Kim Thomas
Moon Valley High School
NBCT, T[3] National Instructor

http://bit.ly/NCTM592

Limits and l'Hôpital – Learning Through Visualization

Calculating limits in calculus is more than following a set of rules and guidelines. Understanding limits, including the application of l'Hôpital's Rule, is made easier using visualization techniques using graphing technology. Assessment and practice for AP Calculus will also be addressed.

NCTM San Diego **Saturday, April 6, 2019** **8:00 – 9:15 AM**

Stewart – Single Variable Calculus 2009, page 291

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Note 1: L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule.

Note 2: L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

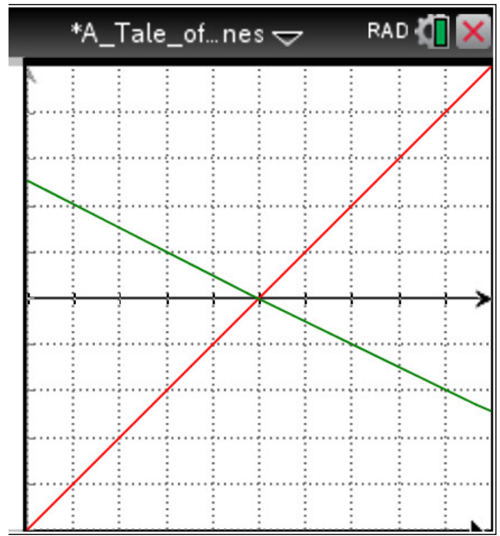
Note 3: For the special case in which $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$, it is easy to see why l'Hospital's Rule is true. In fact, using the alternative form of the definition of a derivative, we have

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\displaystyle\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} = \lim_{x \to a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

Kim.Thomas@guhsdaz.org

A Tale of Two Lines https://goo.gl/LGWnEM



1. Note that the grid markings on the graph represent the same scale for both *x* and *y*. These appear to be the graphs of two linear functions. Call the function with positive slope $\mathbf{f}(x)$ and the function with negative slope $\mathbf{g}(x)$.

   a. What is the slope of the graph of **f?**

   What is the slope of the graph of **g?**

   How do you know?

   b. Suppose the intersection point of the two graphs is (a, 0). Use the slope information from part 1a and the point slope form to find expressions for $\mathbf{f}(x)$ and for $\mathbf{g}(x)$ that would fit these two linear graphs.

2. Again, note that the grid markings on the graph represent the same scale for both *x* and *y*.

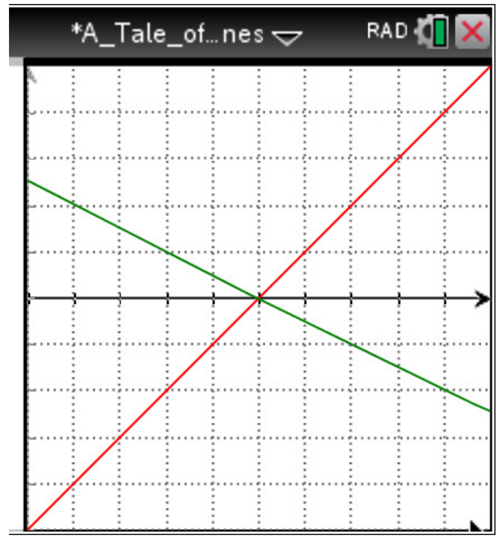
   a. Use the grid to find the ratio $\frac{\mathbf{f}(x)}{\mathbf{g}(x)}$ for 4 values of *x* that are different than $x = a$.

values of x	x =	x =	x = a	x =	x =
ratio $\frac{\mathbf{f}(x)}{\mathbf{g}(x)}$					

   b. Based on your answer to part 2a, what do you think the limit of $\frac{\mathbf{f}(x)}{\mathbf{g}(x)}$ is as x approaches a?

   Explain your reasoning.

3. In question 1, you were asked to find the slopes of the graphs of **f** and **g.** What is the ratio of the slope of the graph of **f** to the slope of the graph of **g?** How does this compare to the limit you found in question 2?



4. Notice the zooming tool to the left of the screen shows that you are "zoomed in" as much as allowed by the tool. Use the slider to zoom out on the graph.

   a. It turns out that these graphs were not straight lines. Why did they look like straight lines when you were zoomed in?

   b. The zoomed-in version of the graph allowed us to approximate the slopes of the graphs near $x = a$. What is another name for the slope of a function at a point?

   c. Why does this suggest that $\lim\limits_{x \to a} \left(\dfrac{\mathbf{f}(x)}{\mathbf{g}(x)} \right)$ is the same as $\lim\limits_{x \to a} \left(\dfrac{\mathbf{f}'(x)}{\mathbf{g}'(x)} \right)$? Explain your reasoning.

   d. Based on your response to part 4b, what is $\lim\limits_{x \to a} \left(\dfrac{\mathbf{f}(x)}{\mathbf{g}(x)} \right)$?

5. What you have seen is the geometry behind something called l'Hôpital's Rule.

   L'Hôpital's Rule states, in part, that for differentiable functions **f** and **g,** with $\lim\limits_{x \to a} \mathbf{f}(x) = 0$ and $\lim\limits_{x \to a} \mathbf{g}(x) = 0$, then $\lim\limits_{x \to a} \dfrac{\mathbf{f}(x)}{\mathbf{g}(x)} = \lim\limits_{x \to a} \dfrac{\mathbf{f}'(x)}{\mathbf{g}'(x)}$ if the limit of the quotients of the derivatives exists. Use l'Hôpital's Rule to find the following limits, if possible.

   a. $\lim\limits_{x \to 1} \dfrac{\ln x}{x - 1}$

   b. $\lim\limits_{x \to -1} \dfrac{x^6 - 1}{x^4 - 1}$

MPAC 1: Reasoning with definitions and theorems

- use definitions and theorems to build arguments, to justify conclusions or answers, and to prove results;
- confirm that hypotheses have been satisfied in order to apply the conclusion of a theorem;
- apply definitions and theorems in the process of solving a problem;
- interpret quantifiers in definitions and theorems (e.g., "for all," "there exists");
- develop conjectures based on exploration with technology; and
- produce examples and counterexamples to clarify understanding of definitions, to investigate whether converses of theorems are true or false, or to test conjectures.

MPAC 2: Connecting concepts

- relate the concept of a limit to all aspects of calculus;
- use the connection between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems;
- connect concepts to their visual representations with and without technology; and
- identify a common underlying structure in problems involving different contextual situations.

MPAC 3: Implementing algebraic/computational processes

- select appropriate mathematical strategies;
- sequence algebraic/computational procedures logically;
- complete algebraic/computational processes correctly;
- apply technology strategically to solve problems;
- attend to precision graphically, numerically, analytically, and verbally and specify units of measure; and
- connect the results of algebraic/computational processes to the question asked.

MPAC 4: Connecting multiple representations

- associate tables, graphs, and symbolic representations of functions;
- develop concepts using graphical, symbolical, or numerical representations with and without technology;
- identify how mathematical characteristics of functions are related in different representations;
- extract and interpret mathematical content from any presentation of a function (e.g., utilize information from a table of values);
- construct one representational form from another (e.g., a table from a graph or a graph from given information); and
- consider multiple representations of a function to select or construct a useful representation for solving a problem.

MPAC 5: Building notational fluency

- know and use a variety of notations;
- connect notation to definitions (e.g., relating the notation for the definite integral to that of the limit of a Riemann sum);
- connect notation to different representations (graphical, numerical, analytical, and verbal); and
- assign meaning to notation, accurately interpreting the notation in a given problem and across different contexts.

MPAC 6: Communicating

- clearly present methods, reasoning, justifications, and conclusions;
- use accurate and precise language and notation;
- explain the meaning of expressions, notation, and results in terms of a context (including units);
- explain the connections among concepts;
- critically interpret and accurately report information provided by technology; and
- analyze, evaluate, and compare the reasoning of others.

https://advancesinap.collegeboard.org/stem/calculus/mathematical-practices

Estimating Limits

Selected values for the function *f* are given in the table.

x	2	3	4	5	6	7	8
f(x)	3.7	4.3	4.9	4.8	5.6	6.2	6.9

The function *f* is increasing everywhere except at $x = 5$, and the limit as x approaches 5 for $f(x)$ exists. Which is a reasonable estimate for $\lim_{x \to 5} (f(x))$?

(a) 4.6 (b) 4.8 (c) 5.3 (d) 5.6

Limits and Tables

$$\lim_{x \to -3} \left(\frac{x^2 - 9}{x + 3} \right)$$

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$$

Extra l'Hôpital Visual Examples

$$\lim_{x \to 0}\left(\frac{\tan^{-1} x}{e^x - 1}\right)$$

$$\lim_{x \to 0^+}\left(\frac{\ln x}{\frac{1}{x}}\right)$$

Connect the MPACs to the Solution Process and Student Work

Show all steps leading to your final answer. Identify your final answer. Make sure your work is clear and organized.

11. Evaluate $\lim_{x \to 0} \frac{e^x + 2e^x - 3}{x + \sin x}$

6 pt

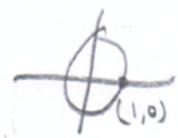
$\lim_{x \to 0} e^x + 2e^x - 3 = 0$

$\lim_{x \to 0} x + \sin x = 0$

l'hôp applies

$\lim_{x \to 0} \frac{e^x + 2e^x}{1 + \cos x} = \frac{1+2}{1+1}$

$\boxed{3/2}$



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12. Evaluate $\lim_{x \to 3} \frac{\int_3^x \cos\left(\frac{\pi t}{3}\right) dt}{x^3 - 27}$

6 pt

$\lim_{x \to 3} \int_3^x \cos\left(\frac{\pi t}{3}\right) dt = 0$

$\lim_{x \to 3} x^3 - 27 = 0$

l'hôp applies

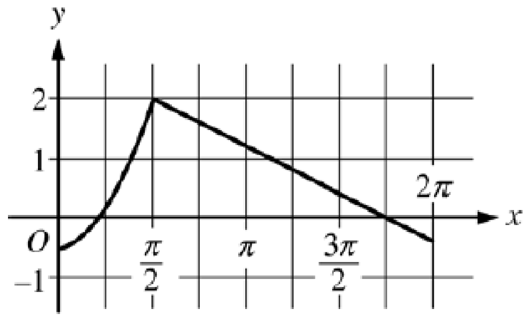
$\lim_{x \to 3} \frac{\cos \frac{\pi x}{3}}{3x^2} = \frac{\cos \pi}{27}$

$\boxed{\frac{-1}{27}}$

2018 AB FRQ 5

5. Let f be the function defined by $f(x) = e^x \cos x$.

(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.



Graph of g'

2016 BC FRQ 4

4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1}\left(\frac{g(x) - 2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

2013 BC FRQ 5

5. Consider the differential equation $\frac{dy}{dx} = y^2(2x + 2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Find $\lim_{x \to 0} \frac{f(x) + 1}{\sin x}$. Show the work that leads to your answer.

2018 AB FRQ 5 Scoring Guideline

(d) $\lim_{x \to \pi/2} f(x) = 0$

Because g is differentiable, g is continuous.

$\lim_{x \to \pi/2} g(x) = g\left(\frac{\pi}{2}\right) = 0$

By L'Hospital's Rule,

$\lim_{x \to \pi/2} \frac{f(x)}{g(x)} = \lim_{x \to \pi/2} \frac{f'(x)}{g'(x)} = \frac{-e^{\pi/2}}{2}.$

$3 : \begin{cases} 1 : g \text{ is continuous at } x = \frac{\pi}{2} \\ \quad \text{and limits equal } 0 \\ 1 : \text{applies L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

Note: max $1/3$ [1-0-0] if no limit notation attached to a ratio of derivatives

2016 BC FRQ 4 Scoring Guideline

(c) $\lim_{x \to -1} (g(x) - 2) = 0$ and $\lim_{x \to -1} 3(x+1)^2 = 0$

Using L'Hospital's Rule,

$\lim_{x \to -1} \left(\frac{g(x) - 2}{3(x+1)^2} \right) = \lim_{x \to -1} \left(\frac{g'(x)}{6(x+1)} \right)$

$\lim_{x \to -1} g'(x) = 0$ and $\lim_{x \to -1} 6(x+1) = 0$

Using L'Hospital's Rule,

$\lim_{x \to -1} \left(\frac{g'(x)}{6(x+1)} \right) = \lim_{x \to -1} \left(\frac{g''(x)}{6} \right) = \frac{-2}{6} = -\frac{1}{3}$

$3 : \begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

2013 BC FRQ 5 Scoring Guideline

(a) $\lim_{x \to 0} (f(x) + 1) = -1 + 1 = 0$ and $\lim_{x \to 0} \sin x = 0$

Using L'Hospital's Rule,

$\lim_{x \to 0} \frac{f(x) + 1}{\sin x} = \lim_{x \to 0} \frac{f'(x)}{\cos x} = \frac{f'(0)}{\cos 0} = \frac{(-1)^2 \cdot 2}{1} = 2$

$2 : \begin{cases} 1 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$