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The following pages are adapted from (1) Heid, M. K., & Wilson, P. W. (with G. W. Blume). (Eds.) (2015). *Mathematical understanding for secondary teaching: A framework and classroom-based situations.* Charlotte, NC: Information Age. and (2) Zbiek, R. M., Blume, G. W., & Heid, M. K. (Eds.) (2018). *Facilitator's guidebook for use of mathematics situations in professional learning.]*

5 minutes:

Read the prompt and talk with a neighbor about what mathematics you would want the student teacher in the prompt to know.

Situation 21. Graphing Quadratic Functions.

When preparing a lesson on graphing quadratic functions, a student teacher found that the textbook for the class claimed that $x = -\frac{b}{2a}$ was the equation for the line of symmetry of a parabola $y = ax^2 + bx + c$. The student teacher wondered about the mathematical basis for the equation.

Following are mathematical foci that you may have identified. Read them.

Mathematical Focus 1. *Knowing the general form of the equation representing the graph of a parabola can enable one to identify the line of symmetry.*

Mathematical Focus 2. *The first derivative of a polynomial function can be used to obtain the coordinates of the relative extrema of the function. In a parabola, this corresponds to the vertex, the* x*-coordinate of which gives the* x*-coordinate of all points on the line of symmetry.*

Mathematical Focus 3. *Using transformations, the graph of the function $y = x^2$ can be mapped to the graph of any quadratic function of the form $y = ax^2 + bx + c$. The graph of the function given by $g(x) = v_1 f\left(h_2(x - h_1)\right) + v_2$ is the image of the graph of* f *under the composition of a horizontal translation, a horizontal stretch or contraction, a vertical stretch or contraction, and a vertical translation related to the values of h_1, h_2, v_1, and v_2, respectively.*

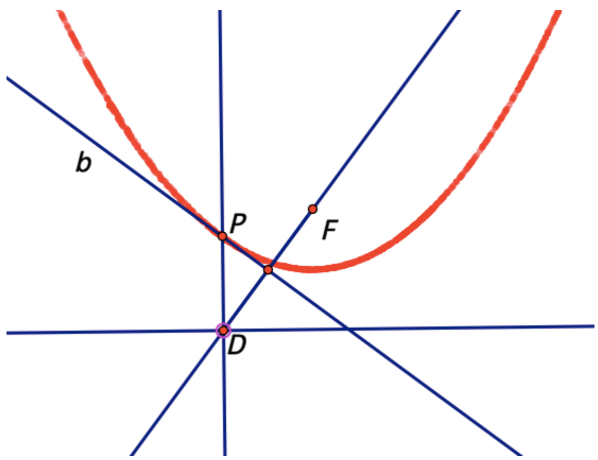
Mathematical Focus 4. *General facts about the roots of polynomial equations, known as Viète's formulas, can quickly yield information about the line of symmetry of a parabola.*

[Given the roots of a quadratic function, r_1 and r_2, a result of François Viète states that $r_1 + r_2 = -\frac{b}{a}$ and $r_1 \cdot r_2 = \frac{c}{a}$.]

Background (For later) Point out that participants will benefit from review of the locus definition of parabola and a definition of line of symmetry. 1 minute

Related terminology: What is meant by each of the following:
parabola, function, quadratic function, and line of symmetry.

Why does the following dynamic geometry construction result in a parabola?
First, using a dynamic geometry environment, construct a line (called a *directrix*) and a point, F (called the *focus*), not on the directrix. Then construct a point D on the directrix and the segment DF. Then construct the perpendicular bisector of segment DF. Call it *b*. Next, construct a perpendicular to the directrix through the point D. Identify the intersection of the perpendicular and *b* as the point P. Finally, drag the point D along the directrix, tracing the point P as D is dragged. The resulting trace is a parabola. Can you explain why?

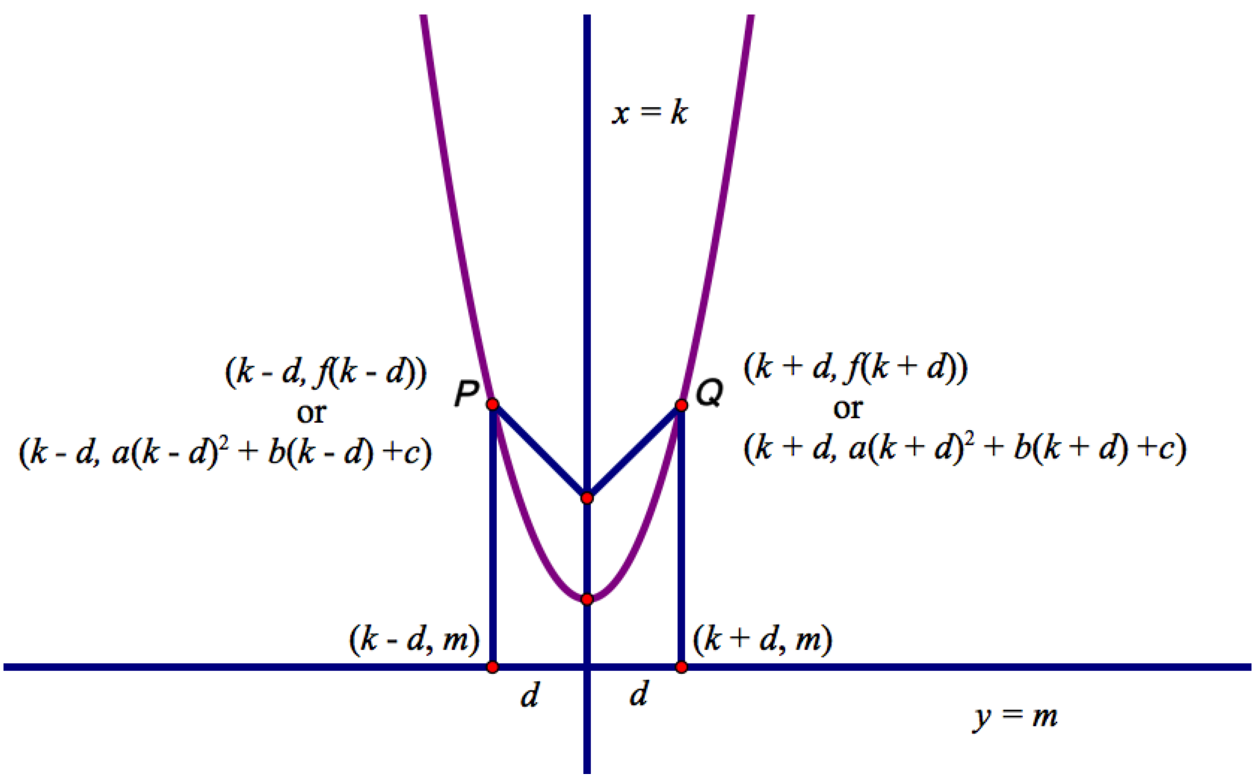


Connecting to the Foci

Something to consider related to Mathematical Focus 1:

Depending on the participants, the following could be done with a specific function rule for *f*. Ask for an explanation of why this approach makes sense.

Consider the graph of $f(x) = ax^2 + bx + c$.
Suppose there is a line of symmetry for *f* at $x = k$.
How can we find the value of *k* (in terms of *a*, *b*, and *c*) for the line of symmetry?



Hint: The parabola is symmetric about $x = k$ iff $f(k - d) = f(k + d)$ for every value of d. Solve this equation and interpret the result.

Something to consider related to Mathematical Focus 3:

Starting with $y = ax^2 + bx + c$, completing the square yields

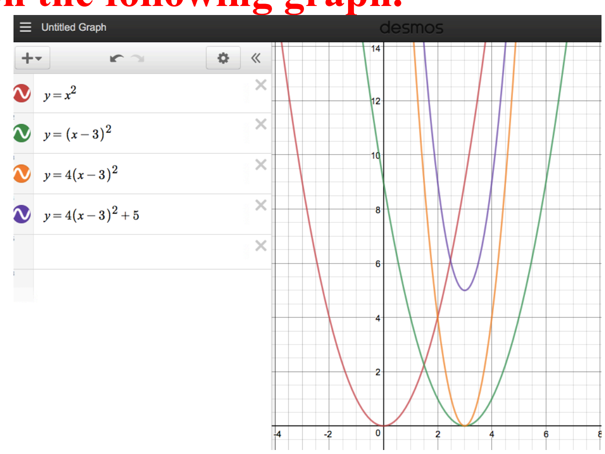
$$y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{2a}\right)$$

How can this be used to show that $x = -\frac{b}{2a}$ is a line of symmetry for the graph of. $y = ax^2 + bx + c$?

Hint: Recognize that the graph of $y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{2a}\right)$ as resulting from a sequence of transformations of the graph of $y = x^2$, that the graph of $y = x^2$ is symmetric about $x = 0$, and that the symmetry of $y = x^2$ is preserved under the given transformations.

Depending on the participants, different strategies might be pursued.

1. **Using an app such as Desmos ask participants to start with a quadratic function of the form $y = x^2$, then graph functions of the form $y = (x + a)^2$, $y = b(x + a)^2$, and $y = b(x + a)^2 + c$. Ask them to compare the symmetry of these functions. An example is on the following graph.**



2. The following could be done with a specific function rule for *f*.

a. Verify that "completing the square" for the function $y = 2x^2 + 6x + 4$ results in the equivalent $\boldsymbol{y = 2\left(x + \frac{3}{2}\right)^2 + \left(4 - \frac{3^2}{4}\right)}$. Ask them to make a transformation argument for why this function has the same line of symmetry as $\boldsymbol{y = x^2}$. If needed, lead participants through the following:

Compare the line of symmetry of $\boldsymbol{y = x^2}$ with the line of symmetry for $\boldsymbol{y = \left(x + \frac{3}{2}\right)^2}$.

Compare the line of symmetry of $\boldsymbol{y = \left(x + \frac{3}{2}\right)^2}$ with the line of symmetry for $\boldsymbol{y = 2\left(x + \frac{3}{2}\right)^2}$.

Compare the line of symmetry of $\boldsymbol{y = 2\left(x + \frac{3}{2}\right)^2}$ with the line of symmetry for $\boldsymbol{y = 2\left(x + \frac{3}{2}\right)^2 + \left(4 - \frac{3^2}{4}\right)}$.

b. Ask them to make the same argument for a quadratic function of their choice.

Something to think about related to Mathematical Focus 4:

This Focus is best pursued having participants work with a CAS calculator. They can then generate the products and experiment with others.

To highlight the role of the factors in forming the products, have participants consider the expansion of $(\boldsymbol{x_1} + 1)(x_2 + 2)(x_3 + 3) = x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot 3 + x_1 \cdot 2 \cdot x_3 + 1 \cdot x_2 \cdot x_3 + x_1 \cdot 2 \cdot 3 + 1 \cdot x_2 \cdot 3 + 1 \cdot 2 \cdot x_3 + 1 \cdot 2 \cdot 3$.

If $\boldsymbol{x_1, x_2,}$ *and* $\boldsymbol{x_3}$ are all the same (call them *x*), then the product is $\boldsymbol{x +} 1)(x + 2)(x + 3) = x \cdot x \cdot x + x \cdot x \cdot 3 + x \cdot 2 \cdot x + 1 \cdot x \cdot x + x \cdot 2 \cdot$

$3 + 1 \cdot x \cdot 3 + 1 \cdot 2 \cdot x = x^3 + (3 + 2 + 1)x^2 + (2 \cdot 3 + 1 \cdot 3 + 1 \cdot 2)x + 1 \cdot 2 \cdot 3$

Ask participants to articulate the pattern they see and extend it to

$$(x - 1)(x - 2)(x - 3)(x - 4)$$

$$(x - 1)(x - 2) = x^2 - 3x + 2$$
$$(x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6$$
$$(x - 1)(x - 2)(x - 3)(x - 4) = x^4 - 10x^3 + 35x^2 - 50x + 24$$

1. What pattern characterizes the coefficients in the expanded form? How does this pattern relate to Viète's formulas for the roots of a quadratic function? What explains the pattern?
2. Can you predict the expanded form of

$$(x - 1)(x - 2)(x - 3)(x - 4)(x - 5)?$$

Hint: Recognize that the expanded product is the sum of products formed by multiplying one term of each factor. Think about each term of the expansion being the sum of terms with the same power of *x*.

This Focus is best pursued having participants work with a CAS calculator. They can then generate the products and experiment with others.

To highlight the role of the factors in forming the products, have participants consider the expansion of $\boldsymbol{x_1} + 1)(x_2 + 2)(x_3 + 3) = x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot 3 + x_1 \cdot 2 \cdot x_3 + 1 \cdot x_2 \cdot x_3 + x_1 \cdot 2 \cdot 3 + 1 \cdot x_2 \cdot 3 + 1 \cdot 2 \cdot x_3 + 1 \cdot 2 \cdot 3.$

If $\boldsymbol{x_1}, \boldsymbol{x_2}, and\ \boldsymbol{x_3}$ are all the same (call them *x*), then the product is $\boldsymbol{x} + 1)(x + 2)(x + 3) = x \cdot x \cdot x + x \cdot x \cdot 3 + x \cdot 2 \cdot x + 1 \cdot x \cdot x + x \cdot 2 \cdot 3 + 1 \cdot x \cdot 3 + 1 \cdot 2 \cdot x = x^3 + (3 + 2 + 1)x^2 + (2 \cdot 3 + 1 \cdot 3 + 1 \cdot 2)x + 1 \cdot 2 \cdot 3$

Ask participants to articulate the pattern they see and extend it to $(x - 1)(x - 2)(x - 3)(x - 4)$.