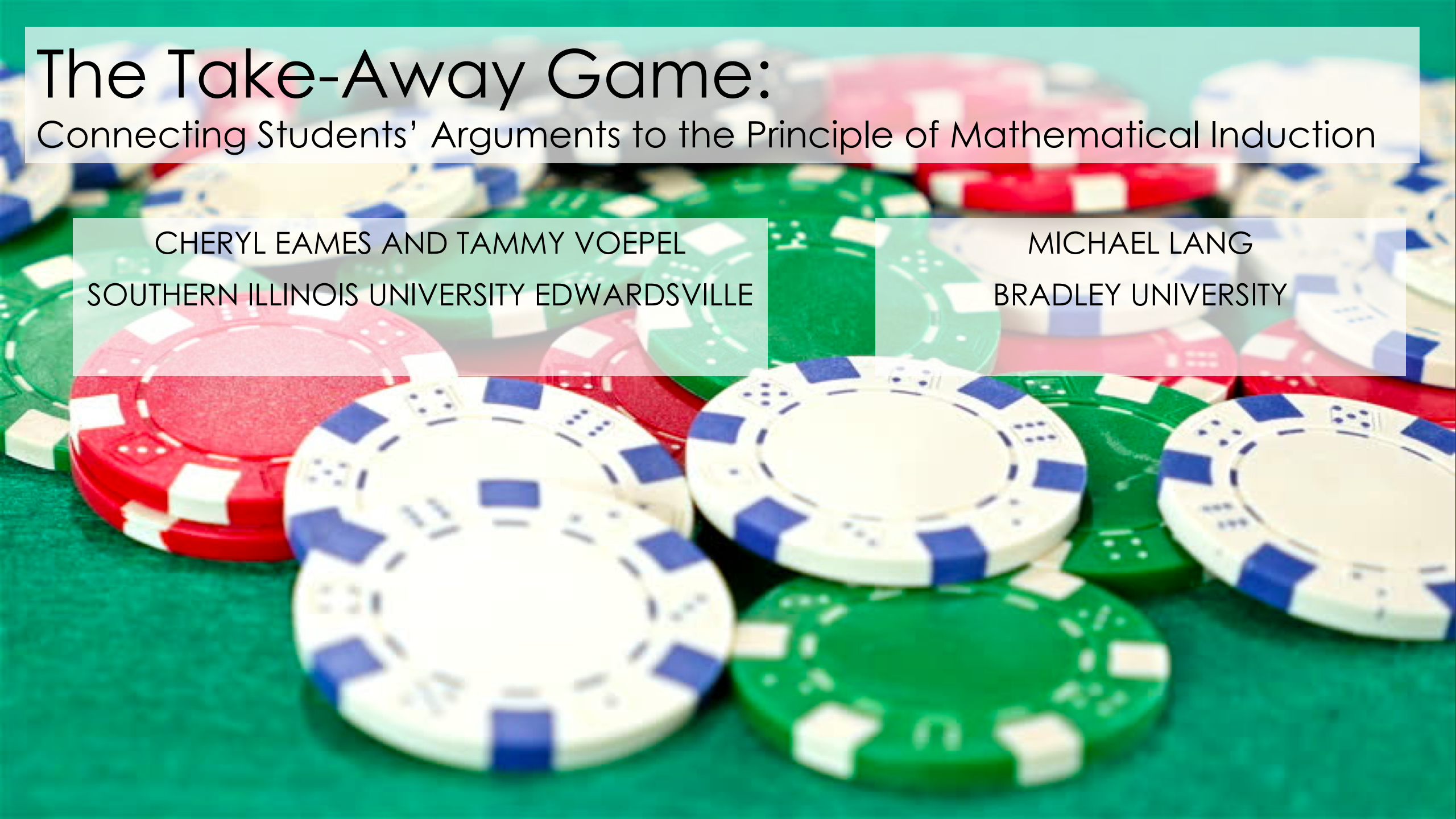


The Take-Away Game:

Connecting Students' Arguments to the Principle of Mathematical Induction

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Principles behind the lesson design

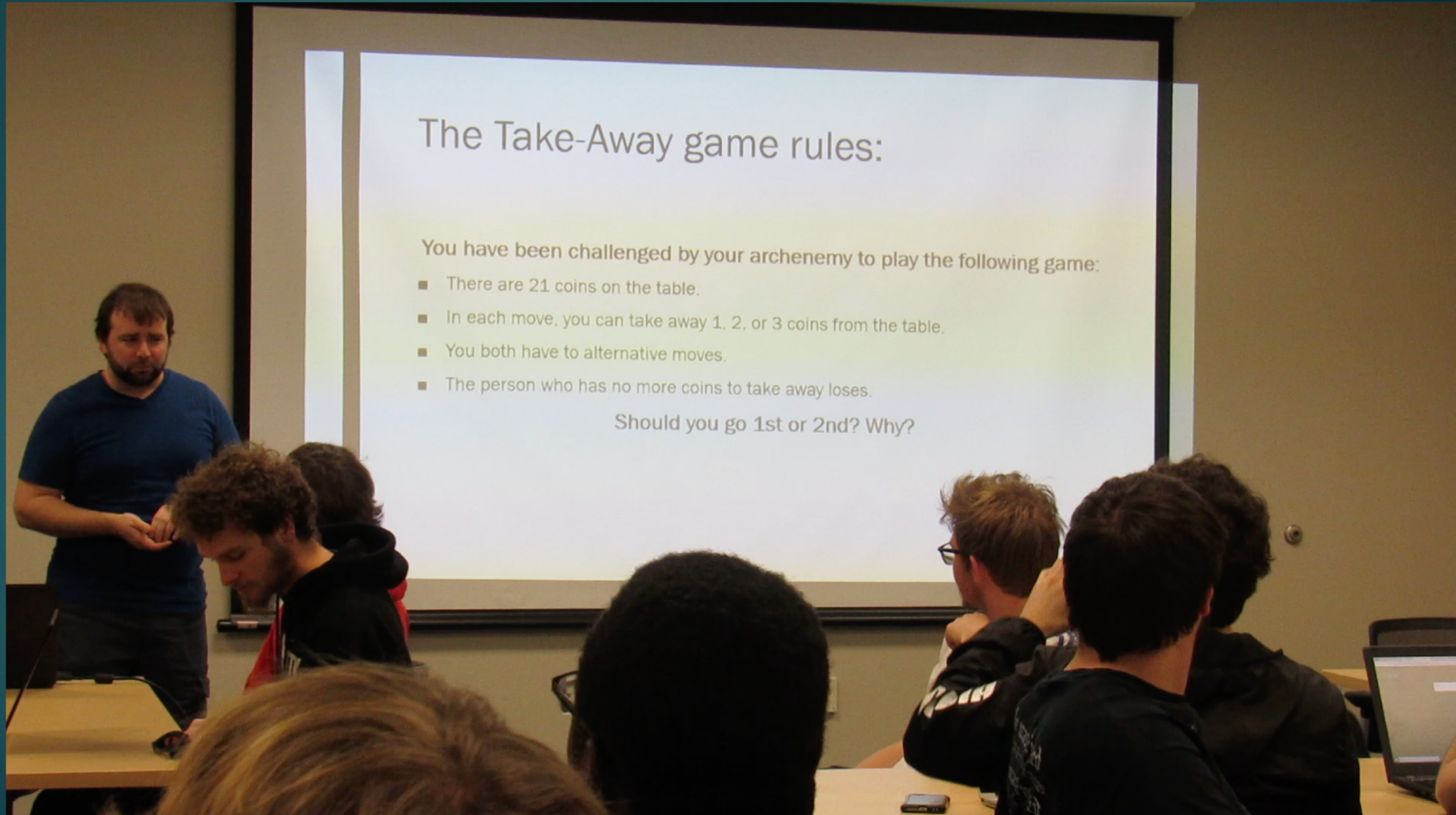
- ▶ Informed by 5 Practices of Orchestrating Productive Math Discussions (Smith & Stein, 2011)
- ▶ Goals:
 - ▶ Provide opportunity to construct viable arguments and critique the reasoning of others (CCSSM Practice 3)
 - ▶ Introduce students to the principle of mathematical induction as a technique for proving a statement that is asserted about every natural number
 - ▶ Engage students in continued practice with processes and notation for modular arithmetic

Students' conceptual difficulties with proof by induction

Students tend to develop an **authoritative proof scheme** for mathematical induction; **they are able to follow the rules and steps prescribed by the principle of mathematical induction without understanding its meaning** (Harel, 2001)

- ▶ To avoid developing an authoritative proof scheme, Harel (2001) suggests:
 - ▶ Creating a scenario in which students can experience a process of pattern generalization
 - ▶ Providing opportunities for students to see how “the principle is born out of a need to solve problems” and “how it is derived from previous, more elementary experience” (p. 194)

The Take-Away Game



The Take-Away game rules:

You have been challenged by your archenemy to play the following game:

- There are 21 coins on the table.
- In each move, you can take away 1, 2, or 3 coins from the table.
- You both have to alternative moves.
- The person who has no more coins to take away loses.

Should you go 1st or 2nd? Why?

5 Practices of Orchestrating Productive Math Discussions (Smith & Stein, 2011)

1. **Anticipating** likely student responses to challenging mathematical tasks
2. **Monitoring** students' actual responses to the task
3. **Selecting** particular students to present their mathematical work in a whole-class discussion
4. **Sequencing** the student responses that will be displayed in a specific order
5. **Connecting** student responses to each other and to key mathematical ideas

Practice 1

Anticipating likely student responses to challenging mathematical tasks

Here is what we thought our students would notice.

- ▶ With only 1, 2, or 3 chips left, the person who draws 1st wins.
- ▶ With only 4 chips left, the person to draw 2nd wins.
- ▶ With 5, 6, or 7 chips on the table, the 1st person to draw can win by creating the "4 chips left" situation again.
- ▶ A pile of chips that is a multiple of 4 is just like the "4 chips left" situation. The person to draw 2nd wins.
- ▶ When there is a pile of chips that is not a multiple of 4, the 1st person to draw can win (by again reducing to the multiple of 4 situation in the 1st play).
So, with 21 chips at the start, I should try to go 1st. If the other person goes 1st, I can only win if he/she "makes a mistake" by not creating a pile of chips that is a multiple of 4.

Practice 2

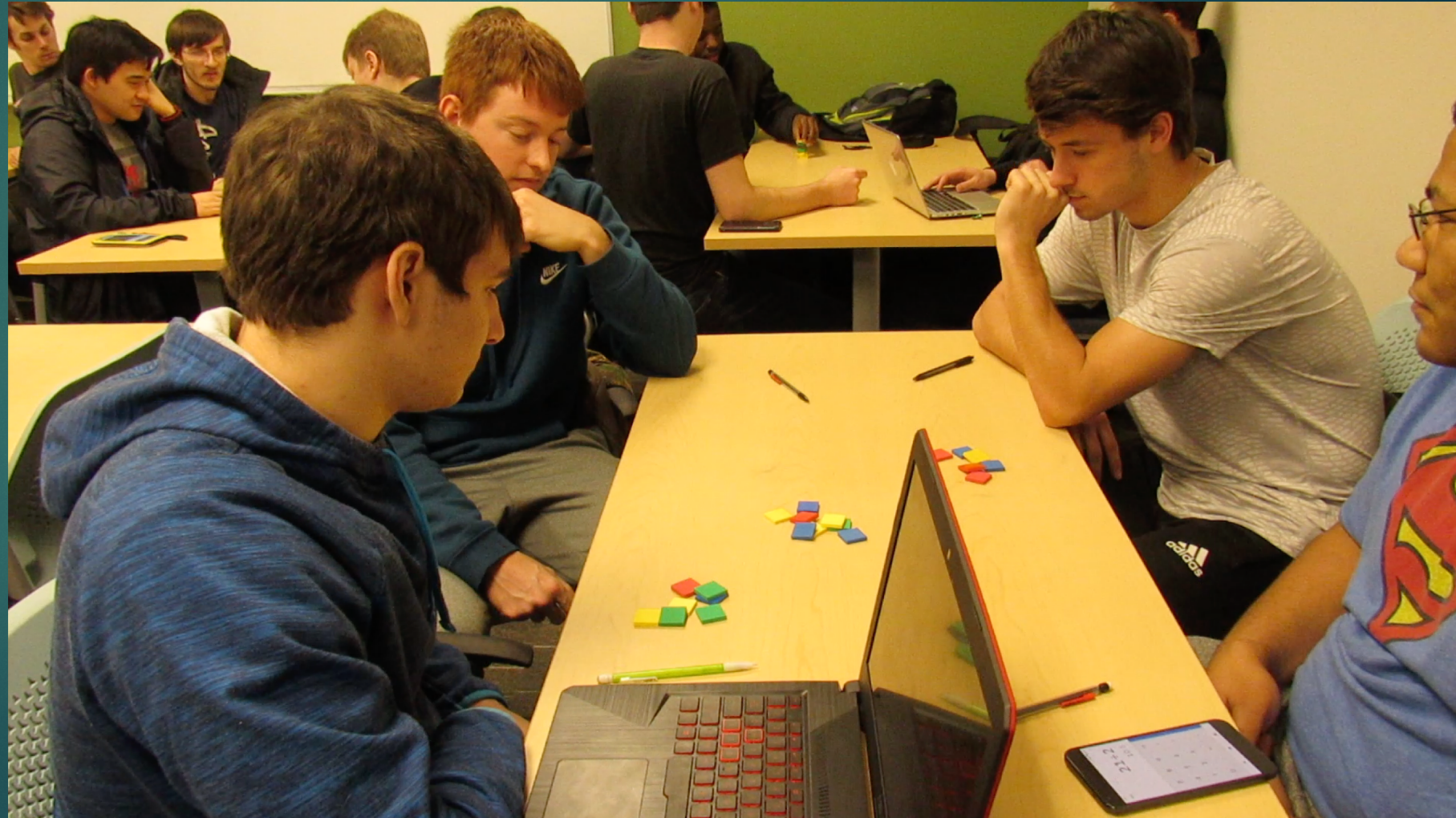
Monitoring students' actual responses to the task

There are 8 chips on the table....

Superman: So, you need to leave 4 left.

Gray tee: Yeah...that's what...you want the other person to have to choose from four.

Superman: Uh huh. Because no matter what they pick, they'll always lose.



Practice 2

Monitoring students' actual responses to the task



Anticipated Student Approaches	Planned Instructor Response
Students play with a trial-and-error strategy.	Small piles might be easier to think about than big piles. What would the smallest possible pile be? How would you handle that? Why?
With only 1, 2, or 3 chips left, the person to draw 1 st wins.	Suppose there are 4 chips on the table. Should you go 1 st or 2 nd ? Why?
With only 4 chips left, the person to draw 2 nd wins.	Suppose there are 5 chips on the table. Should you go 1 st or 2 nd ? Why? 6? 7?
With 5, 6, or 7 chips on the table, the 1 st person to draw can win by creating the "4 chips left" situation again.	Suppose there are 8 chips on the table. Should you go 1 st or 2 nd ? Why?
A pile of chips that is a multiple of 4 is just like the "4 chips left" situation. The person to draw 2 nd wins.	Suppose the number of chips in the pile is not a multiple of 4. Should you go 1 st or 2 nd ? Why?
When there is a pile of chips that is not a multiple of 4, the 1 st person to draw can win (by again reducing to the multiple of 4 situation). So, with 21 chips at the start, I should try to go 1 st . If the other person goes 1 st , I can only win if he/she "makes a mistake" by not creating a pile of chips that is a multiple of 4.	Suppose there are n chips on the table. Should you go 1 st or 2 nd ? Why?

Practice 2

Monitoring students' actual responses to the task

Gray tee: I would say what you have to do...is you want to go first. And then you take...as you noticed it starts at 4. And then like everything after you can control at 7, 10, 13, 16, 19. So, you take 2 away and then you're at 19, and then whatever they do, you do.

Superman: You just try and keep it at 3.

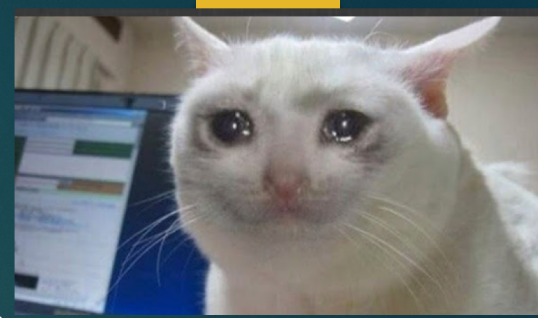
Gray tee: So, like if they take 1, you take 2. If they take 2, you take 1. If they take 3, you have to take 3 as well.

Superman: uh huh



Practices 3 & 4

Selecting particular students to present their mathematical work during the whole-class discussion, and **Sequencing** the student responses that will be displayed in a specific order



yummmm chips

Bruh

Add more chips if you're losing

FOURce the other player to have a multiple of four from which to choose

whyyyyyy^^

Try to eliminate all but 4 to win

As long as the first person takes 2, they won't lose

If you go second, try mirroring the archenemy until you can leave them with only 4

If at the end of your turn, opponent is left with $n = 4k$, you win by definition of a victory.

If at the end of your turn, opponent is left with:

$$n = 4k + 1,$$

$$n = 4k + 2,$$

$$n = 4k + 3$$

, you lose.

where:

k is an integer

n is total number of chips



Practice 5

Connecting different student responses and
connecting the responses to key mathematical ideas



Anticipated Student Response	Connecting to the Principle of Mathematical Induction
With only 1, 2, or 3 chips left, the person to draw 1 st wins. With only 4 chips left, the person to draw 2 nd wins.	<u>Base case</u> If there are 1, 2, or 3 chips on the table, the 1 st player wins. If there are 4 chips, the 2 nd player wins.
A pile of chips that is a multiple of 4 is just like the "4 chips left" situation. The person to draw 2 nd wins. A pile of chips that is not a multiple of 4 is just like the "1, 2, or 3" chips left situation. The 1 st person to draw can win by creating the "multiple of 4 chips left" situation.	<u>Induction hypothesis</u> <u>Suppose</u> that our theorem is true for piles up to size n . That is, for a pile of size i where $1 \leq i \leq n$, <u>suppose</u> that • if i is not a multiple of 4, the player who is about to move can win, and • if i is a multiple of 4, the player who is about to pick will lose.
	<u>Induction Step</u> Now <u>prove</u> that the theorem is true if we face a pile of size $n + 1$. Let's write our $n + 1$ as $4a + b$ where $a \geq 1$, and $b \in \{0, 1, 2, 3\}$. If $b \neq 0$, then $n + 1$ is not a multiple of 4. We should remove b chips from the table; this results in $4a$ chips on the table. From here, by the induction hypothesis, the archenemy loses and we win. If $b = 0$, then $n + 1 = 4a$ is a multiple of 4. No matter how many chips (1, 2, or 3) we pick up, the archenemy is facing a pile whose size is not a multiple of 4. By the induction hypothesis, we lose.

Follow-up problems

- ▶ More games that involve inductive reasoning
 - ▶ Suppose each move consisted of removing either 1 or 3 chips. For what values of n does the 1st player have a winning strategy?
 - ▶ Consider another take-away game, where each move consists of removing at least one chip and at most k chips, for $k \geq 1$. For what size chip piles does the 1st player have a winning strategy?
- ▶ Critiquing invalid “proofs” by induction
- ▶ More classic proof by induction examples

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