,\ldots,m \qquad (1)$$

where, $\mathbf{Y}_k=(y_{1k},y_{2k},\ldots,y_{nk})^T$ is an n×p matrix of the response variables for the k^{th} sample, and $\mathbf{X}_k=[1\ \mathbf{x}_k]$ is an n×2 matrix of the explanatory variables. For simplicity, we consider fixed values for the explanatory variables, and we take the same set of values for each sample. $\mathbf{B}_k=(\boldsymbol{\beta}_{0K},\boldsymbol{\beta}_{1K})^T$ is a 2×p matrix of the model parameters. $\boldsymbol{E_k}=(\mathbf{e}_{1k},\mathbf{e}_{2k},\ldots,\mathbf{e}_{nk})^T$ is n×p matrix of error terms for the k^{th} sample and the vector of regression errors at each point $\mathbf{e}_{ik}$ follows a p-variate Normal distribution with a mean vector 0 and p×p variance-covariance matrix $\sum_k$. The variance-covariance matrix of the error terms is defined as (2), where σ_{ij} represents the covariance between i^{th} and j^{th} error vector terms at each observation.

$$\sum_k=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p}\\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp}\end{bmatrix} \qquad (2)$$

A. M-estimator

M-estimator proposed by Huber; in this method, a function of the error is substituted with the squared of errors in the least squares method [21]. With this replacement, the effect of outliers will be nullified when estimating the model parameters. The parameters are obtained by minimizing the (3),

$$\min\sum_{i=1}^{n}\rho\left(\frac{e_i}{s}\right)=\min\sum_{i=1}^{n}\rho\left(\frac{Y_i-x_i'\beta}{s}\right) \qquad (3)$$

where ρ is a function from a family of Huber or Bisquare functions, S is the robust estimate of σ and e is the residual. Equation (4) is used to calculate S:

$$s=\frac{\text{median }|e_i-\text{median}(e_i)|}{0.6745} \qquad (4)$$

According to Kudraszow and Maronna in a multivariate linear profile, due to the correlation between response variables, Mahalanobis norms of the residuals are replaced by univariate residuals [22]. In this case, for matrixes $B\in R^{p\times k}$ and $\Sigma\ni R^{q\times k}$, Mahalanobis norm of residuals is calculated as:

$$d_i(\mathbf{B},\boldsymbol{\Sigma})=(\hat{\mathbf{u}}_i(\mathbf{B})^T\,\boldsymbol{\Sigma}^{-1}\,\hat{\mathbf{u}}_i(\mathbf{B}))^{\frac{1}{2}} \qquad (5)$$

where, $\hat{\mathbf{u}}_i(\mathbf{B})=\mathbf{y}_i-\mathbf{B}^T\mathbf{x}_i$. Therefore, estimating parameters of multivariate linear profiles can be obtained as follows:

$$\min\sum_{i=1}^{n}\rho\left(\frac{d_i(\hat{\beta})}{s}\right)=\min\sum_{i=1}^{n}\rho\left(\frac{[(y_i-\mathbf{B}'x_i)'\boldsymbol{\Sigma}^{-1}(y_i-\mathbf{B}'x_i)]^{\frac{1}{2}}}{s}\right) \qquad (6)$$

$$s=\frac{\text{median}\left|d_i(\hat{\beta})-\text{median}(d_i(\hat{\beta}))\right|}{0.6745} \qquad (7)$$

Values of $(\mathbf{B})$ and $(\boldsymbol{\Sigma})$ are obtained by calculating the derivatives of (6) with respect to the coefficient and covariance matrices. Then, the Iteratively reweighted least squares (IRLS) method is used to achieve solutions to the optimization problem.

B. Minimum Covariance Determinant (MCD) Regression

The LR-Weighted MCD estimator is a robust estimator that can be used to estimate the parameters of a multivariate linear regression model in the presence of outliers [23]. The algorithm consists of two steps.

- Step 1: Estimation of the Location Vector $(\hat{\mu})$

The MCD estimator is obtained by minimizing the determinant of the covariance matrix subject to a constraint on the number of data points used in the estimation. The MCD estimator is given by:

$$\hat{\mu}=\text{median}(Y_k)+c_p.\ \text{mad}(Y_k)\ ,\ K=1,2,\ldots,m \quad (8)$$

size n, and mad (Y_k) is the median absolute deviation of the m samples. The constant c_p depends on the dimension of the data and the breakdown point of the estimator, which determines the proportion of outliers that can be accommodated.

- Step 2: Estimation of the Regression Parameters

The regression parameters are estimated using a weighted linear regression model. The weights are determined based on the Mahalanobis distance of each observation to the estimated location vector. According to Maronna et al, The Mahalanobis distance is a measure of the distance between a point and a distribution, taking into account the covariance structure of the distribution [24]. The weighted linear regression model is given by:

$$Y_k - \hat{\mu} = X_k B_k + e_k \tag{9}$$

where e_k is the $n \times p$ matrix of the regression errors. The weights are given by:

$$\omega_{ij} = \frac{d_{ij}^{-2/(p+1)}}{\sum_{i=1}^{n} d_{ij}^{-2/(p+1)}} \tag{10}$$

where d_{ij} is the Mahalanobis distance between the i^{th} observation and the estimated location vector, and the sum in the denominator is taken over all n observations. The regression parameters are estimated by minimizing the weighted sum of squares:

$$\widehat{B}_k = \left(X_k^T W_k X_k\right)^{-1} X_k^T W_k (Y_k - \hat{\mu}) \tag{11}$$

where W_k is the n×n diagonal matrix of weights. The LR-Weighted MCD estimator combines the robustness of the MCD estimator with the efficiency of the weighted regression model to provide a robust and efficient estimator of the location vector and regression parameters.

III. Proposed Methods

A. Proposed Robust Capability Indices Based On the First Component of the Capability Vector

Shahriari et al proposed a three-component vector of multivariate process capability indices [C_{PM}, PV, LI] [25]. Ebadi and Amiri developed the multivariate process capability vector into a multivariate linear profile [17]. In the current research, robust estimators are used to estimate parameters to evaluate the robust PCIs. So, the M-estimator and fast LR-Weighted MCD are utilized to estimate the profile parameters. The first vector of the proposed process capability for the i=1,2,…,n and j=1,2,…,p is defined as:

$$\widetilde{C_{pm}} = \left[\frac{\prod_{j=1}^{p}\prod_{i=1}^{n}(USL_{ij} - LSL_{ij})}{\prod_{j=1}^{p}\prod_{i=1}^{n}(\widetilde{UPL}_{ij} - \widetilde{LPL}_{ij})}\right]^{\frac{1}{pn}} \tag{12}$$

where USL_{ij} and LSL_{ij} are upper and lower specification limits for the i^{th} level of the j^{th} response variable. $\widetilde{UPL}_{ij}$ and $\widetilde{LPL}_{ij}$ are modified process regions, which are simplified by Härdle and Simar in (13) [26].

$$\widetilde{UPL}_{ij} = \tilde{\mu}_{ij} + \sqrt{X_{\alpha p}^2 \tilde{\sigma}_{ij}^2}\ , \qquad \widetilde{LPL}_{ij} = \tilde{\mu}_{ij} - \sqrt{X_{\alpha p}^2 \tilde{\sigma}_{ij}^2} \tag{13}$$

where $\tilde{\mu}_{ij}$ and $\tilde{\sigma}_{ij}^2$ are the robust estimates of the mean and variance for the i^{th} level of the $j^{th}$the response variable, respectively.

B. Robust Process Capability Indices Based on Principal Component Analysis

The PCI based on the principal component analysis was proposed by [13]. In this method, a set of correlated response variables is reduced to a set of linear combinations of uncorrelated variables. By recognizing variables, those retain a long portion of the information contained in the original set of data (approximately 80-90% of the observations). This method reduces the initial *P* response variables to the q principal components. The new components resulting from a dimension reduction are considered new response variables, and new relationships between these response variables and independent variables are generated. Therefore, q Principal components will be considered in the further analysis. According to [17] the process capability index for the multivariate linear profile can be estimated by the PCA method, however, their proposed PCI is not robust in the presence of outliers. Hence, in this research, the robust process capability index based on the PCA method is calculated by using eigenvalue ($l_1, l_2, \ldots, l_p$) and eigenvector ($e_1, e_2, \ldots, e_p$) of the variance-covariance matrix $\sum$ as follows:

$$\widetilde{MC_{pc}} = \left(\prod_{j=1}^{p}\prod_{i=1}^{n}\widehat{C}_{p;pcij}\right)^{\frac{1}{qn}} \tag{14}$$

where q is the number of principal components that retain 80-90% of variability, and $\widehat{C}_{p;pcij}$ is a univariate process capability for the i^{th} level of a j^{th} PC.

$$\widetilde{C}_{p;pcij} = \frac{USL_{pcij} - LSL_{pcij}}{6\tilde{S}_{pcj}} \tag{15}$$

where $\tilde{s}_{pcj}$ is a robust estimate of the variance-covariance matrix. Upper and lower specification limits calculate as:

$$S_{pcj}^2 = e_j^T S e_j \tag{16}$$

$$USL_{pcij} = e_j^T USL_i\ ,\ LSL_{pcij} = e_j^T LSL_i \tag{17}$$

Same as the $\widetilde{C_{PM}}$ PCI, the M-estimator, and fast LR-Weighted MCD is used to estimate the profile parameters in this PCI as well.

C. Efficiency of the Estimators

In this research, the efficiency of the estimators (M-estimator and LR-Weighted MCD) is evaluated. Efficiency means the closeness of an estimator to the intended parameter. To find the efficiency, Euclidean norm, and Condition number are used for each estimator. According to [24] to calculate the closeness of the variance-covariance matrix ($\widehat{\Sigma}$) to its actual value (Σ_0), Euclidean norm is defined in (20):

$$\mathrm{Norm}(\hat{\Sigma}\text{-}\Sigma_0)=\|\hat{\Sigma}\text{-}\Sigma_0\| \quad (20)$$

Also, the second efficiency estimator that examines the closeness of the variance-covariance matrix ($\hat{\Sigma}$) to its actual value (Σ_0), by the condition number that is defined as (21):

$$\mathrm{Cond}(\hat{\Sigma})=\frac{\lambda_l(\hat{\Sigma})}{\lambda_p(\hat{\Sigma})} \quad (21)$$

where, λ_l and λ_p are the largest and smallest eigenvalues, respectively. As much as the result of (20) and (21) are smaller, the estimator's performance is better.

IV. SIMULATION STUDIES

In this section, the regression model parameters and proposed PCI by the classic and robust estimators in the presence and absence of outliers are simulated. The model presented by [20] is utilized in this simulation. The proposed model is as follows:

$$Y_1=3+2x+\varepsilon_1 \ , \ Y_2=2+1x+\varepsilon_2 \quad (22)$$

where x_i values are [1,2, 3,…,10] and the error terms vector $(\varepsilon_1,\varepsilon_2)$ is a bivariate Normal random vector with mean vector zero and known covariance matrix $\Sigma_0=\begin{bmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho\\ \sigma_1\sigma_2\rho & \sigma_2^2\end{bmatrix}$. Also, for the sake of simplicity, $\sigma_1^2 = \sigma_2^2 = 1$ and $\rho = 0.5$.

In each simulation run, m = 20 profiles are generated. To consider the results in the presence of outliers, the simulations are conducted under the four following scenarios to estimate and compare the proposed PCIs.

- Absence of contamination
- Presence of contamination in error terms variance-covariance matrix
- Presence of contamination and shift in error terms matrix of the mean
- Presence of contamination and shift in both intercept and slope simultaneously

The above scenarios are compared under different percentages of outliers in data. The contamination percentages are assumed 0.05, 0.1, 0.15, and 0.2. It means that in each profile, δ100% of observations include outliers, and (1- δ)100% of the observations follow the profile model. In 1000 simulation runs, the multivariate linear profile parameters are estimated based on classic, M-estimator, and LR-Weighted MCD in the absence of contamination. As it is shown in Table I, all estimators perform similarly and close to their actual values in the absence of contamination. Small $\mathrm{Cond}(\hat{\Sigma})$ shows the efficiency of each estimator. In the second scenario, the performance of estimators is compared in the presence of contamination in error terms variance-covariance matrix. In this situation, δ100% of observations of each profile have error terms that follow Normal distribution, $NID((0,0),\Sigma_\delta)$, and (1- δ)100% of observations follow the considered profile model. In this case, δ100% of

Table I. Comparison of robust and classic estimates of model parameters in the absence of contamination

Estimators	$\hat{\beta}_{11}$	$\hat{\beta}_{12}$	$\hat{\beta}_{21}$	$\hat{\beta}_{22}$	$\hat{\Sigma}$	$\mathrm{Cond}(\hat{\Sigma})$
M	2.98	2.01	1.99	0.99	$\begin{bmatrix}0.842 & 0.416\\ 0.416 & 0.819\end{bmatrix}$	3.013
LR-weighted MCD	2.95	2.00	2.00	0.99	$\begin{bmatrix}0.926 & 0.455\\ 0.455 & 0.909\end{bmatrix}$	2.967
Classic	2.97	2.01	1.99	0.99	$\begin{bmatrix}1.0094 & 0.502\\ 0.502 & 0.984\end{bmatrix}$	3.036
Actual	3	2	2	2	$\begin{bmatrix}1 & 0.5\\ 0.5 & 1\end{bmatrix}$	3

observation error term variance-covariance is $\Sigma_\delta=\Sigma_0+\Sigma_\lambda$ that contaminated by $\Sigma_\lambda=\begin{bmatrix}2\lambda & \lambda\\ \lambda & 2\lambda\end{bmatrix}$ and λ=1,2,4,6,8 . In figure I, the classic and robust estimators of variance-covariance matrix are compared based on the different contamination percentages and the norm of variance-covariance matrix error terms. The figure shows by increasing the contamination percentage and norm of the variance-covariance matrix error terms, the classic estimator is affected by the contamination drastically, while the two robust estimators are less affected by contamination even in the large contamination percentages. Also, the results of simulation studies for the third and fourth scenarios are similar to the second scenario. In the following, the performance of proposed robust PCIs is investigated for the first and second scenarios. In this simulation study, the specification limits for different levels of the independent variable are considered as (23):

$$USL_{ij}=XB + 4 \ , \ LSL_{ij}=XB - 4 \quad (23)$$

where, $B=\begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix}$, $X=\begin{bmatrix}1 & 1\\ 1 & 2\\ \vdots & \vdots\\ 1 & 10\end{bmatrix}$. The actual PCI is calculated based on the actual parameter values of the profile model. The actual value of PCIs helps to compare the performance and accuracy of proposed robust and classic indices. As presented in Table II, the classic PCI has a better performance and is closer to the actual values than the robust PCIs. The simulation studies' results for scenarios two to four were alike, so only the second scenario's findings are discussed here.

Table II. Comparison of the robust and classic process capability indices in the absence of contamination

Estimators	$\hat{C}_{pm}$	$M\hat{C}_{pc}$	$NM\hat{C}_{pm}$
M	1.2329	1.1487	1.5440
LR-weighted MCD	1.3508	1.2569	1.6934
Classic	1.2865	1.1939	1.6176
Actual	1.2288	1.1630	1.5396

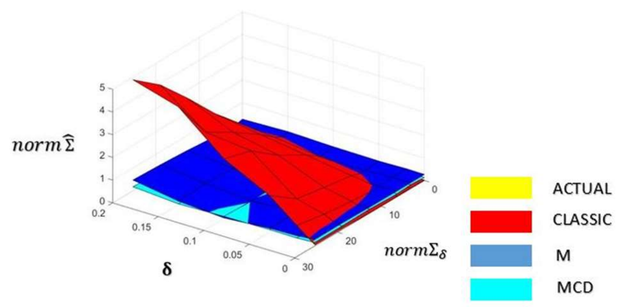

Figure. 1. comparison of the classic and robust estimators in different contamination and norm of variance-covariance matrix error terms of outliers

Considering δ 100% contamination and different error terms variance-covariance values of outliers (Σ_{δ}), the robust, and classic process capability indices are presented in Tables III and IV. In Table III, by increasing the contamination percentage and the norm of error term variance-covariance of outliers, the classic $\widehat{C}_{PM}$ is affected by the outliers and its values are much less than the actual values. However, two robust $\widehat{C}_{PM}$ indices using the M-estimator and LR- Weighted MCD estimate much closer to the actual values and less affected by the outliers. Table III's results shown in Figure 2.

Table III. Comparison of the robust and classic process capability $\hat{C}_{PM}$ in the presence of the δ100% NID((0,0), Σ_{δ}) contamination in error terms variance-covariance matrix

Estimators		classic	M	LR-weighted MCD
δ	norm(Σ_{δ})	$\widehat{C}_{PM}$	$\widehat{C}_{PM}$	$\widehat{C}_{PM}$
0.05	3.16	1.1835	1.3159	1.2792
	6.32	1.1268	1.2924	1.2642
	12.64	1.0754	1.3008	1.2723
	18.97	1.0368	1.3251	1.2957
	25.29	0.8954	1.3019	1.2781
0.10	3.16	1.1590	1.2977	1.2646
	6.32	1.0493	1.2440	1.2380
	12.64	0.9158	1.2261	1.2483
	18.97	0.7993	1.2243	1.2306
	25.29	0.7716	1.2333	1.2643
0.15	3.16	1.0915	1.2337	1.2268
	6.32	0.9830	1.1748	1.1893
	12.64	0.8572	1.1807	1.2226
	18.97	0.7778	1.1670	1.2271
	25.29	0.6494	1.1795	1.2346
0.20	3.16	1.0452	1.1921	1.1777
	6.32	0.9217	1.1509	1.1891
	12.64	0.7617	1.1171	1.1959
	18.97	0.6638	1.1118	1.2011
	25.29	0.6222	1.0947	1.1829

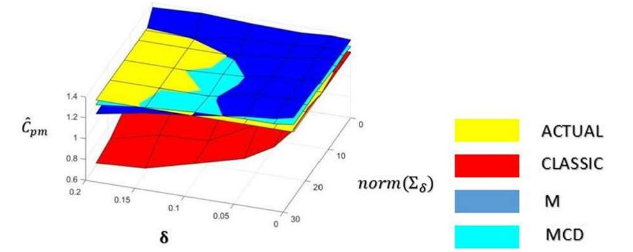

Figure. 2. comparison of the robust $\widehat{C}_{PM}$ indices and classic $\widehat{C}_{PM}$ index based on the presence of contamination in error terms variance-covariance matrix

Proposed robust process capability indices are more accurate and perform closer to actual values than the classic index, even with different levels of contamination. Table IV presents simulation results for estimated process capability using the multivariate capability vector and PCA method, varying Σ_{δ} and contamination percentages. The proposed process capability index, $M\widehat{C}_{pc}$ using LR-Weighted MCD performs better than $M\widehat{C}_{pc}$ using M-estimator in higher contamination percentages and large values of norm(Σ_{δ}). Figure 3 compares robust and classic $M\widehat{C}_{pc}$ indices' performance based on error term contamination in the variance-covariance matrix.

Table IV. Comparison of the robust and classic process capability $M\hat{C}_{pc}$ in the presence of the δ100% NID((0,0), Σ_{δ}) contamination in the error terms variance-covariance matrix

Estimators		classic	M	LR-weighted MCD
δ	norm(Σ_{δ})	$M\widehat{C}_{pc}$	$M\widehat{C}_{pc}$	$M\widehat{C}_{pc}$
0.05	1.4950	1.6646	1.6150	1.4950
	1.4257	1.6243	1.5845	1.4257
	1.3152	1.6199	1.5888	1.3152
	1.2652	1.6248	1.5921	1.2652
	1.1938	1.6282	1.5988	1.1938
0.10	1.4200	1.5977	1.5551	1.4200
	1.3014	1.5414	1.5455	1.3014
	1.1591	1.5197	1.5402	1.1591
	1.0510	1.5459	1.5626	1.0510
	0.9882	1.5628	1.5705	0.9882
0.15	1.3768	1.5582	1.5327	1.3768
	1.2217	1.4718	1.5037	1.2217
	1.0578	1.4608	1.5346	1.0578
	0.9618	1.4493	1.5215	0.9618
	0.8741	1.4995	1.5784	0.8741
0.20	1.3123	1.4945	1.4746	1.3123
	1.1658	1.4223	1.4519	1.1658
	0.9740	1.3840	1.4801	0.9740
	0.8727	1.3860	1.5081	0.8727
	1.4950	1.6646	1.6150	1.4950

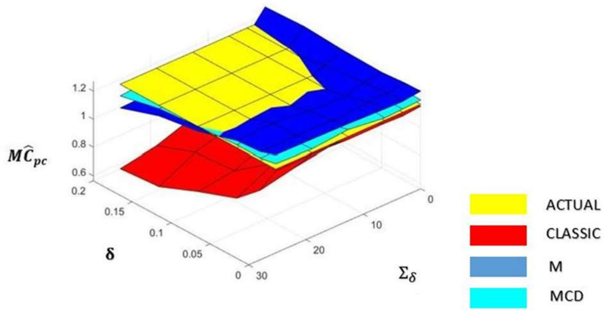

Figure. 3. Comparison of the robust $M\hat{C}_{pc}$ indices, and classic $M\hat{C}_{pc}$ based on the presence of contamination in the error terms variance-covariance matrix

V. Conclusion

In this research, Multivariate linear profiles were evaluated in presence of the outliers. To estimate the parameters of this type of profile, a classic method (Least squares) and two robust estimators, (M-estimator and LR-Weighted MCD) were compared by using simulation studies. Simulation results show that all three methods estimated the parameters very close to the actual value in absence of outliers, however, in presence of outliers in error terms variance-covariance matrix, as the percentages in contamination and shift in error term variance-covariance increased, the classic method was affected by the outliers, and the robust methods outperformed the classic method. Two robust process capability indices for multivariate linear profiles were proposed. Using the simulation studies, the performance of these PCIs was investigated. Both robust and classic PCIs were estimated close to their actual values in the absence of outliers. However, in the presence of outliers, the classic PCI was affected by the contamination and its performance was not close to the actual value. on the contrary, the robust PCIs still performed well in the presence of contaminations. According to the results of the simulation, the proposed process capability indices using LR-Weighted MCD estimators performed better than the process capability indices using the M-estimator in different percentages of contamination as well as different error terms variance-covariance values of outliers. Proposing robust process capability indices for multivariate multiple profiles is an issue for the future research.

References

1. L. Kang and S. L. Albin, "On-line monitoring when the process yields a linear profile," *Journal of quality Technology,* vol. 32, no. 4, pp. 418-426, 2000.
2. M. Ebadi and H. Shahriari, "Robust estimation of parameters in simple linear profiles using M-estimators," *Communications in Statistics-Theory and Methods,* vol. 43, no. 20, pp. 4308-4323, 2014.
3. M. Kordestani, F. Hassanvand, Y. Samimi, and H. Shahriari, "Monitoring multivariate simple linear profiles using robust estimators," *Communications in Statistics-Theory and Methods,* vol. 49, no. 12, pp. 2964-2989, 2020.
4. F. Hassanvand, Y. Samimi, and H. Shahriari, "A robust control chart for simple linear profiles in two□stage processes," *Quality and Reliability Engineering International,* vol. 35, no. 8, pp. 2749-2773, 2019.
5. F. Mahpouya, H. Shahriari, and E. Roghanian, "Design of a robust T2 control chart, a re□sampling approach," *Quality and Reliability Engineering International,* vol. 38, no. 2, pp. 924-940, 2022.
6. M. J. Kang, P. Mobtahej, A. Sedaghat, and M. Hamidi, "A Soft Optimization Model to Solve Space Allocation Problems in Breakbulk Terminals," *Computational Research Progress in Applied Science & Engineering (CRPASE),* vol. 7, no. 4, 2021.
7. M. Rabbani, F. Navazi, N. Eskandari, and H. Farrokhi-Asl, "A green transportation location-inventory-routing problem by dynamic regional pricing," *Journal of Industrial Engineering and Management Studies,* vol. 7, no. 1, pp. 35-58, 2020.
8. A. Sedaghat, M. Rabbani, and H. Farrokhi-Asl, "A Sustainable Transportation Location Inventory Routing Problem," *Computational Research Progress in Applied Science & Engineering (CRPASE),* vol. 8, no. 3, 2022.
9. M. M. Ahmadi and H. Shahriari, "Robust Monitoring of Simple Linear Profiles Using M-estimators," Computational Research Progress in Applied Science & Engineering (CRPASE), vol. 8, pp. 1-7, 2022.
10. T. Khatibi and P. Dezyani, "Proposing novel methods for gynecologic surgical action recognition on laparoscopic videos," *Multimedia Tools and Applications,* vol. 79, pp. 30111-30133, 2020.
11. F. Navazi, Y. Yuan, and N. Archer, "The effect of the Ontario stay-at-home order on Covid-19 third wave infections including vaccination considerations: An interrupted time series analysis," *Plos one,* vol. 17, no. 4, p. e0265549, 2022.
12. N. Saraei, M. Khanal, and M. Tizghadam, "Removing Acidic Yellow Dye from Wastewater Using Moringa Peregrina."
13. F. Wang and J. C. Chen, "Capability index using principal components analysis," *Quality engineering,* vol. 11, no. 1, pp. 21-27, 1998.
14. H. Shahriari and M. Sarrafian, "Assessment of process capability in linear profiles," in *Proceedings of the 6th international industrial engineering conference, Tehran, Iran (in Farsi)*, 2009.
15. M. R. Niavarani, R. Noorossana, and B. Abbasi, "Three new multivariate process capability indices," *Communications in Statistics-Theory and Methods,* vol. 41, no. 2, pp. 341-356, 2012.
16. M. Ebadi and H. Shahriari, "A process capability index for simple linear profile," *The International Journal of Advanced Manufacturing Technology,* vol. 64, pp. 857-865, 2013.
17. M. Ebadi and A. Amiri, "Evaluation of process capability in multivariate simple linear profiles," *Scientia Iranica,* vol. 19, no. 6, pp. 1960-1968, 2012.
18. M. Karimi Ghartemani, R. Noorossana, and S. T. A. Niaki, "A new approach in capability analysis of processes monitored by a simple linear regression profile," *Quality and Reliability Engineering International,* vol. 32, no. 1, pp. 209-221, 2016.
19. S. Mehri, M. M. Ahmadi, H. Shahriari, and A. Aghaie, "Robust process capability indices for multiple linear profiles," *Quality and Reliability Engineering International,* vol. 37, no. 8, pp. 3568-3579, 2021.
20. R. Noorossana, M. Eyvazian, A. Amiri, and M. A. Mahmoud, "Statistical monitoring of multivariate multiple linear regression profiles in phase I with calibration application," *Quality and Reliability Engineering International,* vol. 26, no. 3, pp. 291-303, 2010.
21. P. J. Huber and E. M. Ronchetti, "Robust statistics john wiley & sons," *New York,* vol. 1, no. 1, 1981.
22. N. L. Kudraszow and R. A. Maronna, "Estimates of MM type for the multivariate linear model," *Journal of Multivariate Analysis,* vol. 102, no. 9, pp. 1280-1292, 2011.
23. P. J. Rousseeuw and K. V. Driessen, "A fast algorithm for the minimum covariance determinant estimator," *Technometrics,* vol. 41, no. 3, pp. 212-223, 1999.
24. R. A. Maronna, R. D. Martin, V. J. Yohai, and M. Salibián-Barrera, *Robust statistics: theory and methods (with R)*. John Wiley & Sons, 2019.
25. H. Shahriari, N. Hubele, and F. P. Lawrence, "A multivariate process capability vector," *in Proceedings of the 4th Industrial Engineering Research Conference, Institute of Industrial Engineers*, 1995, pp. 304-309.
26. W. K. Härdle and L. Simar, *Applied multivariate statistical analysis*. Springer Nature, 2019.