Parallels between the Progressions and Modern Mathematics

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Part 0

About me

Calling #1: I'm an active, leading mathematical researcher, in algebraic and geometric topology.

Topologists study shapes and their interplay through continuous functions.

About me

Calling #2: I'm devoted to the authentic practice of mathematics, by all.

https://blogs.uoregon.edu/practiceofmathematics/

About me

On my own campus, that has meant creating or revising

- "bridge" course to proof-based mathematics
- pre-service elementary math
- entry-level mathematical modeling.

See my presentation for the MIT Electronic Seminar in Mathematics Education.

About me

And I happily partner with anyone who has a similar vision, working at any level.

Illustrative Mathematics · NCTM National Council of Teachers of Mathematics · Oregon Department of Education · Lane ESD · Oregon Math Network · Smarter Balanced Assessment Consortium · MAA Mathematical Association of America · University of Oregon

Part 1

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First exercise

Iterate the following functions. (That is, apply them repeatedly, to various starting values.)

- $a(x) = x^2$ (last name A-F)
- $b(x) = \frac{1}{2} x - 3$ (last name G-L)
- $c(x) = 3x - 4$ (last name M-S)
- $d(x) = \cos(x)$ (in radians; T-Z)

Then share your findings with your neighbors and discuss.

The contraction mapping theorem

… says that if a function shrinks distances, it has a unique fixed point.

Analogy(-proof): consider the "function" which sends a point in San Diego to the corresponding point in a (marauder's) map of San Diego sitting on the ground.

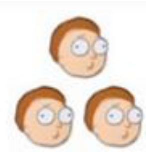
The contraction mapping theorem

Can be applied in very abstract contexts, such as when inputs to functions are not numbers but "pictures" and the functions can be defined geometrically through shrinking and union.

The contraction mapping theorem, applied to pictures

Credit: UConn Math Club

The contraction mapping theorem, applied to pictures



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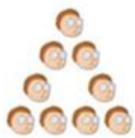
The contraction mapping theorem, applied to pictures



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The contraction mapping theorem, applied to pictures

A Sierpinski Triangle!

Credit: UConn Math Club

Key step for proofs

To set the foundation for this approach to fractal geometry, one considers the “space of all pictures,” which requires formulating a distance between pictures.

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The geometric process of transforming pictures is then a function!

“Spaces (or collections) of”

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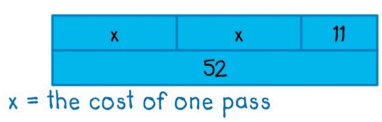
Functions (or other structures) on those spaces then operate globally on all possibilities, and can be leveraged to find special points (e.g. energy on the space of all paths).

“Spaces of”

The earliest in K-12, and most profound (ca 1600 historically), is the step to take the collection of numbers and recognize their geometry, through the number line. (The “space of” numbers.)

The wonder of the number line



“Is it even allowed to solve a problem that way?”

The wonder of the number line

“Is it even allowed to solve a problem that way?”

… Absolutely!

And modern mathematicians solve many problems by taking objects from calculus, algebra, combinatorics… collecting them, and arguing geometrically. (And conversely)

The wonder of the number line

Important note: the number line is introduced in second grade (2.MD.6) not to help students solve problems at hand but so they have experience with it when they do need it for further math, especially fractions.

(analogy: first one learns to read, then one reads to learn.)

“Spaces of”

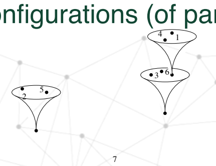
In the HS Progressions, we see graphs as spaces of input-output pairs, or more generally see curves as spaces of solutions of some equations.

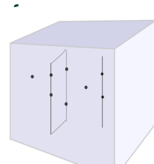
And we see the unit circle as the space of all angles.

Personal story

The space of pictures and its role in constructing fractals captured my imagination fairly early in my development.

Now I’m a world-renowned expert on “spaces of knots” and “spaces configurations (of particles)”.





Part 2

Second exercise

Can you add (or subtract, or multiply, or divide) "numbers" such as …56565656 and …137137 and …5356295141?

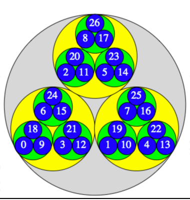
The 10-adics

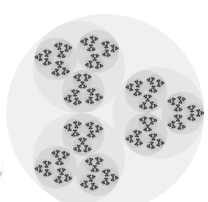
These numbers form a perfectly good system (a "ring"), akin to rational, real, or complex numbers, or polynomials!

Their "base p" version, the p-adics, are more often considered.

The "number fractal"

Believe it or not, there is some geometry which can be associated with these numbers (3-adics here). And it doesn't "go out to infinity"!





Peter Scholze

A 2018 Fields medalist at the age of 29, uses the p-adics so much, he finds them easier to think about than the standard number line!

But he has also carried them further than anyone previously.

Complex numbers

Are a miracle! We add solutions to one algebraic equation that lacked them over the real numbers, namely $x^2 + 1 = 0$, and now all polynomials have as many solutions as possible!!

Complex numbers

Contrast with what would happen if we love the rational numbers and polynomials so much that we "skipped over" including numbers such as pi and e, and just threw in solutions to polynomial equations…

… what a mess!

Scholze's perfectoid numbers

In order to extend our ability to study "shapes" of solutions of equations, Scholze developed a new system:

- Take these p-adics
- Make fractions from them
- Toss in all the solutions to the equations $x^n = p$, where $n = p^k$
- "Complete" again!

Scholze's perfectoid numbers

Sholze's insight is that these numbers behave like Taylor series(with mod-p coefficients)!

Believe it or not, mathematicians generally find Taylor series as "simpler" than numbers.

Scholze's perfectoid numbers

What's crucial here is not that he could imagine new numbers. Most highly trained mathematicians (including graduate students and many undergraduates) could "generalize" numbers in many ways. What's crucial is that he found new numbers which are remarkably useful (e.g. square roots arrive to utilize the Pythagorean theorem)

The number system

The progression of the development of numbers is the most important strand in K-12 mathematics.

I would like to see greater emphasis on NQ standards in high-school curricula. (e.g.

- To the nearest 1/16", how long should this diagonal support be?
- Prove algebraically that the (weighted) average of two numbers sits in between them.
- Compare and contrast the products 37 x 43 and
- (3x + 7) x (4x + 3)

Part 3

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Third exercise

Why should the product of two negative numbers be positive?

Why should $2^{1/3}$ be the cube root of two?

Third exercise

My suggestion (consistent with the university mathematical community) for the first question:

(10 + (-1)) x (10 + (-1)) = …

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"There is only one choice consistent with properties."

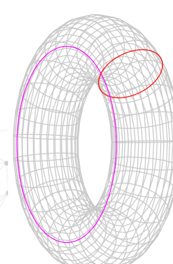
The story of (good) fibrations

Early (20th century) topologists saw the importance of "fiber bundles", which are "(possibly) twisted products" of geometric objects.

The story of (good) fibrations

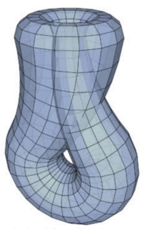
For example R^2 is the "product" of two number lines.

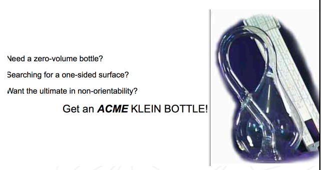
The torus (surface of a donut) is the product of two circles.



The story of (good) fibrations

A Klein bottle is a twisted product of two circles.





The story of (good) fibrations

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So in his 1951 PhD thesis, Serre generalized by defining a fibration to be anything with this property!

The story of (good) fibrations

Serre, Grothendieck and others in the mid-20th century led a shift from the study of mathematics through direct description to description through interrelationships.

Definitions through properties

Middle school mathematics has my vote for the most sophisticated definitions, as the development of negative numbers and of exponentiation requires a shift from direct verification of properties to "the only way we can keep having our favorite property is to define things this way…"

Definitions through properties

These definitions are worth time and attention!

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Suggestion: teach that the product of negative numbers is positive having a "mock trial" (akin to Model UN). Students work in different groups each to find arguments for a different putative answer, then have them cross-examine etc. (Also for 0/0 etc)

Part 4

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Exit ticket

Summarize the main points of each part of the presentation.
Optional frame: “One thing research mathematicians do is ________; we see that move as important in K-12 mathematics when _______.”

1.
2.
3.

Exit ticket

1.

Exit ticket

1. One thing research mathematicians do is take collections of mathematical objects and study them through geometry of the collection. We see this move in K-12 mathematics through the development of the number line, as well as topics such as graphs of functions.

Exit ticket

2.

Exit ticket

2. Another thing research mathematicians do is define new number systems in order to solve problems. We see this in the key thread through K-12 of the development of the standard real number system, where each new addition should be made slowly, carefully and with ample motivation.

Exit ticket

3.

Exit ticket

3. Another things research mathematicians do is define mathematical objects solely through their properties (interrelationships). We see this strategy as needed for subtle definitions introduced in middle school.

Thanks!

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