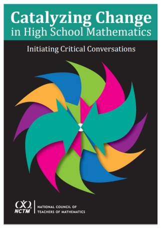


Catalyzing Change in High School Mathematics: Initiating Critical Conversations

Mathematical Practices, Processes, & Essential Concepts

Key Recommendation: Each and every student should learn the Essential Concepts in order to expand professional opportunities, understand and critique the world, and experience the joy, wonder, and beauty of mathematics.

Students should leave high school with the quantitative literacy and critical thinking processes needed to make wise decisions in their personal lives. Students should be able to determine whether or not claims made in scientific, economic, social, or political arenas are valid. Students should have an appreciation for the beauty and usefulness of mathematics and statistics. And students should see themselves as capable lifelong learners and confident doers of mathematics and statistics. Never have the broader aims of mathematics education been more important than they are today, when mathematics underlies much of the fabric of society, from polling and data mining in politics, to algorithms targeting advertisements to groups of people on social media, to complex mathematical models of financial instruments and policies that affect the lives of millions of people.

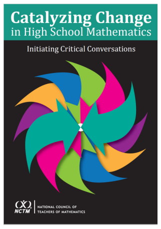
To support these purposes for learning mathematics, *Catalyzing Change* offers forty-one Essential Concepts in the domains of number, algebra and functions, statistics and probability, and geometry and measurement that each and every student should learn. As shown in figure 1, the Essential Concepts are organized into areas of focus within the domains, except in the domain of number, since at the high school level, concepts in number are typically woven into content and instruction in the other domains.

Essential Concepts in Number Essential Concepts in Algebra and Functions Focus 1: Algebra Focus 2: Connecting Algebra to Functions Focus 3: Functions Essential Concepts in Statistics and Probability Focus 1: Quantitative Literacy Focus 2: Visualizing and Summarizing Data Focus 3: Statistical Inference Focus 4: Probability	Essential Concepts in Geometry and Measurement Focus 1: Measurement Focus 2: Transformations Focus 3: Geometric Arguments, Reasoning, and Proof Focus 4: Solving Applied Problems and Modeling in Geometry

Fig. 1. Areas of focus in the Essential Concepts within the domains of high school mathematics

The Essential Concepts do not represent yet another set of standards or a list of disconnected topics to be covered. The Essential Concepts represent a distillation of the critical concepts and skills that, regardless of a state's, province's, or district's standards, students should acquire, retain, and be able to use long after high school. In outlining this critical content, *Catalyzing Change* highlights the roles of technology, reasoning and proof, and modeling, as well as the connections among the content areas. The Essential Concepts specifically address the widespread concern that high school mathematics standards lack the focus of the K–8 mathematics standards—a shortcoming that increases the difficulty that teachers have in supporting students in developing a deep foundational understanding of mathematics.

From: *Catalyzing Change in High School Mathematics: Initiating Crucial Conversations*, NCTM (2018).



Developing Mathematical Practices and Processes

Learning mathematics involves more than just acquiring content and carrying out procedures. Mathematics primarily consists of solving problems. Over time, mathematical concepts have been developed to solve particular types of problems, and, in turn, problems have been invented to shed light on the concepts that have been invented. The mathematical practices and processes that students engage in as they solve problems deepens their understanding of key mathematical concepts. Students' development as problem solvers is as important as students' acquisition of key concepts.

Mathematical modes of thought, sometimes referred to as mathematical practices or mathematical habits of mind (NCTM 2009), "are useful for reasoning about the world from a quantitative or spatial perspective and for reasoning about the mathematical content itself, both within and across mathematical fields" (Levasseur and Cuoco 2003, p. 27). Numerous lists of mathematical practices and processes have appeared in standards documents, including the Standards for Mathematical Practice in the Common Core State Standards for Mathematics (NGA Center and CCSSO 2010a), the Process Standards from *Principles and Standards for School Mathematics* (NCTM 2000), and the reasoning habits in *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM 2009).

Two mathematical practices—reasoning and proof, considered together as a single practice, and mathematical modeling—represent broader approaches that cut across the content areas and provide opportunities for teachers to enhance understanding of the Essential Concepts, as well as to employ each of the eight Mathematics Teaching Practices (NCTM 2014) and their connections with equitable teaching. These two practices are described below and highlighted, when appropriate, within each content domain.

Reasoning and Proof and the Cycle of Inquiry and Justification

Mathematical knowledge often grows in a cycle of inquiry and justification. As Bass (2015) describes, knowledge often progresses through a trajectory marked by five phases: exploration, discovery, conjecture, proof, and certification. The first three phases involve the reasoning of inquiry, and the last two phases involve the reasoning of justification. Exploration and experimentation are activities in which students discover patterns and form and reject and/or refine conjectures. Discovery and conjecture often involve inductive reasoning—making inferences from specific examples to general statements. Proof, by contrast, involves determining when and why a conjecture does or does not hold by using deductive reasoning—applying general theorems or statistical methods to specific instances. The inductive aspect of developing mathematical knowledge is critical in facilitating the deductive aspect. Together, these inductive and deductive activities form a cycle because the proof process may lead to rejecting or refining conjectures, starting the process over again. The proof process may also help students to generalize discoveries and deepen their understanding enough to help them remember an important claim and how to apply it.

The idea that logical conclusions can be established by using reasoning and proof is a standard of knowledge that is quite special to mathematics. Unlike other fields, where new knowledge may invalidate old knowledge, statements in mathematics are not easily overturned by new knowledge. A geometry fact or an algebraic identity or the validity of a statistical method can be established, for all time and all cases, by reasoning from given definitions and assumptions, whereas even the reliability of a statistical claim has a quantifiable level of uncertainty.

Students should recognize the process of establishing valid mathematical statements as the central act of doing mathematics, and they should see this process as the entire cycle of inquiry and justification.

Mathematical Modeling and the Modeling Cycle

A mathematical model is a mathematical representation of a particular real-world process or phenomenon that is under examination, in an attempt to describe, explore, or understand it. When students engage in mathematical modeling, they often have the opportunity to leverage mathematics to understand and critique the world. Mathematical modeling is the creative, often collaborative, process of developing these representations. Modeling always requires decision making that involves determining which aspects of the phenomenon to include in the model and which to suppress or ignore and what kind of mathematical representation to use.

The mathematical modeling cycle begins with a real problem and involves a number of steps (NGA Center and CCSSO 2010a; Consortium for Mathematics and Its Applications and Society for Industrial and Applied Mathematics [COMAP and SIAM] 2016):

- Formulating the problem or question
- Stating assumptions (often requiring simplifications of the real situation) and defining variables
- Restating the problem or question mathematically
- Solving the problem in the mathematical model
- Analyzing and assessing the solution and the mathematical model
- Refining the model, going back to the first steps if necessary
- Reporting the results

The “messiness” of many authentic modeling problems is also a critical part of the process. In mathematical modeling, students develop techniques, use tools, and employ different perspectives as they apply concepts to create models based on problems drawn from their own lives or from situations likely to be encountered later in life.

Modeling, both in small activities and through the full cycle, should be employed within and across each of the content domains to investigate how different phenomena might perform or behave or different events might unfold under given constraints or assumptions. Such models provide opportunities to predict what might, could, or probably will happen and, as a consequence, can empower students as decision makers in informed and productive ways that directly contribute to their ability to understand and critique the world.

Mathematical modeling is central and essential to providing high school students with the knowledge, skills, and dispositions needed to make greater sense of the world. Modeling mathematics and statistics should be key components throughout any high school mathematics program.

Technology as a Driver of Change in Student Engagement
Finally, with respect to practices and processes, it is important to recognize that technology is driving changes that must be reflected in the high school mathematics curriculum and students' engagement with the Essential Concepts. Digital technology can serve three main functions (Drijvers, Boon, and Van Reeuwijk 2011):

1. As a tool for doing mathematics (e.g., when the purpose of a task is not to develop computational or symbolic manipulation expertise)
2. As a learning environment for fostering the development of conceptual understanding (e.g., illustrating the connection between functions and their graphs in a dynamic environment)
3. As a learning environment for practicing skills

Students should have opportunities to use mathematical action technologies in all content domains to explore mathematical relationships and deepen their understanding of Essential Concepts, to interpret mathematical representations, and to employ complex manipulations necessary to solve problems. Mastery of skills should not be a prerequisite for using technology in any content area; rather, the focus when using technology should be on developing understanding and interpreting the results (Roschelle et al. 2000; Sacristán et al. 2010).

Essential Concepts

Essential Concepts in Number

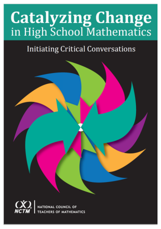
- Together, irrational numbers and rational numbers complete the real number system, representing all points on the number line.
- Quantitative reasoning includes, and mathematical modeling requires, attention to units of measurement.

Essential Concepts in Algebra and Functions

Focus 1: Essential Concepts in Algebra

- Expressions can be rewritten in equivalent forms by using algebraic properties, including properties of addition, multiplication and exponentiation, to make different characteristics or features visible.
- Finding solutions to an equation, inequality or system of equations or inequalities requires the checking of candidate solutions, whether generated analytically or graphically to ensure that solutions are found and that those found are not extraneous.
- The structure of an equation or an inequality (including, but not limited to, one-variable linear and quadratic equations, inequalities and systems of linear equations in two variables) can be purposefully analyzed (with and without technology) to determine an efficient strategy to find a solution, if one exists, and then to justify the solution.
- Expressions, equations and inequalities can be used to analyze and make predictions, both within mathematics and as mathematics is applied in different contexts – in particular, contexts that arise in relation to linear, quadratic and exponential situations.

Focus 2: Essential Concepts in Connecting Algebra to Functions



- Functions shift the emphasis from a point-by-point relationship between two variables (input/output) to considering an entire set of ordered pairs (where each first element is paired with exactly one second element) as an entity with its own features and characteristics.
- Graphs can be used to obtain exact or approximate solutions of equations, inequalities, and systems of equations and inequalities – including systems of linear equations in two variables and systems of linear and quadratic equations (given or obtained by using technology).

Focus 3: Essential Concepts in Functions

- Functions can be described by using a variety of representations: mapping diagrams, function notation (e.g., $f(x) = x^2$), recursive definitions, tables and graphs.
- Functions that are members of the same family have distinguishing attributes (structure) common to all functions within that family.
- Functions can be represented graphically, and key features of the graphs, including zeros, intercepts, and, when relevant, rate of change, and maximum/minimum values, can be associated with and interpreted in terms of the equivalent symbolic representation.
- Functions model a wide variety of real situations and can help students understand the processes of making and changing assumptions, assigning variables, and finding solutions to contextual problems.

Essential Concepts in Statistics and Probability

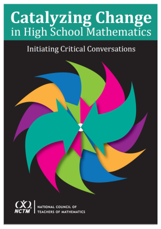
Focus 1: Essential Concepts in Quantitative Literacy

- Mathematical and statistical reasoning about data can be used to evaluate conclusions and assess risks.
- Making and defending informed data-based decisions is a characteristic of a quantitatively literate person.

Focus 2: Essential Concepts in Visualizing and Summarizing Data

- Data arise from a context and come in two types: quantitative (continuous or discrete) and categorical. Technology can be used to "clean" and organize data including very large data sets, into a useful and manageable structure – a first step in any analysis of data.
- Distributions of quantitative (continuous or discrete) in one variable should be described in the context of the data with respect to what is typical (the shape with appropriate measures of center and variability, including standard deviation) and what is not (outliers), and these characteristics can be used to compare two or more subgroups with respect to a variable.
- The association between two categorical variables is typically represented by using two-way tables and segmented bar graphs.

Focus 2: Essential Concepts in Visualizing and Summarizing Data, cont.



- Scatterplots, including plots over time, can reveal patterns, trends, clusters, and gaps that are useful in analyzing the association between two contextual variables.
- Analyzing the association between two quantitative variables should involve statistical procedures, such as examining (with technology) the sum of squared deviations in fitting a linear model, and analyzing residuals for patterns, generating a least-squares regression line and finding a correlation coefficient, and differentiating between correlation and causation.
- Data-analysis techniques can be used to develop models of contextual situations and to generate and evaluate possible solutions to real problems involving those contexts.

Focus 3: Essential Concepts in Statistical Inference

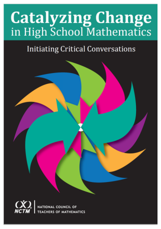
- Study designs are of three main types: sample survey, experiment, and observational study.
- The role of randomization is different in randomly selecting samples and in randomly assigning subjects to experimental treatment groups.
- The scope and validity of statistical inferences are dependent on the role of randomization in the study design.
- Bias, such as sampling, response or nonresponse bias, may occur in surveys, yielding results that are not representative of the population of interest.
- The larger the sample size, the less the expected variability in the sampling distribution of a sample statistic.
- The sampling distribution of a sample statistic formed from repeated samples for a given sample size drawn from a population can be used to identify typical behavior for that statistic. Examining several such sampling distributions leads to estimating a set of plausible values for the population parameter, using the margin of error as a measure that describes the sampling variability.
- Simulation of sampling distributions by hand or with technology can be used to determine whether a statistic (or statistical difference) is significant in a statistical sense or whether it is surprising or unlikely to happen under the assumption that outcomes are occurring by random chance.

Focus 4: Essential Concepts in Probability

- Two events are independent if the occurrence of one event does not affect the probability of the other event. Determining whether the two events are independent can be used for finding and understanding probabilities.
- Conditional Probabilities – that is, those probabilities that are "conditioned" by some known information- can be computed from data organized in contingency tables. Conditions or assumptions may affect the computation of a probability.

Essential Concepts in Geometry and Measurement

Focus 1: Essential Concepts in Measurement



- Areas and volumes of figures can be computed by determining how the figure might be obtained from simpler figures by dissection and recombining.
- Constructing approximations of measurements with different tools, including technology, can support and understanding of measurement.
- When an object is the image of a known object under a similarity transformation, a length, area, or volume on the image can be computed by using proportional relationships.

Focus 2: Essential Concepts in Transformations
- Applying geometric transformations to figures provides opportunities for describing the attributes of the figures preserved by the transformation and for describing symmetries by examining when a figure can be mapped onto itself.
- Showing that two figures are *congruent* involves showing that there is a rigid transformation (translation, rotation, reflection, or glide reflection) or, equivalently, a sequence of rigid motions that maps on figure to the other.
- Show that two figures are *similar* involves finding a *similarity transformation* (dilation or composite of a dilation with a rigid motion) or, equivalently, a sequence of similarity transformations that maps one figure onto the other.
- Transformations in geometry serve as a connection with algebra, both through the concept of functions and through the analysis of graphs or functions as geometric figures.

Focus 3: Essential Concepts in Geometric Arguments, Reasoning and Proof
- Proof is the means by which we demonstrate whether a statement is true or false mathematically, and proofs can be communicated in a variety of ways (e.g., two-column, paragraph).
- Using technology to construct and explore figures with constraints provides an opportunity to explore the independence and dependence of assumptions and conjectures.
- Proofs of theorems can sometimes be made with transformations, coordinates or algebra; all approaches can be useful, and in some cases one may provide a more accessible or understandable argument than another.

Focus 4: Essential Concepts in Solving Applied Problems and Modeling in Geometry
- Recognizing congruence, similarity, symmetry, measurement opportunities, and other geometric ideas, including right triangle trigonometry in real-world contexts, provides a means of building understanding of these concepts and is a powerful tool for solving problems related to the physical world in which we live.
- Experiencing the mathematical modeling cycle in problems involving geometric concepts, from simplification of the real problem through the solving of the simplified problem, the interpretation of its solution, and the checking of the solution's feasibility, introduces geometric techniques, tools, and points of view that are valuable to problem solving.