

		
<small>Volume 110, Number 2</small>		<small>February 2003</small>
R. Michael Range	Complex Analysis: A Brief Tour into Higher Dimensions	89
Harold L. Schoen Christian R. Hirsch	Responding to Calls for Change in High School Mathematics: Implications for Collegiate Mathematics	109
David J. Hunter Paul T. von Hippel J. Peter May	How Rare Is Symmetry in Musical 12-Tone Rows? Mursah's Proof of the Nullstellensatz	124 133
NOTES		
J. M. Aldaz A. Bravo	Euclid's Argument on the Infinitude of Primes	141
Jack B. Brown George Kosłowski Bang-Yen Chen	Smooth Interpolation, Hölder Continuity, and the Takagi-van der Waerden Function When Does the Position Vector of a Space Curve Always Lie in Its Rectifying Plane?	142 147
Xingzhi Zhan	The Sharp Rado Theorem for Majorizations	152
Gabriel Navarro	On the Fundamental Theorem of Finite Abelian Groups	153
PROBLEMS AND SOLUTIONS		
REVIEWS		
M. Victor Wickerhauser	Ripples in Mathematics: The Discrete Wavelet Transform. By Arne Jensen and Anders I. Coum-Harbo A First Course in Wavelets with Fourier Analysis. By Albert Boggess and Francis J. Narcowich	163
Steven G. Krantz	From Holomorphic Functions to Complex Manifolds. By Karl Fritzsche and Hans Grauert	167

AN OFFICIAL PUBLICATION OF THE MATHEMATICAL ASSOCIATION OF AMERICA

RESOURCE

Schoen, H. L., & Hirsch, C. R. (2003). Responding to calls for change in high school mathematics: Implications for collegiate mathematics. *American Mathematical Monthly*, (110)2, 109–123.

ABSTRACT

This article briefly discusses the recent history of mathematics reform in high school, and then reports on research evidence for one of the new NSF-funded curriculum projects, the Core-Plus Mathematics Project (CPMP). Implications for collegiate mathematics are also discussed.

A summary of the results of several studies using a range of achievement measures comparing CPMP students to comparable students in more conventional high school mathematics curricula is given on page 114:

“Thus, research to date indicates that CPMP students perform particularly well [and better than the comparison students] on measures of conceptual understanding, interpretation of mathematical representations and calculations, and problem solving in applied contexts. Their performance is also relatively strong in content areas like statistics and probability that are emphasized in the curriculum. On measures of algebraic manipulative skills, CPMP students usually, but not always, score as well as students in more conventional curricula.”

A study of student performance on a mathematics placement test used at a major university is summarized on page 116:

“On the algebra subtest, the means of the [conventional] precalculus students and the CPMP Course 4 students were virtually identical. On the intermediate algebra subtest, the mean of the precalculus group was greater than that of the Course 4 group. The only statistically significant difference in means was on the calculus readiness subtest ($t = 4.93, p < 0.01$). That difference favored the CPMP students.”

Responding to Calls for Change in High School Mathematics: Implications for Collegiate Mathematics

Harold L. Schoen and Christian R. Hirsch

1. INTRODUCTION. Nearly all mathematicians, mathematics educators, teachers, parents, and students agree on the need to improve the mathematical learning of students in our nation's schools. Yet there are widely differing viewpoints concerning appropriate directions for school mathematics curricula that might help accomplish this goal. With its publication of the *Curriculum and Evaluation Standards for School Mathematics* in 1989 [5] and subsequent supporting documents on standards for teaching [20] and for assessment [2], the National Council of Teachers of Mathematics (NCTM) provided a vision of school mathematics that was both broadly embraced [18] and intensely debated (see, for example, [13], [26], or [27]). In the case of curriculum recommendations for high school mathematics reform, the NCTM Standards reflected earlier expressed views of the mathematics community. In particular, in its 1983 report "The Mathematical Sciences Curriculum K–12: What Is Still Fundamental and What Is Not" [15], the Conference Board of the Mathematical Sciences (CBMS) recommended that the secondary school curriculum be streamlined to make room for new topics and techniques from discrete mathematics, statistics, and probability, and that the content, emphases, and approaches in algebra, geometry, and precalculus be re-examined in the light of emerging computing technologies. The report suggested that "technology provides an opportunity to devote less time to traditional [manipulative] techniques while boosting understanding and allowing more time for more complex, realistic problem-solving," but cautioned that "there is little research data on the feasibility of such changes, and there are almost no prototype school curricula embodying the new priorities" [15, p. 5].

The broad CBMS recommendations received further elaboration in the 1983 National Science Board Commission on Precollege Education in Mathematics, Science, and Technology report "Educating Americans for the 21st Century" [8]. In addition, that report challenged the sequence of separate year-long courses in algebra, geometry, and precalculus topics and called for serious consideration of the development of an integrated secondary school mathematical sciences curriculum. This report, like the CBMS report that preceded it and the NCTM Standards that followed, sketched directions for change. These reports did not provide guidance for the specifics of day-to-day, week-to-week, or even year-to-year practice. That burden was left to curriculum developers and teachers.

The decade of the nineties was characterized by efforts funded by the National Science Foundation to develop curricula that interpreted the broad recommendations for change and to evaluate their effects in schools. This period was also characterized by continuing debates on mathematics curriculum and teaching issues from various perspectives—mathematical, historical, societal needs, research on teaching and learning, and personal experience (see [1], [10]). Open, reasoned debates of these issues are an important requisite for real progress. However, we agree with learning researchers Anderson, Greeno, Reder, and Simon [1] that continuing debates should be informed

by the accumulation of empirical evidence that describes levels of important learning outcomes consistently achieved by students in particular curricula.

The Core-Plus Mathematics Project (CPMP) is one of four high school mathematics curriculum development projects funded in 1992 by NSF to develop Standards-oriented, integrated high school mathematics curricula. The CPMP curriculum, which has become a focal point of some of the debates on high school mathematics reform, was developed in consultation with mathematicians, mathematics education researchers, instructional and evaluation specialists, and classroom teachers. The development of the curriculum was further shaped by empirical evidence gathered each year from teachers and students in a diverse set of schools as they participated in the piloting and field testing of each course. Data sources include student achievement pretests and posttests, student attitude surveys, teacher surveys assessing classroom practices and concerns, structured classroom observations by CPMP evaluation staff, and written and oral feedback from teachers concerning the curriculum's strengths and weaknesses.

These data were collected mainly to monitor the impact of the curriculum over the four-year process of developing each course and to help guide revisions between pilot and field-test versions and between field-test and published versions of each course. We believe that some of our empirical findings, especially those concerning what students know and are able to do mathematically when they graduate from high school, will be of interest to the college mathematics community. We also agree with Day and Kalman [6] that college mathematics faculty would benefit from being more informed about the changing curriculum of secondary schools. In this paper, following a brief overview of how CPMP responded to recommendations for change, we describe the different patterns of mathematical achievement of students in the curriculum compared with students in more traditional high school mathematics curricula, with particular attention to students' preparedness for college mathematics.

2. CURRICULAR CONTENT AND ORGANIZATION. The content and organization of the curriculum reflects the project's interpretation of curriculum, teaching, and assessment recommendations in the NCTM Standards documents and in the earlier CBMS and NSB Commission reports. The curriculum consists of a three-year core mathematics program intended for a wide range of high school students, plus a flexible fourth-year course continuing the preparation of students for college mathematics. The curriculum is published under the title *Contemporary Mathematics in Context: A Unified Approach* [4]. Unit titles for the four-year curriculum are given in the following table. Information on the goals and topics of the units can be found at www.wmich.edu/cpmp.

Some of the features of the curriculum that distinguish it from more traditional curricula are the following.

- Each course advances students' understanding of mathematics along interwoven strands of algebra and functions, statistics and probability, geometry and trigonometry, and discrete mathematics.
- These mathematical strands are developed in coherent, focused units that are connected by fundamental ideas such as function, symmetry, and data analysis; and by mathematical habits of mind such as visual thinking, recursive thinking, and searching for and explaining patterns.
- Mathematics is developed in context with an emphasis on problem solving and mathematical modeling.

TABLE 1. The CPMP curriculum

Course 1		Course 2	
1	Patterns in Data	1	Matrix Models
2	Patterns of Change	2	Patterns of Location, Shape, and Size
3	Linear Models	3	Patterns of Association
4	Graph Models	4	Power Models
5	Patterns In Space and Visualization	5	Network Optimization
6	Exponential Models	6	Geometric Form and Its Function
7	Simulation Models	7	Patterns in Chance
CAPSTONE	Planning a Benefits Carnival	CAPSTONE	Forests, the Environment, and Mathematics
Course 3		Course 4	
1	Multiple-Variable Models	1	Rates of Change
2	Modeling Public Opinion	2	Modeling Motion
3	Symbol Sense and Algebraic Reasoning	3	Logarithmic Functions and Data Models
4	Shapes and Geometric Reasoning	4	Counting Models
5	Patterns in Variation	5	Binomial Distributions and Statistical Inference
6	Families of Functions	6	Polynomial and Rational Functions
7	Discrete Models of Change	7	Functions and Symbolic Reasoning
CAPSTONE	Making the Best of It: Optimal Forms and Strategies	8	Space Geometry
		9	Informatics
		10	Problem Solving, Algorithms, and Spreadsheets

- Graphing calculators are used as tools for developing mathematical understanding and for solving authentic problems.
- Instructional materials promote active learning and teaching centered around collaborative small-group investigations of problem situations followed by whole-class summarizing activities that lead to analysis, abstraction, and further application of underlying mathematical structures.
- Conceptual understanding, reasoning with multiple representations, and oral and written communication are emphasized.
- Mathematical thinking and reasoning are central to all courses; with formal proof developed “semilocally” in Courses 3 and 4.
- The design of Course 4 permits tailoring of seven-unit courses around core units (1–4) plus options so as to keep all college-bound students in the mathematics pipeline, whether their intended undergraduate program is calculus-based or not.
- Assessment of students is an integral part of the curriculum and instruction.

Students with aptitude and interest in mathematics are often accelerated into Course 1 in eighth grade (or earlier), enabling them to complete Course 4 and, if desired, AP Statistics in eleventh grade and AP Calculus in twelfth grade.

3. PATTERNS OF MATHEMATICAL LEARNING. The main goal of the curriculum is to improve students' understanding of mathematical concepts and processes and their ability to use mathematics effectively in realistic problem solving. Our research has focused on how well the curriculum is meeting that goal. One of the most important overriding questions is:

How does the pattern of mathematics learning that students attain in the CPMP curriculum differ from that attained by comparable students in more traditional curricula?

Relative to this question, over the past eight years we conducted or cooperated in the conduct of various studies of mathematics achievement in CPMP classrooms. Most of these studies were part of CPMP's national field test carried out in thirty-six schools over four years. A broad cross-section of students from urban, suburban, and rural communities with ethnic and cultural diversity was represented. Data from the administration of a range of achievement measures to CPMP students and comparable students in more traditional high school mathematics curricula were collected during these studies.

Since random assignment of students to classes is impossible in school-based studies, we established the comparability of groups with respect to baseline achievement measures such as eighth-grade standardized mathematics achievement test scores or pretest scores from the beginning of grade nine. In nearly every comparison, there was a great deal of overlap of the two score distributions and plenty of room for improvement by both groups. What is most interesting is the consistent pattern of outcomes across the studies. CPMP students almost always performed better than comparison students on measures of conceptual understanding, interpretation of mathematical representations and calculations, and problem solving in applied contexts, but sometimes not as well on measures of algebraic manipulation skills. Some results of studies from Courses 1, 2, and 3 are briefly summarized as follows:

- On the Educational Testing Service's (ETS) *Algebra End-of-Course Examination*, the subtest means of Course 2 students in three high schools that use CPMP with all students were higher at the end of the year than those of the national cohort of algebra students who completed this test.¹ The order of subtest mean differences²

¹The CPMP sample included the full range of students in three different schools that together serve a diverse population. At the beginning of grade nine prior to using the CPMP curriculum, the mean of students in these three schools on the ITED-Q was at the 56th student percentile. However, the CPMP sample is not nationally representative, and its comparability to the ETS sample cannot be determined precisely. Thus, the subtest means in the ETS sample reported here should be viewed as benchmarks. From this perspective, the ETS results show that CPMP students have areas of relative strength that differ from those of the algebra students who completed this test nationally.

²Percentages correct on subtests are given here for ease of interpretation, but we recognize that these figures mask the variability of subtest scores. The differences on ETS subtests as numbers of standard deviations (S.D.) of the CPMP group are Concepts (0.45 S.D.), Processes (0.32 S.D.), and Skills (0.05 S.D.). On the NAEP content subtests, the differences are Statistics & Probability (0.92 S.D.), Measurement (0.67 S.D.), Algebra & Functions (0.54 S.D.), and Numbers & Operations (0.38 S.D.). On the NAEP process subtests, the differences are Concepts (0.90 S.D.), Problem Solving (0.65 S.D.) and Procedures (0.44 S.D.). Mean differences are statistically significant ($p < 0.05$) on all subtests except ETS Skills.

favoring the CPMP students were Concepts (50% to 41%), Processes (39% to 32%), and Skills (43% to 42%).

- On a test consisting of thirty released items from the National Assessment of Educational Progress (NAEP), the means of CPMP students (mostly juniors) in twenty-two schools at the end of Course 3 were higher than those of NAEP's nationally representative sample of 1992 beginning-of-year seniors on the five content and three process subtests.³ The content subtests ordered by mean differences were Data, Statistics & Probability (67% to 45%), Measurement (59% to 43%), Algebra & Functions (53% to 42%), Geometry (60% to 49%), and Numbers & Operations (44% to 34%). The process subtests ordered by mean differences were Concepts (61% to 44%), Problem Solving (53% to 40%), and Procedures (56% to 45%). For more details, see [23].
- In a six-school study of algebraic skill and understanding across the domain of algebra and advanced algebra at the end of Course 3, CPMP students performed significantly better ($p < 0.05$) than a matched group of advanced algebra students on concept and application tasks. The advanced algebra students scored significantly better ($p < 0.05$) on a measure of paper-and-pencil algebraic manipulation skills. For a detailed task-by-task analysis, see [11]. Similar results were found in separate comparisons of CPMP Course 1 and CPMP Course 2 students with algebra and geometry students, respectively, except that in the Course 2 study there was no significant difference in means on algebraic skills between the CPMP and comparison groups [23].
- A comparison of SAT I Mathematics scores⁴ of CPMP Course 3 students versus advanced algebra students from eight schools showed no significant difference. (SAT Verbal scores were used to statistically equate groups.) A CPMP Course 4 versus precalculus comparison of SAT Mathematics scores showed a significant mean difference ($p < 0.05$) in favor of the CPMP students.
- A comparison of ACT scores of CPMP Course 3 students versus advanced algebra students from fifteen schools showed a significant difference ($p < 0.05$) in ACT Mathematics means favoring the advanced algebra students. This was offset by an even larger difference favoring CPMP students in ACT Science Reasoning. (ACT English scores were used to statistically equate groups.) A CPMP Course 4 versus precalculus comparison showed no significant differences in Mathematics or Science Reasoning means. (In alignment with some of CPMP's goals, the ACT Science Reasoning test requires students to retrieve information from graphs and tables, draw conclusions and predict results based on summaries of described experiments, to compare two opposing views, and to draw conclusions about those ideas.) In spite of these differences in subtest scores, ACT Composite (average of Mathematics, Science Reasoning, English, and Reading) means of CPMP and comparison groups

³The CPMP sample was broadly representative of school types and regions of the country. At the beginning of grade nine prior to using the CPMP curriculum, the mean of students in these twenty-two schools on the ITED-Q was at the 55th student percentile, with school means ranging from the 8th to the 80th student percentiles. The NAEP sample was composed of beginning-of-year seniors, whereas the CPMP students were at the end-of-year juniors and are not a nationally representative sample. These facts along with likely differences in administration procedures and ordering of test items suggest that the subtest means in the NAEP sample reported here should be viewed as benchmarks. From this perspective, the NAEP results show that CPMP students have areas of relative strength that differ from those of typical beginning-of-year twelfth-grade students.

⁴The use of SAT and ACT scores for comparisons has the advantage that one can be confident that students gave their best efforts on the tests. The main disadvantage is that the sample of students who take these tests is self-selected rather than formed in a systematic way. The results described here are for all scores from those 1998–1999 field-test schools that complied with our request for ACT and SAT scores, a request that was made of the entire set of field-test schools.

were virtually identical. SAT and ACT comparisons are described further in [23] and [24].

Thus, research to date indicates that CPMP students perform particularly well on measures of conceptual understanding, interpretation of mathematical representations and calculations, and problem solving in applied contexts. Their performance is also relatively strong in content areas like statistics and probability that are emphasized in the curriculum. On measures of algebraic manipulative skills, CPMP students usually, but not always, score as well as students in more traditional curricula. The nature of the differences between the mathematical competence of students who complete the CPMP curriculum and those who complete more traditional high school curricula is further elucidated by the following comparisons of university mathematics placement test results.

4. PREPARATION FOR COLLEGE MATHEMATICS. Comparatively weaker algebra skills of CPMP field-test students at the end of Course 1 and Course 3 as reported in the previous section prompted several revisions in the published version of Courses 1 through 3 and in the field-test and published versions of Course 4. By design, greater emphasis was devoted to symbolic reasoning and manipulation in Course 4. In addition, a feature was added to each unit in Course 4 called Preparing for Undergraduate Mathematics Placement (PUMP). Each PUMP section consists of ten multiple-choice practice test items similar to those typically found on university mathematics department placement tests. The intent of these revisions was to improve students' fluency in algebraic manipulation skills, while maintaining their strength on measures of understanding.

Following these revisions, a study was conducted during the Course 4 field test to determine how well students were prepared for university mathematics courses when they completed Course 4. Specifically, the research question was the following.

How does the level of preparation for calculus and other undergraduate mathematics courses that students attain in the CPMP curriculum differ from that attained by comparable students in more traditional curricula?

Content of CPMP Course 4. Course 4 consists of a core of four units intended for all college-bound students plus specialized units for students planning to major in the mathematical, physical, and biological sciences or engineering, and other units for students planning to major in business and the social or the health sciences. The core units are the first four units in Table 2, although some topics in these units (for example, parametric equations in Unit 2 and mathematical induction in Unit 4) are typically not included in courses for the latter group of students. The Course 4 sequence of interest in this study is the calculus-preparatory sequence for mathematics, engineering, and the physical or biological sciences. The broad content of that seven-unit course is outlined in Table 2. (More specific topics in each of these units can be found at the CPMP website.) As is the case in traditional precalculus courses, some of this content has already been introduced in previous courses but is dealt with more formally and deeply in Course 4. As in all courses, the topics are usually developed in the context of modeling realistic problem situations and then examined in terms of their underlying mathematical structure. Although use of graphing calculators is assumed, increased attention is given in Course 4 to analysis of symbolic representations of functions and associated symbolic manipulation and reasoning skills.

TABLE 2. Focus of Course 4 calculus-preparatory units

1	Rates of Change
	Instantaneous Rates of Change
	Rates of Change for Familiar Functions
	Accumulation at Variable Rates
2	Modeling Motion
	Modeling Linear Motion—Vectors
	Simulating Linear and Nonlinear Motion—Parametric Equations
3	Logarithmic Functions and Data Models
	Inverses of Functions
	Logarithmic Functions
	Linearizing Data
4	Counting Models
	Methods of Counting
	Counting Throughout Mathematics
	The Principle of Mathematical Induction
5	Polynomial and Rational Functions
	Polynomial Functions
	Polynomials and Factoring
	Rational Functions
6	Functions and Symbolic Reasoning
	Reasoning with Exponential and Logarithmic Functions
	Reasoning with Trigonometric Functions
	Solving Trigonometric Equations
	The Geometry of Complex Numbers
7	Space Geometry
	Representing Three-Dimensional Shapes
	Equations for Surfaces

Sample. The sixth of the seven CPMP calculus-preparatory units, “Functions and Symbolic Reasoning,” features content that is especially crucial for calculus and is prominent on most placement tests. It includes both conceptual and symbolic manipulation work with logarithms, verifying trigonometric identities, solving trigonometric equations, and reasoning and calculating with complex numbers in trigonometric form. In this comparison, the CPMP group consisted of all students ($N = 164$) who completed at least six of these seven units, including “Functions and Symbolic Reasoning,” as part of their four-year study of the CPMP program. The precalculus students ($N = 177$) comprised all college-intending students in the field test who were at the end of a traditional precalculus course that was the fourth course in a college-preparatory sequence that included algebra, geometry, and advanced algebra. Both groups were composed of students who fell mainly between the 75th and 95th national percentile, on average about the 85th percentile for each group, on standardized mathematical achievement tests at the beginning of or just prior to high school. The very best mathematics students in these schools were likely enrolled in an AP Calculus course as seniors.

Instrument. A placement test used at a major university was administered to students in field-test schools at the end of CPMP Course 4 or at the end of traditional precalculus. This multiple-choice test is used to make recommendations to entering freshmen

concerning the college mathematics course that would be best for them. The test was compiled from a bank of placement items that until recently was available from the MAA. A graphing calculator (such as the TI-82 or TI-83) that does not have symbol manipulation capability is allowed on this test.

This test contains three subtests—Algebra (15 items), Intermediate Algebra (15 items), and Calculus Readiness (20 items). The first two subtests consist almost entirely of algebraic manipulation tasks such as simplifying and factoring algebraic expressions, solving equations and inequalities, and finding equations for lines given sufficient conditions. The third subtest measures some of the important concepts and processes that underlie calculus, such as reasoning with logarithmic and exponential equations, trigonometric functions and identities, composition of functions, rational functions and their domains, systems of nonlinear equations, and area under a curve. Items similar to selected items from each subtest are presented in a later section.

Group differences by mathematical subtest. Placement subtest means and standard deviations for each group are given in Table 3. While there is room for improvement by both groups on all three subtests, we focus here on group differences on the subtests. On the algebra subtest, the means of the precalculus and CPMP Course 4 groups were virtually identical. On the intermediate algebra subtest, the mean of the precalculus group was greater than that of the Course 4 group. The only statistically significant difference in means was on the calculus readiness subtest ($t = 4.93$, $p < 0.01$). That difference favored the CPMP students.

TABLE 3. Results by group and subtest

Group	N	Algebra		Intermediate Algebra		Calculus Readiness	
		Mean	SD	Mean	SD	Mean	SD
CPMP	164	11.5	2.6	9.2	3.4	12.9	4.7
Precalculus	177	11.4	2.3	9.6	3.2	10.5	4.3

To examine further the group differences in performance by mathematical content, we ran for each item a t -test that compared the mean performance for CPMP Course 4 students with that of the precalculus students. This allowed us to identify all items for which the mean for the two groups differed substantially (0.01 level of significance). These items are analyzed next.

Algebraic test items. All algebra and intermediate algebra test items for which Course 4 and precalculus means differed at the 0.01 level of significance are given in Table 4. Group item means differed significantly on only two of the fifteen algebra test items, and the mean differences were significant on seven of the fifteen intermediate algebra items (in all, four in favor of the CPMP students and five in the other direction).

Different emphases of the CPMP and traditional curricula help to explain most of the differences in item means. The treatment of geometry every year and the emphasis on connections between algebra and geometry in the CPMP curriculum may explain why students were better able to find an equation of a line through two given points (BC 1). A similar explanation may apply for IC 2 where students have first to identify the opposite vertices of a rectangle in order to find the length of the diagonal. The other two items favoring CPMP involved algebraic manipulation, but some conceptual understanding may help students avoid common errors. In IC 1, many students are

TABLE 4. Algebra test items*

Algebra Test Items	
CPMP > Precalculus ($p < .01$)	Precalculus > CPMP ($p < .01$)
<i>BC 1. CPMP 91%; Precalculus 76%</i> What is the equation of the line which goes through the points (0, 3) and (1, 5)?	<i>BP 1. CPMP 78%; Precalculus 91%</i> The inequality $5x - 4 < 2x + 6$ is equivalent to:
Intermediate Algebra Test Items	
CPMP > Precalculus ($p < .01$)	Precalculus > CPMP ($p < .01$)
<i>IC 1. CPMP 55%; Precalculus 38%</i> If $x > 0$, then $\sqrt{25x^2 - 9x^2} = ?$	<i>IP 1. CPMP 59%; Precalculus 73%</i> Subtract: $\frac{1}{b} - \frac{4}{a} = ?$
<i>IC 2. CPMP 90%; Precalculus 78%</i> If a rectangle has vertices (0, 0), (4, 0), (0, 3) and (4, 3), then the length of a diagonal is approximately:	<i>IP 2. CPMP 55%; Precalculus 69%</i> Add: $\frac{b}{2a} + \frac{b}{3a} = ?$
<i>IC 3. CPMP 65%; Precalculus 50%</i> Which of the following best approximates the positive solution of the equation: $x^2 - 5x = 4$?	<i>IP 3. CPMP 64%; Precalculus 79%</i> $\sqrt{27x^6y^9} = ?$
	<i>IP 4. CPMP 44%; Precalculus 62%</i> One of the factors of $15x^2 + 7x - 2$ is:

*These items are parallel to the actual test items, but with answer choices removed.


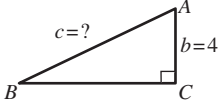
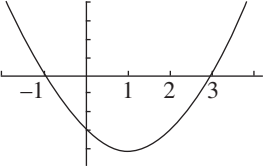
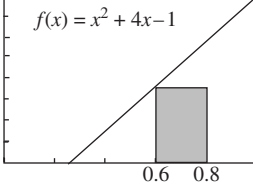
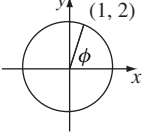
tempted to take the square root of each term in the difference under the radical sign. Failure to rewrite the quadratic equation with all terms on one side is the common error in IC 3. It is likely that most CPMP students solved such an equation using either the graph- or table-building capabilities of their calculators, thereby avoiding inappropriate reasoning about factors or pitfalls inherent in remembering or using the quadratic formula.

All the items on which the precalculus students scored higher involved uses of symbol manipulation procedures that are commonly emphasized in traditional curricula. These include operations with rational expressions (IP 1 and IP 2) and factoring trinomials with leading coefficient greater than one (IP 4), topics that are emphasized less in the CPMP curriculum in order to devote more time to developing conceptual understanding of polynomial and rational functions and their uses as mathematical models. Another of these items required an answer in simplest radical form (IP 3), a topic that receives less attention in the calculator-enhanced CPMP curriculum.

Calculus readiness test items. All calculus readiness test items for which the CPMP Course 4 and precalculus mean percent correct differed at the 0.01 level of significance are given in Table 5. Group item means differed significantly on twelve of the twenty calculus readiness items, eleven in favor of the CPMP students and one in the other direction.

Consistent with evaluation findings described earlier, CPMP students performed at a higher level than precalculus students on measures of conceptual and application out-

TABLE 5. Calculus readiness test items*

CPMP > Precalculus ($p < .01$)	
<p><i>RC 1. CPMP 65%; Precalculus 45%</i></p> <p>The box pictured below has a square base and a closed top. Express its surface area in terms of b and h.</p> 	<p><i>RC 7. CPMP 66%; Precalculus 50%</i></p> <p>In the right triangle shown: $\sin B = 0.47$ and $b = 4$. Find c.</p> 
<p><i>RC 2. CPMP 73%; Precalculus 48%</i></p> <p>If $f(x)$ is a function whose graph is the parabola shown, then $f(x) > 0$ whenever:</p> 	<p><i>RC 8. CPMP 77%; Precalculus 45%</i></p> <p>The area of the rectangle pictured below is:</p> 
<p><i>RC 3. CPMP 63%; Precalculus 40%</i></p> <p>A certain deer population increases by a factor of 1.2 each year. (For example, if there are 100 deer now, a year from now there will be 120.) Over a 12-year period, by what factor does the deer population increase?</p>	<p><i>RC 9. CPMP 73%; Precalculus 48%</i></p> <p>The point of intersection, in the first quadrant, of the line $y = 3x + 1$ and the curve $y = 2x^2$ has the x-coordinate equal to:</p>
<p><i>RC 4. CPMP 74%; Precalculus 56%</i></p> <p>If $\frac{(3x - 1)(x + 1)}{(x - 1)} = 0$, then $x = ?$</p>	<p><i>RC 10. CPMP 67%; Precalculus 47%</i></p> <p>$\cos(90^\circ - \phi) = ?$</p>
<p><i>RC 5. CPMP 77%; Precalculus 62%</i></p> <p>If $f(x) = 3x - 2$ and $g(x) = x^2$, then $g(f(x)) = ?$</p>	<p><i>RC 11. CPMP 53%; Precalculus 37%</i></p> <p>The measure, in radians, of the angle ϕ shown below is:</p> 
<p><i>RC 6. CPMP 49%; Precalculus 29%</i></p> <p>If $2^{6,000} = 10^x$, then x is</p>	
Precalculus > CPMP ($p < .01$)	
<p><i>RP 1. CPMP 40%; Precalculus 56%</i></p> <p>$\cos \phi \cot \phi \sec^2 \phi = ?$</p>	

*These items are parallel to the actual test items, but with answer choices removed.

comes. A perusal of the group item data and the items themselves illustrates the nature of the differences. Virtually all the items on which CPMP students did better measure conceptual understanding or applications of mathematics. Many items on which group means differed most involve graphical or diagram interpretation and/or verbally-stated applications (RC 1, RC 2, RC 3, RC 7, RC 8, RC 9, and RC 11), both areas of emphasis in the CPMP curriculum. The large difference in understanding of exponents indicated by item RC 6 may be due to the early, conceptual introduction to exponents in CPMP Course 1 and frequent revisiting of situations involving exponential growth and exponential functions throughout the curriculum. The other calculus readiness items on which CPMP students did better (RC 4, RC 5, and RC 10) are symbolic, but they require understanding of key ideas and not just recall of procedures.

The item that favored precalculus students, RP 1 in Table 5, involves recall of trigonometric definitions followed by simplifying a product of three trigonometric fractions. This difference in means is not unexpected. Because of the emphasis on circular functions as mathematical models, students worked less in precalculus classes with secant, cosecant, and cotangent functions.

Course placements. The main educational significance of this test lies in its use as a tool to help place entering freshmen in beginning college mathematics courses. To facilitate course placement, mathematics departments establish criteria like the following for the test in the present study: (1) Calculus I is recommended if a student has a total score of 35 or higher; (2) a precalculus course is recommended if the combined score on the algebra and intermediate algebra subtests is at least 20; and (3) a more basic course is recommended if neither of criteria 1 and 2 is met.

Notice that the calculus readiness subtest score does not enter into criterion 2. Based on these typical criteria, the algebraic skills of students placed in the same course are likely to be more or less homogeneous, but their levels of conceptual understanding as measured by the calculus readiness subtest may be very different. In fact, that is the case in the present study. Table 6 gives the calculus readiness mean and standard deviation for the subsets of the high school curriculum groups that met each of the above placement criteria.

TABLE 6. Calculus readiness results by group and criterion met

Group	Criterion 1			Criterion 2			Criterion 3		
	N	Mean	SD	N	Mean	SD	N	Mean	SD
CPMP	83	16.6	2.3	25	9.9	3.1	56	8.8	3.4
Precalculus	69	14.8	2.6	44	8.7	2.5	64	7.2	2.8

For those meeting each of the three criteria, the calculus readiness mean of CPMP Course 4 students is one-third to over one-half a standard deviation higher than the precalculus students' mean. In fact, the calculus readiness mean of Course 4 students in the criterion 3 group is essentially the same as that of traditional precalculus students in the criterion 2 group.

5. SUMMING UP. In this paper, we have provided a broad overview of a high school mathematics curriculum that embodies the recommendations for change and new priorities called for in major school mathematics policy documents of the 1980s and early 1990s. We have also described patterns of performance of students who experienced

the curriculum and examined how those patterns differ from the performance of students who complete more traditional curricula. In line with recommendations of the CBMS report [15], the evaluation has been ongoing for an extended period of time and in a variety of school settings. Nonetheless, we agree with McKnight and his colleagues [17, pp. 1–2] that educational research does not provide proof in a mathematical sense concerning the research questions it addresses. Rather, as related educational research studies accumulate, they provide a body of evidence that can be used to help inform educational issues and decisions. In that spirit, we have presented our findings, and in the same spirit we offer the following observations.

The reported findings are consistent with an earlier finding by Begle [3]—recently reaffirmed in the 1995 Third International Mathematics and Science Study [22], in a study of the impact of systemic initiatives in high school mathematics classes conducted by the RAND Corporation [16], and in a research review by Schoenfeld [25]—that the curricular materials used in schools affect student learning in important ways. On the whole, the evidence suggests that it is possible to streamline the traditional components of high school mathematics and incorporate important concepts and methods of statistics, probability, and discrete mathematics, while significantly improving students’ understanding of the mathematical content and its applications. A trade-off in somewhat lower traditional paper-and-pencil algebraic skills may result, although the revisions in the Course 4 field-test material appear to have reduced the deficit.

Because of equity considerations, the CPMP curriculum capitalized on the capabilities of graphing calculators rather than computers. Still, in 1983 CBMS raised the possibility that “In the future, students and adults may not need to do much algebraic manipulation—software like muMath will do it for them—but they will still need to recognize which forms they have and which forms they want” [15, p. 4]. Capabilities and user-friendliness of computer algebra systems (CASs) have advanced considerably since the 1981 version of muMath. Today CASs are available for handheld devices as well as for desktop platforms. Future curriculum development and research efforts need to focus carefully on how best to address manipulative skills that remain important, particularly in a CAS world, at the same time maintaining gains in mathematical understanding and problem solving.

The CPMP curriculum and other similar NSF-funded high school mathematics curricula share many of the content themes of reform calculus courses: greater emphasis on conceptual understanding; better balance among verbal, graphical, numerical, and symbolic representations; and greater attention to realistic applications and mathematical modeling, including data analysis. They also share many of the pedagogical themes, including: greater emphasis on active learning and teaching, often involving collaborative group work; more experience in communicating mathematics, both orally and in writing; use of technology, particularly graphing calculators, for exploration and problem solving; and more varied methods of assessment [12], [21]. In addition, the increased attention in reform high school curricula to statistical ideas and probabilistic reasoning and to discrete mathematical modeling and matrices is consistent with recent core curriculum initiatives in undergraduate mathematics [7]. In principle, these commonalities could provide a smooth transition between high school and college mathematics. In practice, current placement tests seldom assess the new mathematical knowledge and abilities that students from reform programs bring to our campuses. This fact is one of the reasons why the MAA Board of Governors recently voted to discontinue support for the use of their placement test program.

A continuing central issue in the development of mathematics curriculum materials is how to properly balance conceptual understanding, procedural skill, and problem solving. Traditional mathematics curricula have often been organized and taught as a

sequence of techniques ordered from basic to more complex. The assumption underlying this organization seems to be that technical proficiency should (even must) be developed before focusing curriculum and instruction on conceptual understanding, problem solving, and applications. This assumption also underlies placement tests that often screen students on manipulative skills first, with only those who pass through the skill filter assessed on conceptual understanding and reasoning. On the other hand, curricula like CPMP draw on recent research on teaching and learning mathematics that suggests there is not a strict linear ordering in the learning of skills, concepts, and applications, especially in the presence of today's technological tools [9]. These curricula are organized so that conceptual understanding, procedural skill, and problem solving develop together, largely through problem-based activities in which students engage in making sense of mathematical situations. As a consequence of these competing viewpoints, students from reform high school curricula may be penalized depending on the interpretation of scores on traditionally organized placement tests. Interpretations of scores on such placement tests are likely to be even more problematic when, unlike in this study, calculator use is prohibited.

In conclusion, we believe that the results presented here provide evidence in support of the feasibility of curricula that embody the recommendations for change in [5], [8], [15], and more recently in NCTM's *Principles and Standards for School Mathematics* [19]. The results are also consistent with the emerging research on NCTM Standards-oriented curricula reviewed by Schoenfeld, who noted that "There is substantial and mounting evidence that when teachers are adequately prepared to help students work through these curricula, the students learn not only skills and procedures, but also concepts and problem solving as well" [25, p. 22]. More research is needed to study the effect of the final versions of these new curricula on student achievement outcomes in high school and post-high school settings. Ideally, such research would involve schools that have faithfully implemented the curriculum for at least a few years so that (1) teachers understand and take advantage of the curriculum's full scope and sequence and (2) both teachers and students are accustomed to the expectations of the classroom environment. With guidance from research and with support from a broad range of stakeholders, including mathematicians, high school curriculum developers will be in a position to build on what is now known and work toward even stronger patterns of student outcomes in the future.

6. A FINAL NOTE. Just as CPMP students in the reported study were relatively better prepared for concepts that underlie calculus than for formal algebraic manipulation, one might expect that they would be better prepared for a reform undergraduate calculus course than for a traditional one. Two studies of the transition to college mathematics of students who experienced at least three years of the pilot- or field-test versions of the CPMP curriculum have recently been completed.

In one study, four years of college mathematics grade data were analyzed for all graduates of a nearby suburban high school who attended the University of Michigan, which has a reform calculus program [23]. This high school used a traditional mathematics program in the first two years of the study and a pilot version of CPMP with all but the accelerated students in the third year and with all students in the fourth year. During each year of the study, accelerated students typically took AP Calculus as seniors. Over the four years, the numbers of graduates of this high school who matriculated at the University of Michigan were 50, 74, 87, and 72, respectively. Over these same four years, freshman students completing Calculus I or a higher mathematics course at the University of Michigan among graduates of this high school as a percent of the total number of matriculants were 70%, 60%, 68%, and 65%. (If a

student completed two such courses, the student was counted twice.) Average grades on a 4.0 scale for these students in Calculus I and higher mathematics courses were 2.71, 2.98, 3.13, and 3.08, respectively. Obviously many factors other than a student's high school mathematics program play a role in success or failure in college mathematics. With that caveat, however, it is clear that this four-year trend is consistent with a conjecture that the CPMP curriculum did no harm to either the percent of students enrolling in these courses or their course grades—and may have helped the latter.

In a preliminary report of a second study, the experiences of seven students who had completed three years and, in some cases, a fourth year of the pilot- or field-test version of the CPMP curriculum were carefully monitored as they entered Michigan State University [14]. Experiencing varying degrees of difficulty with the transition to traditional university content and teaching, three of these students passed traditional Calculus I with grades of B or higher, two others passed with lower grades, and a sixth student withdrew before the end of the course. The seventh student encountered difficulty with traditional Calculus I, seeing the course as dramatically different from her high school CPMP classes. She transferred after two class periods to Applied Calculus, an alternative track in which a Harvard reform textbook is used. She went on to earn grades of 'A' in both Applied Calculus I and II. According to the researchers, "She quickly found that course [Applied Calculus I] and its successor . . . more to her liking. The content fit perfectly with her Core experiences, the problems were the familiar situational type . . . She cruised through both semesters of Applied Calculus, reporting few difficulties" [14, p. 21].

ACKNOWLEDGMENTS. We would like to thank Paul Eenigenburg, Jim Fey, Kathy Heid, Ken Ross, and Ann Watkins for reviewing drafts of this paper and for their helpful comments. The work reported in this paper was supported, in part, by grants from the National Science Foundation (MDR-9255257, ESI-9618193). Views and conclusions expressed are those of the authors and not necessarily those of the Foundation.

REFERENCES

1. J. R. Anderson, J. G. Greeno, L. M. Reder, and H. Simon, Perspectives on learning, thinking, and activity, *Educ. Researcher* **29** (1997) 11–13.
2. *Assessment Standards for School Mathematics*, National Council of Teachers of Mathematics, Reston, VA, 1995.
3. E. G. Begle, Some lessons learned by SMSG, *Math. Teacher* **66** (1973) 207–214.
4. A. F. Coxford, J. T. Fey, C. R. Hirsch, H. L. Schoen, G. Burrill, E. W. Hart, B. A. Keller, and A. E. Watkins, *Contemporary Mathematics In Context: A Unified Approach* (Courses 1–4), Glencoe/McGraw-Hill, Columbus, OH, 1997–2001.
5. *Curriculum and Evaluation Standards for School Mathematics*, National Council of Teachers of Mathematics, Reston, VA, 1989.
6. J. Day and D. Kalman, Teaching linear algebra: issues and resources, *College Math. J.* **32** (2001) 162–168.
7. J. A. Dossey, ed., *Confronting the Core Curriculum: Considering Change in the Undergraduate Mathematics Major*, MAA Notes, no. 45, Mathematical Association of America, Washington, D.C., 1998.
8. *Educating Americans for the 21st Century*, National Science Board, Washington, D.C., 1983.
9. M. K. Heid, The technological revolution and the reform of school mathematics, *Amer. J. of Educ.* **106** (1997) 5–61.
10. R. Howe, The AMS and mathematics education: the revision of the "NCTM Standards," *Notices Amer. Math. Soc.* **45** (1998) 243–247.
11. M. A. Huntley, C. L. Rasmussen, R. S. Villarubi, J. Sangtong, and J. T. Fey, Effects of standards-based mathematics education: a study of the Core-Plus Mathematics Project algebra and functions strand, *J. for Res. in Math. Educ.* **31** (2000) 328–361.
12. J. F. Hurley, U. Koehn, and S. L. Ganter, Effects of calculus reform: local and national, this MONTHLY **106** (1999) 800–811.
13. J. Kilpatrick, Confronting reform, this MONTHLY **104** (1997) 955–961.

14. G. Lewis, V. Lazarovici, and J. Smith, Meeting the demands of calculus and college life: the mathematical experiences of graduates of some reform-based high school programs, paper presented at the Annual Meeting of the American Educational Research Association, April 2001.
15. *The Mathematical Sciences Curriculum K–12: What Is Still Fundamental and What Is Not*, Conference Board of the Mathematical Sciences, Washington, D.C., 1983.
16. D. F. McCaffrey, L. S. Hamilton, B. M. Stecher, S. P. Klein, D. Bugliari, and A. Robyn, Interactions among instructional practices, curriculum, and student achievement: the case of standards-based high school mathematics, *J. for Res. In Math. Educ.* **32** (2001) 493–517.
17. C. McKnight, A. Magid, T. J. Murphy, and M. McKnight, *Mathematics Education Research: A Guide for the Research Mathematician*, American Mathematical Society, Providence, 2000.
18. D. B. McLeod, R. E. Stake, B. P. Schappelle, M. Mellissinos, and M. J. Gierl, Setting the standards: NCTM's role in the reform of mathematics education, in *Bold Ventures*, vol. 3, *Case Studies of U.S. Innovations in Mathematics Education*, S. A. Raizen and E. D. Britton, eds., Kluwer, Dordrecht, The Netherlands, 1996, pp. 13–132.
19. *Principles and Standards for School Mathematics*, National Council of Teachers of Mathematics, Reston, VA, 2000.
20. *Professional Standards for Teaching Mathematics*, National Council of Teachers of Mathematics, Reston, VA, 1991.
21. A. W. Roberts, ed., *Calculus: The Dynamics of Change*, MAA Notes, no. 39, Mathematical Association of America, Washington, D.C., 1996.
22. W. H. Schmidt, C. C. McKnight, L. S. Cogan, P. M. Jakwerth, and R. T. Houang, *Facing the Consequences: Using TIMSS for a Closer Look at U.S. Mathematics and Science Education*, Kluwer, Dordrecht, The Netherlands, 1999.
23. H. L. Schoen and C. R. Hirsch, The Core-Plus Mathematics Project: perspectives and student achievement, in *Standards-Based School Mathematics Curricula: What Are They? What Do Students Learn?*, S. Senk and D. Thompson, eds., Lawrence Erlbaum Assoc., Mahwah, NJ, 2001, pp. 311–343.
24. H. L. Schoen, K. J. Cebulla, and M. S. Winsor, Preparation of students in a standards-oriented mathematics curriculum for college entrance tests, placement tests, and beginning mathematics courses, paper presented at the annual meeting of the American Educational Research Association, April 2001; *Resources in Education*, ED 453 217.
25. A. H. Schoenfeld, Making mathematics work for all children: issues of standards, testing, and equity, *Educ. Researcher* **31** (2002) 13–25.
26. H. Wu, The Mathematician and the mathematics education reform, *Notices Amer. Math. Soc.* **43** (1996) 1531–1537.
27. H. Wu, The mathematics education reform: why you should be concerned and what you can do, this MONTHLY **104** (1997) 946–954.

HAL SCHOEN received his M.S. in mathematics at the University of Dayton and his Ph.D. in mathematics education at Ohio State University under the direction of F. Joe Crosswhite. He is professor of mathematics education at the University of Iowa, with a joint appointment in the Mathematics Department and Curriculum & Instruction. He was a member of the writing team of the 1989 NCTM *Curriculum and Evaluation Standards for School Mathematics*. He also served as Co-Chair of the American Educational Research Association's Special Interest Group for Research in Mathematics Education. Since 1992, he has been codirector of the Core-Plus Mathematics Project.

University of Iowa, N287 Lindquist Center, Iowa City, IA 52242
harold-schoen@uiowa.edu

CHRIS HIRSCH received his M.S. in mathematics at the University of Illinois and his Ph.D. in mathematics (with specialization in mathematics education) at the University of Iowa under the direction of H. Vernon Price. He is a professor of mathematics at Western Michigan University, where he teaches courses in mathematics for teachers and in mathematics education. He was a member of NCTM's first Commission on Standards for School Mathematics and chaired the Curriculum Working Group for Grades 9–12. Since 1992, he has been codirector of the Core-Plus Mathematics Project.

Western Michigan University, Department of Mathematics, Kalamazoo, MI 49008
christian.hirsch@wmich.edu