Math Modeling: Our Favorite Ready-To-Use Problems That Encourage Communication and Collaboration

Lauren Shareshian
Greta Mills
Cheryl Gann

http://tinyurl.com/ModelingNCTM2019

Introducing Modeling - Setting the Stage

- Existing problems can be opened up to help create a modeling mindset in your students.
- These problems provide a great entry into mathematizing real-world problems without being overwhelming.
- The best introductory problems can be revisited and refined throughout the course.

http://tinyurl.com/ModelingNCTM2019

Example: The Flagpole Problem

After Hurricane Irma hit West Palm Beach in September 2017, Oxbridge's Buildings and Grounds Department needed to order a replacement flag for the flagpole behind the school.

Standard flag sizing recommends that the length of the flag is at least 1/4 the height of the flagpole, and the width of the flag is approximately 2/3 the length of the flag.

In addition, flag protocol requires that the flag should never touch the ground, even when flying at half-staff.

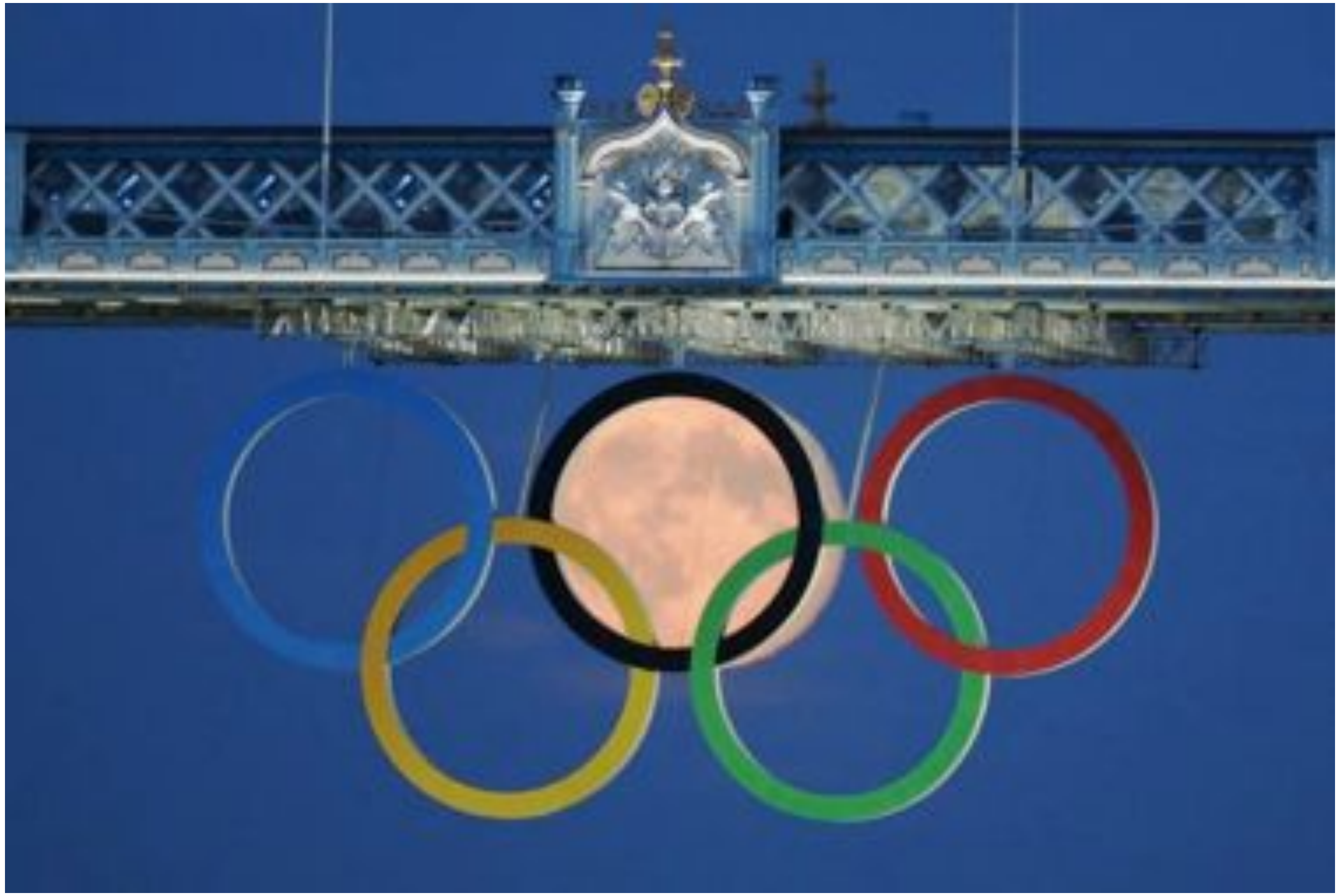
Example: The Flagpole Problem

Using only standard measuring tools, determine the height of the flagpole behind Oxbridge Academy, and use your results to give the minimum and maximum dimensions of the replacement flag.

This introductory example sets the stage for further exploration of indirect measurement...

Example: The Olympic Rings

During the 2012 Summer Olympics, photographer Luke MacGregor (Reuters) took these photos of the moon behind the Olympic Rings:

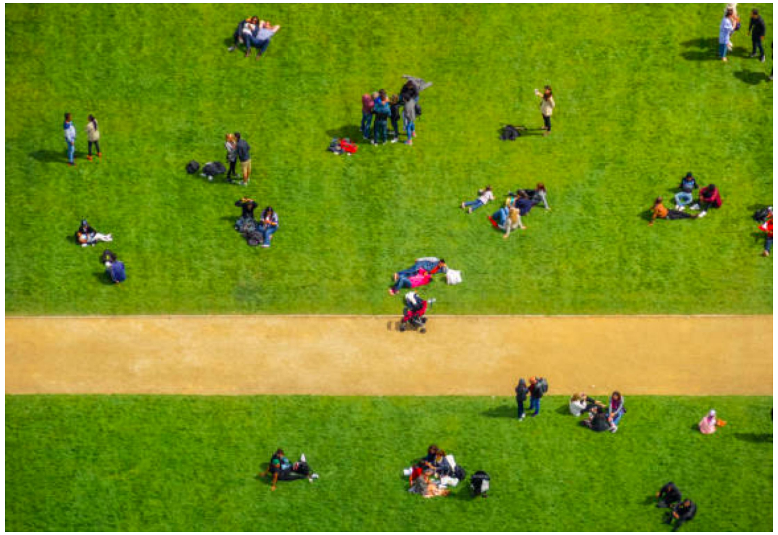


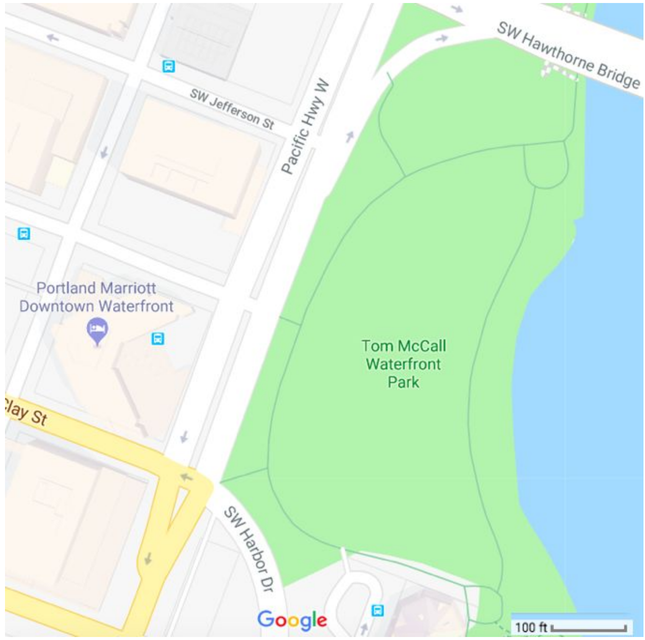
Example: The Olympic Rings

How far was the photographer from the bridge?

Example: Population Estimation

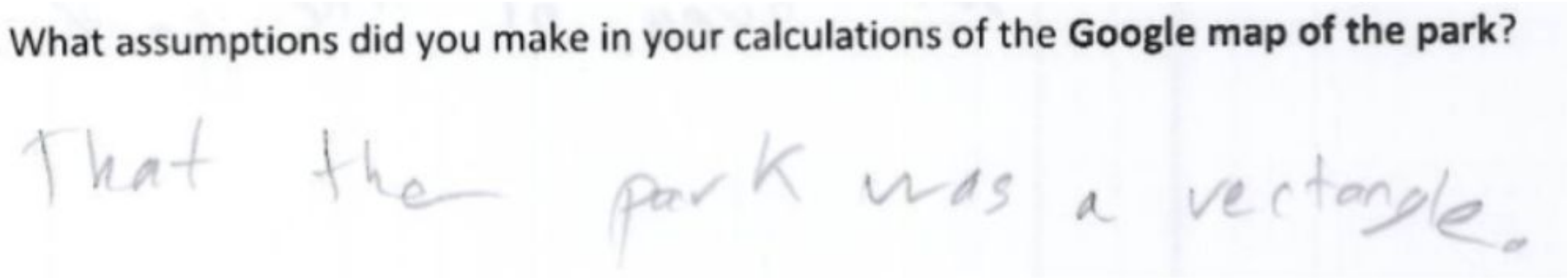
Portland City Government is wondering how many people typically hang out at Waterfront Park on a sunny day. They take an aerial photo and look at a Google map of the area. Provide city officials with an estimate for the number of people in the park.





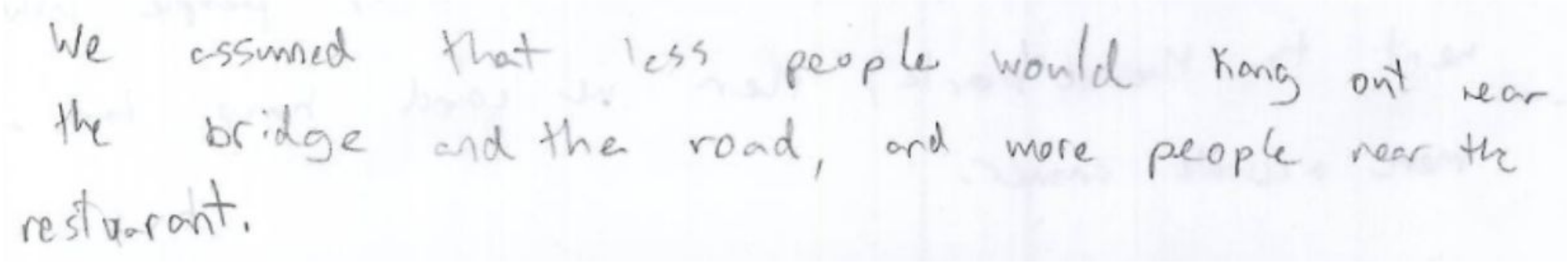
Even mini-modeling activities get all kids involved

- Simpler assumptions…


What assumptions did you make in your calculations of the **Google map of the park?**

That the park was a rectangle.

- To the more complex...


We assumed that less people would hang out near the bridge and the road, and more people near the restuarant.

Even mini-modeling activities increase understanding

- MANY students used the perimeter instead of area formula
- Tons of area/perimeter book problems hadn’t made the concept stick
- They needed a real world example of when to use which formula

7. Discuss any mistakes you made when you first started and how you corrected them.

We found the perrimeter of the google map Photo instead of the area and we fixed it by doing the area instead.

The Bridge Program

- Students who have not historically been successful at traditional algebraic content can flourish during the modeling process.
- Students invited to participate in the NCSSM summer bridge program often have differences in preparation that could make it harder to experience success at NCSSM.
- Modeling is introduced to the students on the first day of the summer bridge program and incorporated throughout the two week math course.

The Bridge Program

- Some summer bridge students expressed their unease and frustration as they first approached the problem and explained how solving the problem gave them a great sense of accomplishment:
- “I learned to persevere and create a solution ‘that worked’ even though it wasn’t the ‘best answer’”.
- “I will keep this lesson with me and apply it to my classes in the following years as well as the rest of my life.”
- “If I'm looking to work in a STEM field one day, it is crucial to understand the applications of the different things we learn.”

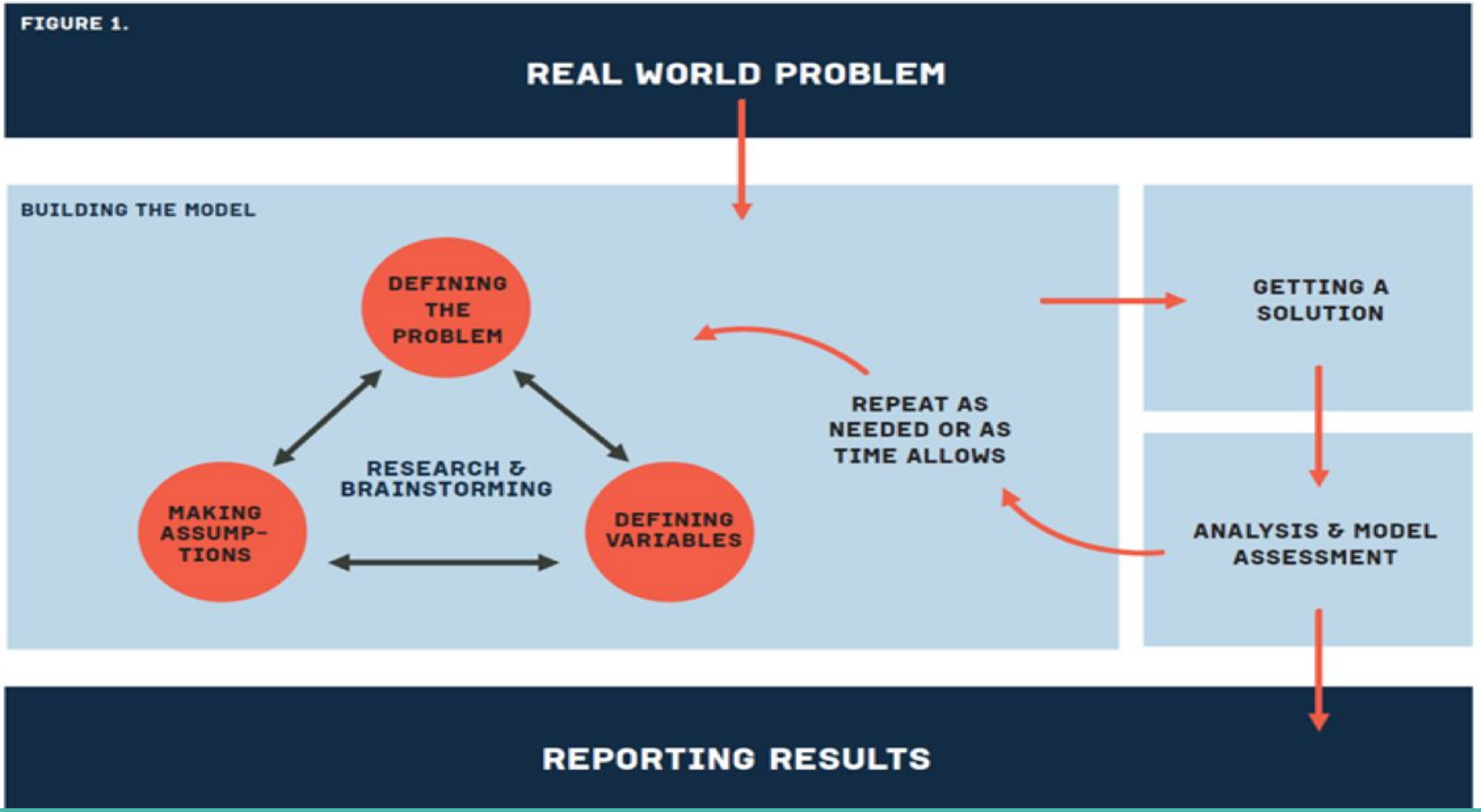


FIGURE 1.
REAL WORLD PROBLEM
BUILDING THE MODEL
DEFINING THE PROBLEM
MAKING ASSUMP-TIONS
DEFINING VARIABLES
RESEARCH & BRAINSTORMING
REPEAT AS NEEDED OR AS TIME ALLOWS
GETTING A SOLUTION
ANALYSIS & MODEL ASSESSMENT
REPORTING RESULTS

Beginning to Model - Grouping Students

When forming groups, some of the questions that a teacher may consider:

- Should I give students input on group formation?
- What do I do with a student who other kids don’t want to work with?
- What are some considerations in how I form groups?
- Should the groups be homogeneous or heterogeneous?

What are useful skills to spread out?

- Big thinkers
- Detail oriented kids
- Strong coders & statisticians
- Clear presentation slide makers
- Good public speakers
- Good time managers
- The kid who will immediately opens up a Google Doc to keep everyone organized and delegate

Heterogeneous vs. Homogeneous Groups

- Heterogeneous groups are great when there are a variety of skills needed.

- Homogeneous groups are good when you don’t want the most advanced student to dominate within a group.

Homogeneous Example: The Midge Problem

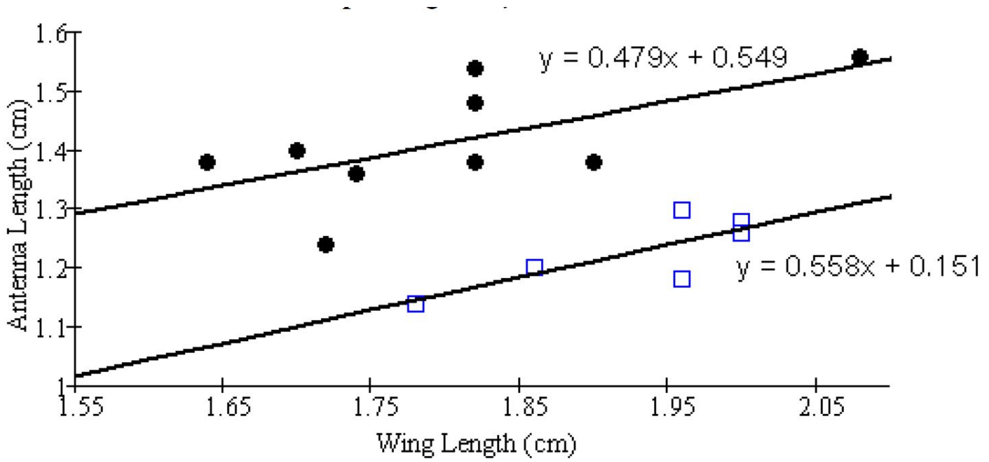


Figure 2: Linear Least-Squares Line Fit to Each Data Set

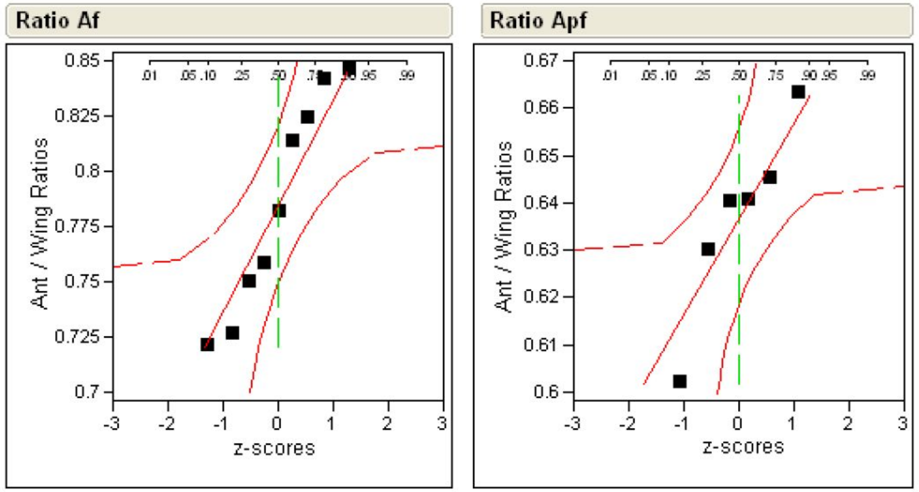


Figure 7: Normal Probability Plots

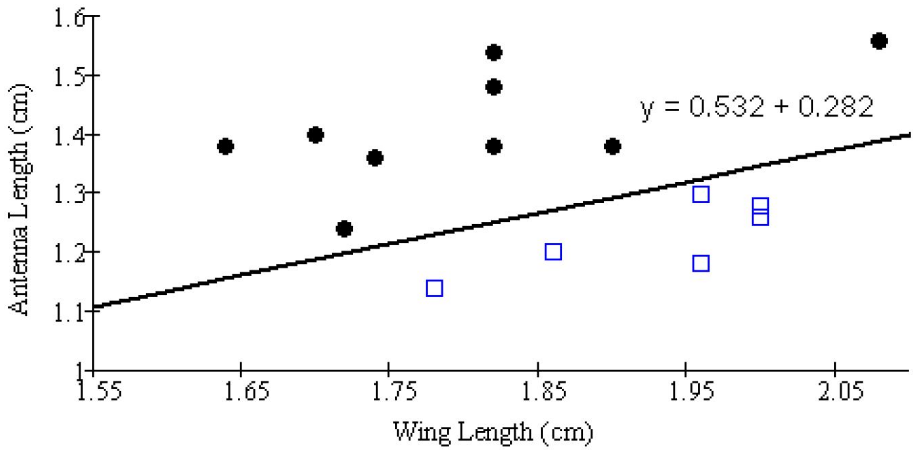


Figure 12: Equi-Probable Boundary and Linear Fits

Heterogenous Example: Rollercoasters

Problem: Create an objective roller coaster ranking system given a spreadsheet of data about each rollercoaster.

Question: What to do with the missing data?

The computer scientist dropped all missing values

In [6]:

```
import pandas as pd
coasters = pd.read_csv('rollercoaster.csv', encoding = "ISO-8859-1")
coasters.dropna(subset = ['Year/Date Opened', 'Height (feet)', 'Speed (mph)', 'Length (feet)', 'Number of Inversion
```

Out[6]:

	Name	Park	City/Region	City/State/Region	Country/Region	Geographic Region	Construction	Type	Status	Year/Date Opened	Height (feet)	Spee (mph
0	10 Inversion Roller Coaster	Chimelong Paradise	Panyu	Guangzhou, Guangdong	China	Asia	Steel	Sit Down	Operating	2006.0	98.4	45.
1	Abismo	Parque de Atracciones de Madrid	Madrid	Madrid	Spain	Europe	Steel	Sit Down	Operating	2006.0	151.6	65.
2	Adrenaline Peak	Oaks Amusement Park	Portland	Oregon	United States	North America	Steel	Sit Down	Operating	2018.0	72.0	45.
3	Afterburn	Carowinds	Charlotte	North Carolina	United States	North America	Steel	Inverted	Operating	1999.0	113.0	62.
4	Alpengeist	Busch Gardens Williamsburg	Williamsburg	Virginia	United States	North America	Steel	Inverted	Operating	1997.0	195.0	67.
5	Alpina Blitz	Nigloland	Dolancourt	Champagne-	France	Europe	Steel	Sit Down	Operating	2014.0	108.3	61.

The most advanced math student used machine learning to fill in the missing values

I. Cost Function of the Linear Regression Model

$$J(\theta, X, y) = \frac{1}{2m} \sum_{i=1}^{m} (h(\theta, X_i) - y_i)^2$$

II. Gradient Descent Functions of the Linear Regression Model

$$\theta_1' = \theta_1 - \alpha \frac{\partial J(\theta)}{\partial \theta_1} = \theta_1 - \frac{\alpha}{m} \sum_{i=1}^{m} (h(\theta, X_i) - y_i)$$

For $j \geq 2$,

$$\theta_j' = \theta_j - \alpha \frac{\partial J(\theta)}{\partial \theta_j} = \theta_j - \frac{\alpha}{m} \sum_{i=1}^{m} (h(\theta, X_i) - y_i) X_{i,j}$$

where $X_{i,j}$ is the value in i-th row and j-th column of matrix X and α is the learning rate.

The multiple linear regression with gradient descent is applied to predict the missing data

The most resourceful student used the basics

“After doing a lot of internet research, I saw that a video of Montana Rusa in Ecuador was similar to that of Manta in San Diego, so the ratio of height of the two was used to calculate the speed of the Montana Rusa in Ecuador.”

All contributions are useful

While the machine learning student research steepest gradient descent, his partner pondered the more qualitative questions of:

“In addition to not being as long or as fast as a standard roller coaster, water rides cannot have inversions. Can we rank water rides using the same metric?”

“Few people want to spend excessive amounts of money for airplane tickets around the world. Thus, distance to the park should be taken into our metric.”

Beginning to Model - Expectations & Scaffolding

When transitioning from smaller doses of modeling into more extensive problems, the teacher can help students gain proficiency in mathematical modeling.

This can be achieved by sharing information about the modeling process, communicating expectations for the work and outcomes, and providing scaffolding to help students who are new to modeling.

Modeling Process Rubric - Sample Rubric 1

MODELING COMPONENT	QUESTIONS ABOUT YOUR MODEL AND HOW YOU MADE IT
DEFINING THE PROBLEM	What is the specific problem your model is going to solve? (My model will tell you . . .)
MAKING ASSUMPTIONS	What have you assumed in order to solve the problem? Why did you make these choices?
DEFINING VARIABLES	Where did you find the numbers that you used in your model?
GETTING A SOLUTION	What pictures, diagrams or graphs might help people understand your information, model, and results?
ANALYSIS AND MODEL ASSESSSMENT	How do you know you have a good/useful model? Why does your model make sense?
REPORTING RESULTS	What are the 5 most important things for your audience/client to understand about your model and/or solution?

Portion of GAIMME Modeling Assessment Rubric from GAIMME 2016, p.197; adapted from Rachel Levy, IMMERSION program.

Scaffolding Example - Foraging for Food

Bees can carry around 15 mg of pollen, but flowers don’t produce that much, so bees have to flit from flower to flower to collect enough pollen to fill their sacs before returning to the hive. In this problem you will explore the relationship between the availability of pollen and a bee’s optimal foraging strategy.

Scaffolding Example - Foraging for Food

How this can be used to provide scaffolding:

- Students work in small groups on parts of the problem then throughout we come back together to share ideas and another small task is given.
- This small group to full class discussion is repeated throughout the work on the problem.
- Student groups are then assigned portions of the paper to draft.
- We then discuss edits to the paper and students are given a final draft of the paper as an example for future writing assignments.

Example - Driving for Gas

A friend tells you she buys her gas at a station several miles off your normal route where the prices are cheaper. Would it be more economical for you to drive the extra distance for the less expensive gas than to purchase gas along your route?

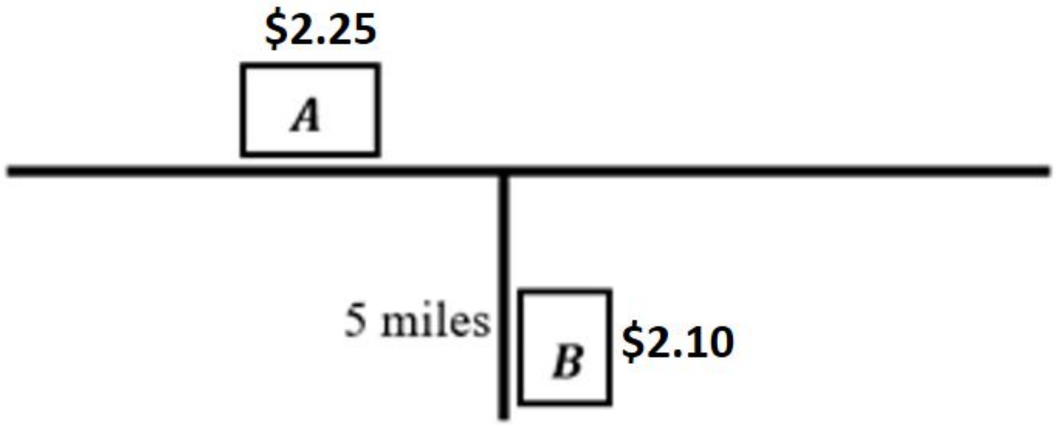
kcd.com

Identifying Variables - Gas Example

- The price of gas at each station.
- The extra miles to the other station.
- The amount of gas I need to buy.
- The fuel efficiency of my car.

Ask an easier question first or create a specific example

Station A is on your normal route and sells gas for $2.25 a gallon, while Station B is 5 miles off your route sells gas for $2.10 a gallon. Your car gets 32 miles per gallon, and your friend's car only gets 15 miles per gallon. Should either of you travel the extra distance to buy gas at Station B?

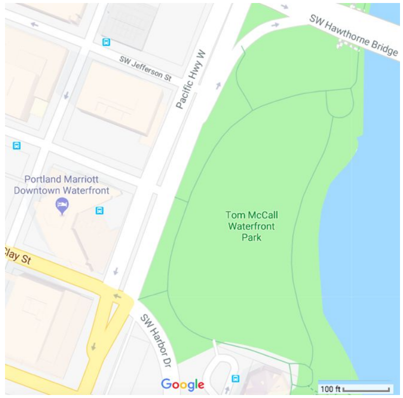


Identifying Variables and Making Assumptions

- Variables in Textbook Problem: all quantities are given
- Variable in Modeling Problem: which quantities should we include?
    - Play around with it first!
    - Ask an easier question first.
    - Create a specific example.
- Assumptions in Modeling Problem: finding the right balance
    - Are we assuming so much that it oversimplifies the problem? Or assuming too little and we can't proceed
    - Justify why the assumptions you are making are reasonable.
    - Save early work - it might be useful for future steps.
    - Add complexity step by step.

Iteration allows for differentiation

- Some groups remain on first iteration while more advanced groups refine their solution
    - In the population problem, did you get one estimate? Now find upper and lower bounds.
    - Did you use a single rectangle to approximate area? Now use smaller rectangles or more intricate shapes.



Iteration in the gas problem

- Did you oversimplify? Add more complexity step by step.

Cost of a tank of gas: $T \cdot P$

Number of useable gallons: $\left(T - \frac{2D}{M}\right)$

Cost per useable gallon: $\frac{T \cdot P}{\left(T - \frac{2D}{M}\right)}$

Our cost index:

$$I = \frac{T \cdot P}{\left(T - \frac{2D}{M}\right)} = \frac{M \cdot T \cdot P}{M \cdot T - 2D}$$

We should buy gas at the distant station if $I < P^*$.

Opportunities for Differentiation

The modeling process provides many opportunities for differentiation, within a specific class or across multiple classes / levels.

Modeling problems have multiples entry points for students from early Algebra 2 through calculus, using a variety of tools from spreadsheets to differential equations.

The following example comes from the 2014 HiMCM contest and can be modified or extended.

Example: The Next Plague

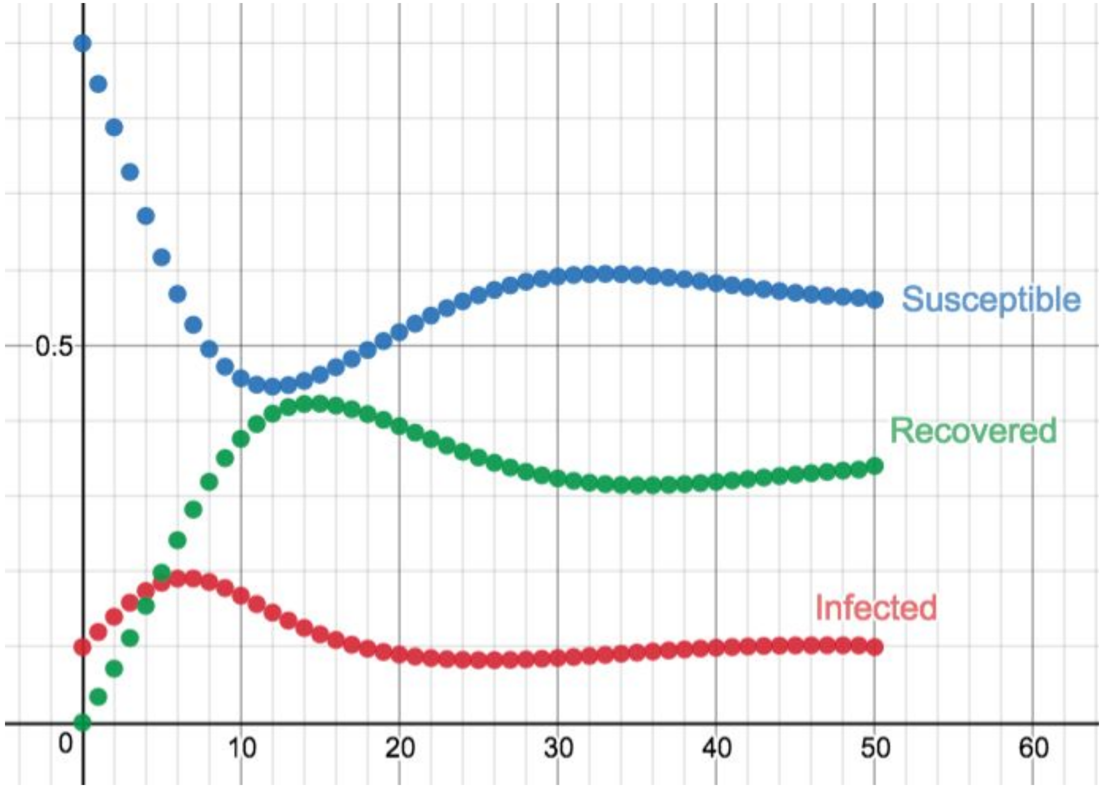
In a small island village almost half of its 300 inhabitants are showing similar symptoms of illness. In the past week, 15 of the "infected" have died. Your modeling team works for the International World Health Organization.

Determine and classify the type and severity of the spread of the disease.

The role of technology

- Students use whatever mathematical tools they have.
- Simple to advanced use of technology and mathematical content.
- From spreadsheets and Desmos...

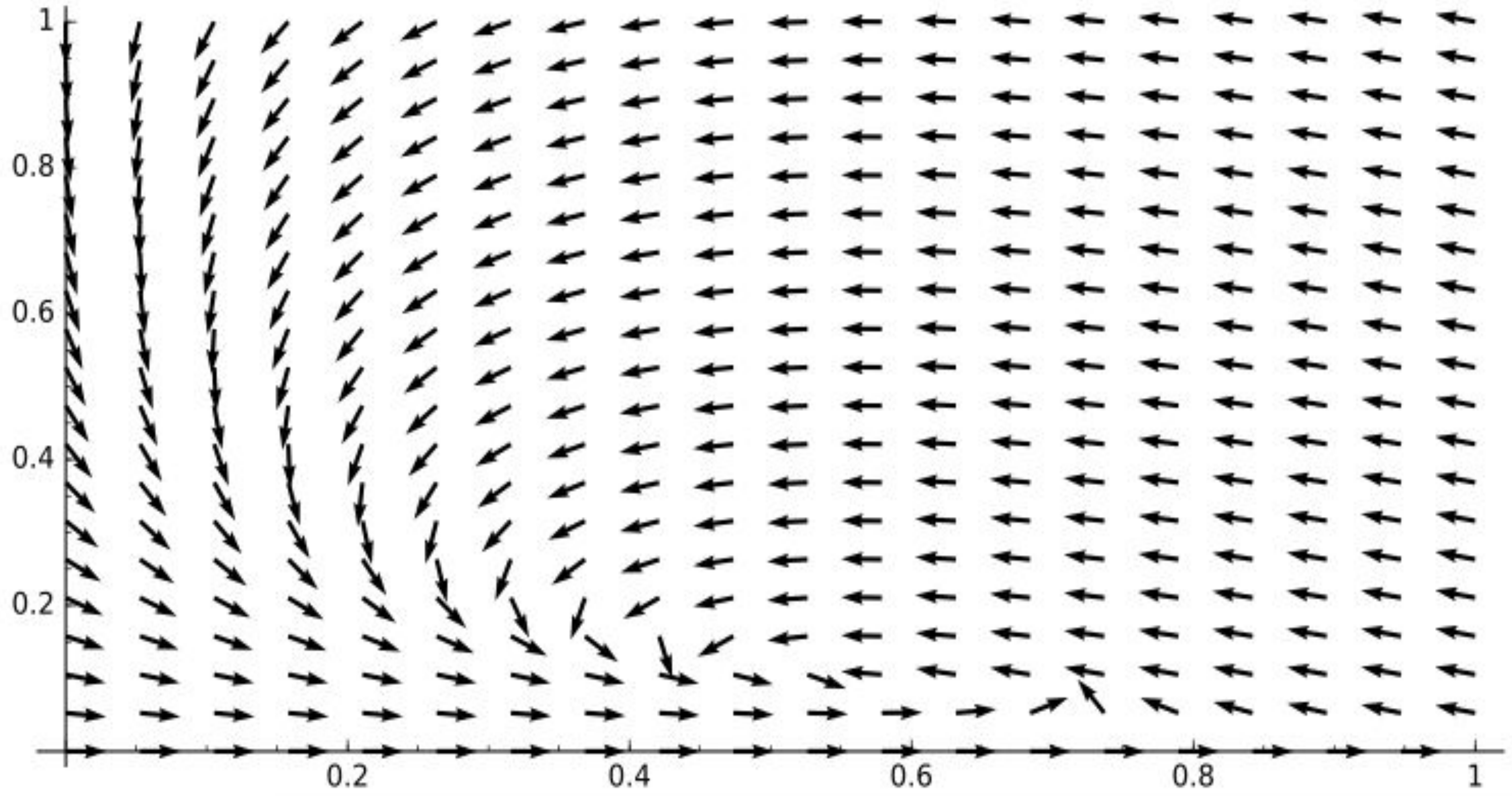
A	B	C	D
time	susceptible	infected	recovered
0	0.9	0.1	0
1	0.846	0.12	0.034
2	0.788488	0.140112	0.0714
3	0.72934202	0.1587599	0.11189808
4	0.67105767	0.17425569	0.15468664
5	0.61636496	0.18517013	0.19846491
6	0.56773203	0.19069171	0.24157626
7	[illegible]	[illegible]	[illegible]



Technology Ebola Example

- To using SageMath…

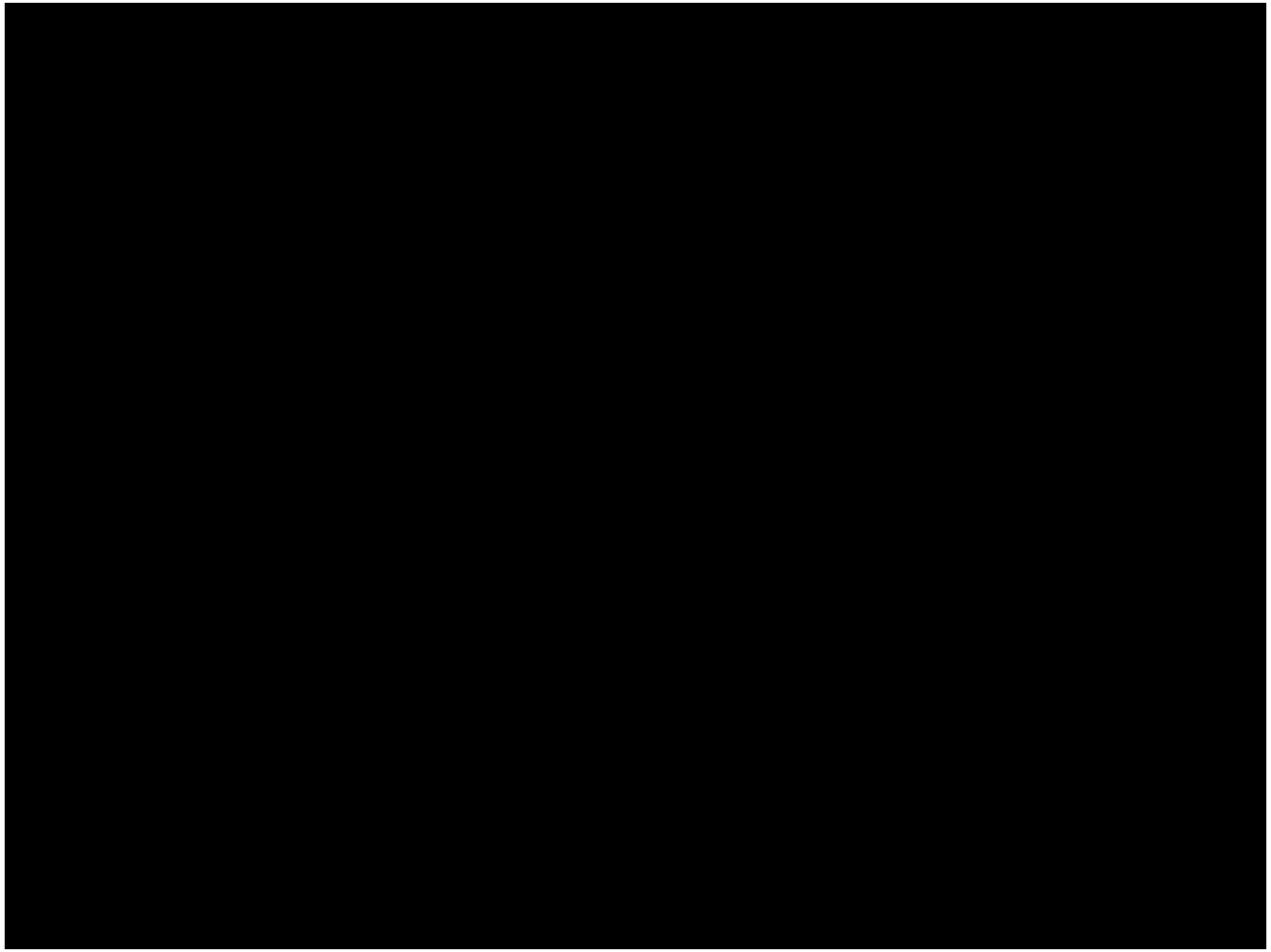
```
x,y=var('x y')
f=-0.6*x*y+0.1*(1-x-y)
g=0.6*x*y-0.34*y
VF=plot_vector_field(  (f/sqrt(f^2+g^2),   g/sqrt(f^2+g^2)),(x,0,1),(y,0,1))
plot(VF)
```



Technology Ebola Example

- To using Netlogo…

```
globals[
  k ;factor for determining arrest probability
  threshold ;by how much must illness factors overcome resistance?
]

students-own [
  resistance
  active?
  jail-term ;how many turns at home remain? if 0, student is at school
]

patches-own [ neighborhood ]

to setup
  ;; (for this model to work with NetLogo's new plotting features,
  ;; __clear-all-and-reset-ticks should be replaced with clear-all at
  ;; the beginning of your setup procedure and reset-ticks at the end
  ;; of the procedure.)
  __clear-all-and-reset-ticks

  set k 2.3
  set threshold 0.01

  ask patches[
    set pcolor gray - 1
    set neighborhood patches in-radius cough-range
  ]

  create-teachers round (initial-teacher-density * 0.01 * count patches) [
    move-to one-of patches with [not any? turtles-here]
    display-teacher
  ]

  create-students round (initial-student-density * 0.01 * count patches) [
    move-to one-of patches with [not any? turtles-here]
    set heading 0
    set resistance random-float 1.0
    if (resistance < initial-infections) [set active? true]
    if (resistance >= initial-infections) [set active? false]
    set jail-term 0
    display-student
  ]
  my-update-plots
end
```

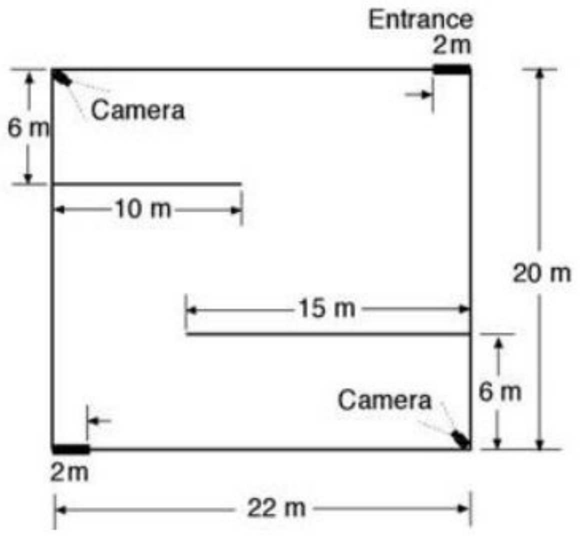


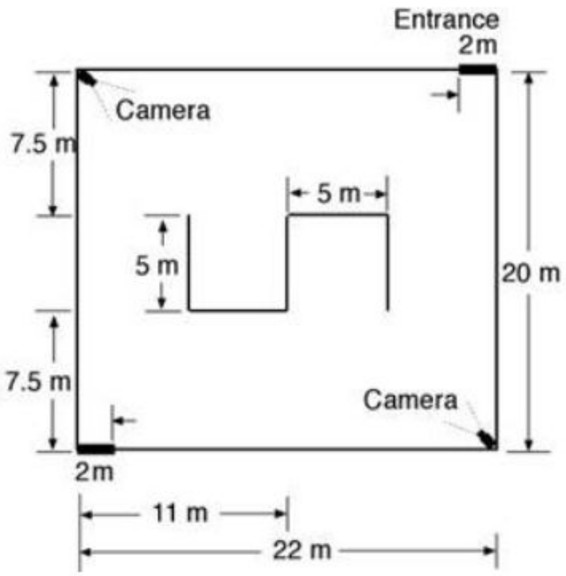
Technology allows for differentiation

- Students use whatever math they are most comfortable with.
- Leads to a variety of solutions, even among students in the same class.
- The next set of student slides all comes from the same modeling class.

Example: The Art Gallery Problem

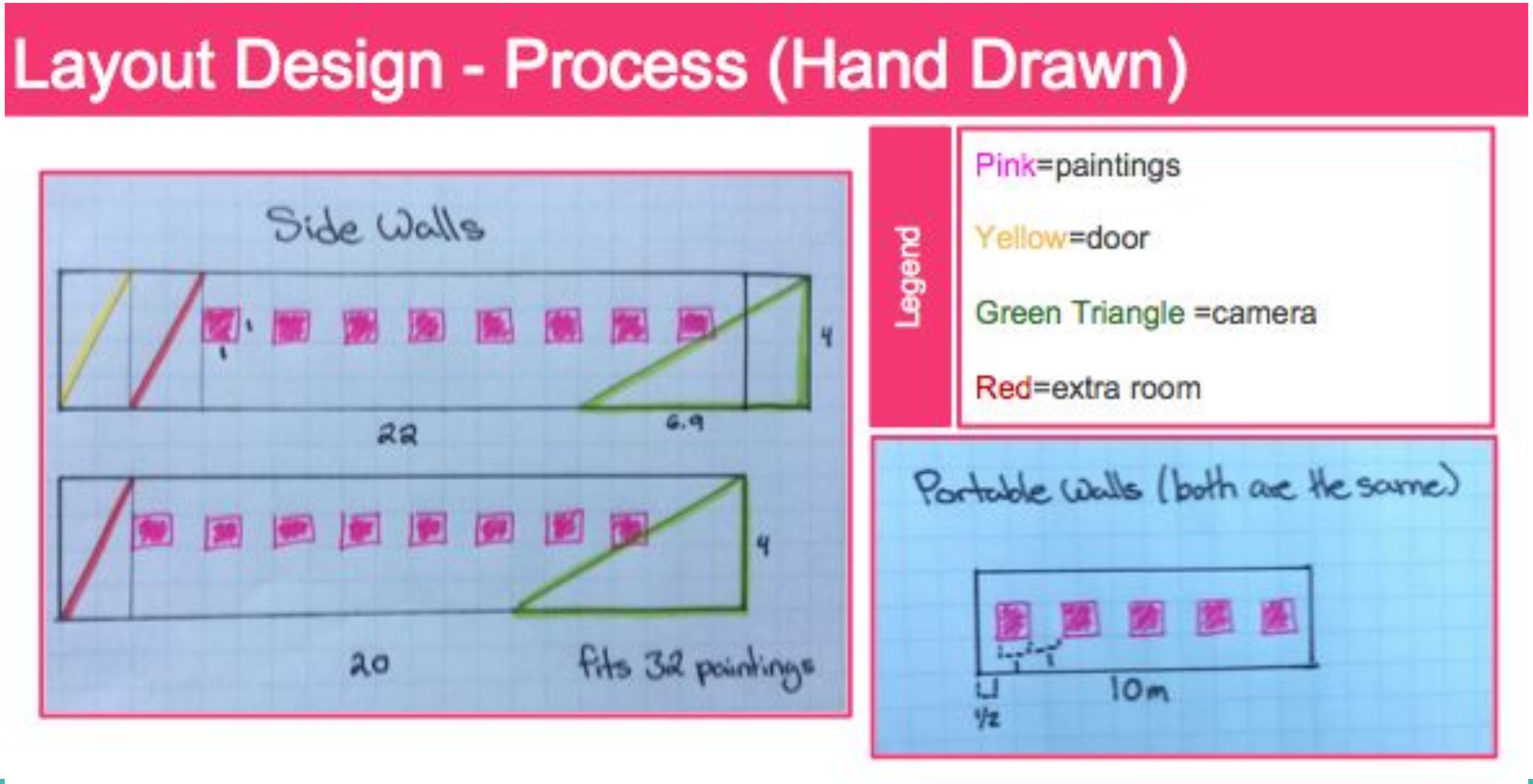
An art gallery needs to hold 50 paintings. Security cameras rotate at opposite ends of the room. The outer walls and portable walls have particular measurements, and the layout should be aesthetically pleasing. Where should you place the paintings in order to maximize security?





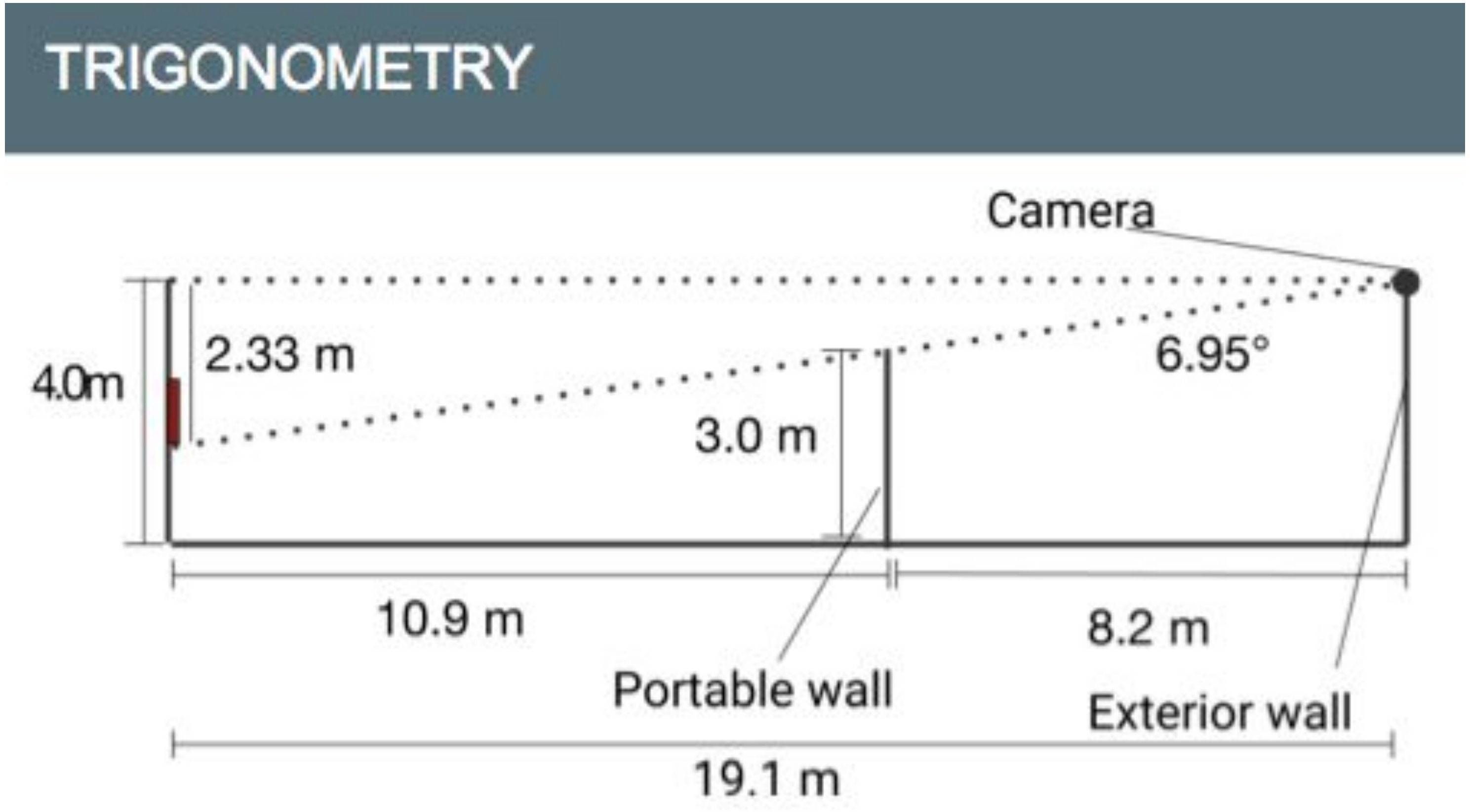
Differentiation – Art Gallery #1

- Low-tech but effective (cardboard box and pencil and paper)



Differentiation – Art Gallery #2

- Another group made such detailed trigonometric diagrams that you could put them in a precalculus textbook!



Differentiation - Art Gallery #3: Calculus

Another group used calculus concepts to quantify security

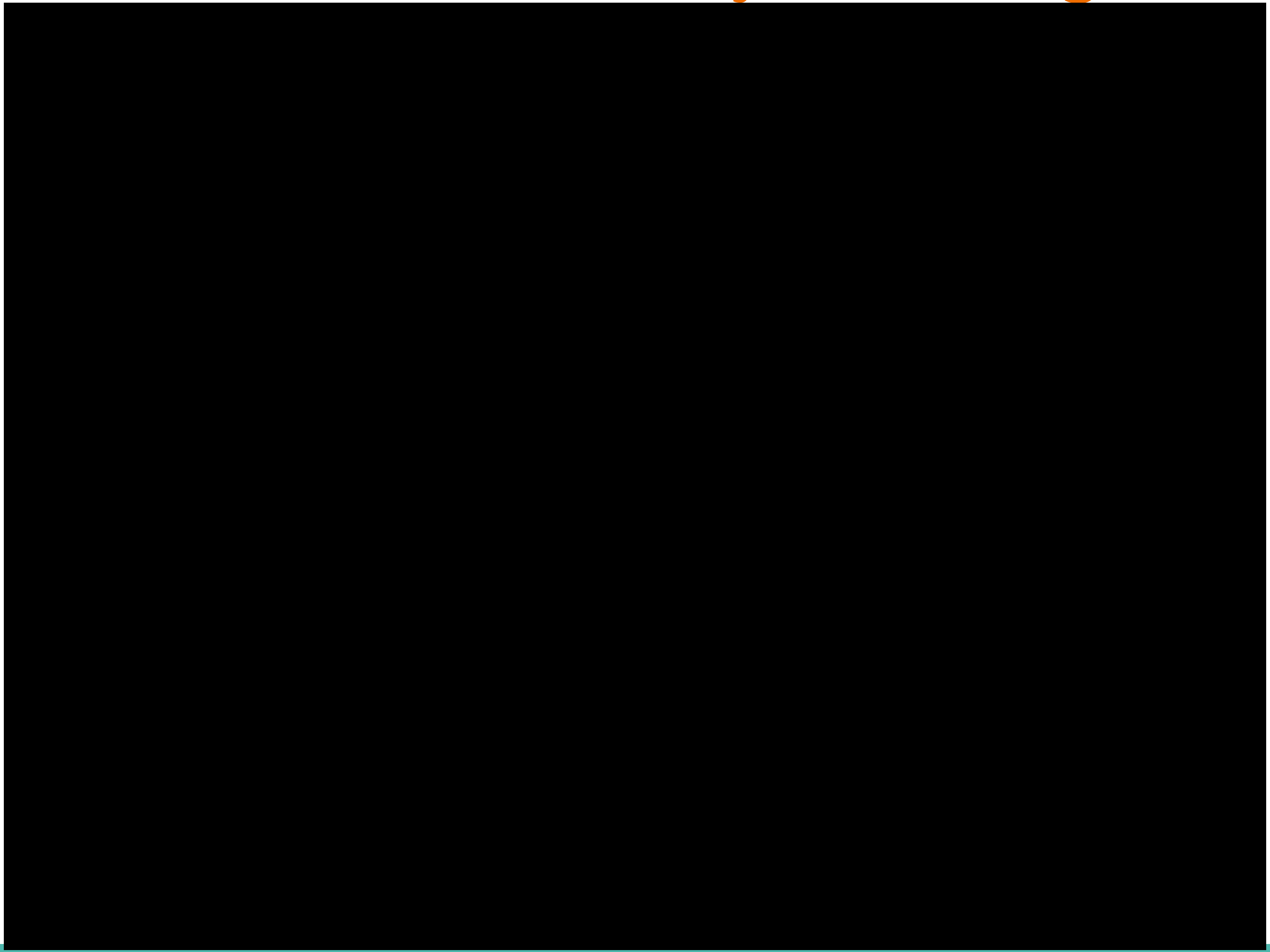
Let R be a given room, and C be a given timing for the cameras

- $V_{R,C}(t)$ is visible painting count at time t given R and C
- Security is therefore

$$S_R = \lim_{t_{max} \to \infty} \max_{C} \frac{1}{t_{max}} \int_0^{t_{max}} V_{R,C}(t)dt$$

- This is the average number of paintings seen per second as time goes to infinity

Differentiation – Art Gallery #4: Geogebra



Differentiation – Art Gallery #5: Google Sketchup

Analyzing and Assessing the Final Solution

- Be able to communicate WHY your solution makes sense.
    - Is it practical?
    - Is it reasonable?
    - How confident are you?
- Reflect on your mistakes
- Be aware of your model's strengths and weaknesses
- Sensitivity analysis
    - Where does the model fall apart?
    - What are the extremes?

Assessing solutions: Middle School Example

Low quality justification....

7. Discuss how confident you are in your result. Does your answer seem reasonable? Why or why not?

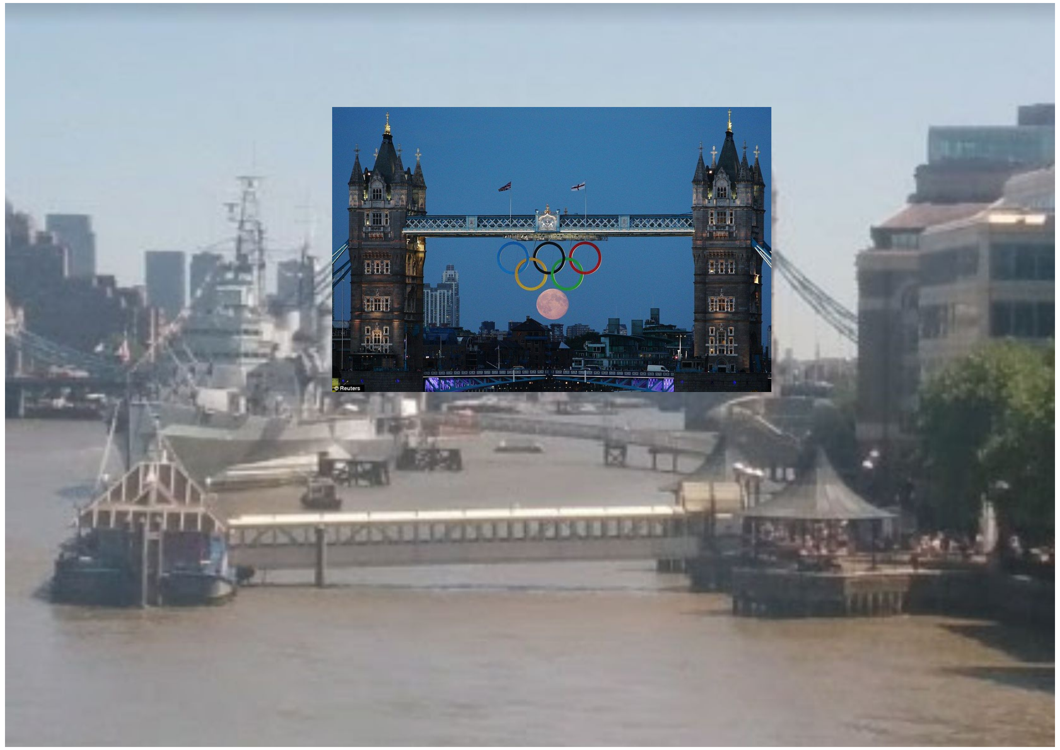
I'm pretty confident because everything made sense, and we checked our answers most of the time.

High quality justification...

I'm confident in my results because 2,116 people in one big park seems like a good number. I wouldn't make sense if only 50 people were in the whole park because the park is huuugeee.

Assessing solutions: High School Example

- **Google Maps!**



Assessing solutions: High School Example

- **Error Analysis**

Mills, Greta 11/11/12

to luke

Thank you so much for your reply. I have contacted Lynne Bundy directly.

We used the photo to try to figure out how far your were from the Tower Bridge - would you be willing to confirm? We think you stood on the southern end of the London Bridge, putting you to the right of the Olympic Rings. My students would really get a kick out of hearing whether or not they were on target.

Thanks!

luke macgregor 11/12/12

to me

Spot on Greta. You can congratulate your students.
Pleased to be of use in the education of others.
Best regards

...

Grading The Final Product

Component of Score	Points Possible	*Examples* of scoring
Introduction	1	0: introduction is absent or doesn't accurately describe the problem. 1: introduction makes clear all aspects of the problem that are relevant to the solution
Model development: Accuracy	6	0: model contains multiple gross inaccuracies, such as multiplying instead of dividing, that compromise the entire model. 3: model uses strong assumptions that are unrealistic, leading to a solution that isn't useful. 6: model is reasonable and assumptions are reasonable
Model development: Flexibility	1	0: model makes unsupported guesses as to parameter values and allows for no other possible values; model makes strong assumptions that limit utility; 1: model includes flexibility in parameters and interprets their impact on the final solution.
Model development: Presentation	4	0: report contains numerous grammatical errors, doesn't flow in a logical way, has poor graphs, and is generally unpleasant to read. 2: Report contains no grammatical errors and flows logically from the beginning to the end; but it includes numerous awkward sentences whose meaning is difficult to understand; the conclusion is written in a mathematical way and is not "translated"; graphs lack labeled axes 4: paper flows smoothly and logically; graphs are neat and appropriately-sized and very easy to read; conclusion is an easy-to-read interpretation of the mathematical solution.
Solution: Accuracy	5	0: the solution process was inappropriate (*e.g.*, instead of finding zeros of f', the "solution" finds zeros of f) 1: the solution process was appropriate but contained critical errors that invalidate the solution (*e.g.*, a derivative was determined incorrectly) 3: the solution contained minor errors that do not compromise the solution entirely but do make it less accurate 5: the solution is accurate for the given model
Solution: Presentation	3	0: the solution is never translated into a meaningful statement in the context of the problem 1: the solution is presented sloppily or incompletely (*e.g.*, by failing to interpret parameters' effects) 2: the solution is clear and complete 3: the solution is clear and complete and implications of the solution(s) are discussed
Total possible points	20	

Self Evaluation Questions

- What strengths did you bring to the group?
- Which tactics (graphical, numerical, or analytical) provided you with a greater insight into the problem?
- What was the greatest moment in your work, the greatest source of pride, or the greatest challenge?
- What specific contributions did you make to the work?

Group/Problem Evaluation Questions:

- How well did you work together as a group?
- How well do you feel your group used class time in working on this?
- What difficulties did you encounter in your work as a group?
- Would you recommend this problem to your peers? What tips would you give them as they set to work?

Ensuring that everyone contributes

The Team (20%) - developing collaborative skills (individual grade)

	Not at all		Average		Exceptional
The team member is consistent in focus and effort, and is self directed	1	2	3	4	5
The team member follows through on assigned tasks and is prepared for each meeting	1	2	3	4	5
The team member respectfully listens to, interacts with, and discusses the work with all team members	1	2	3	4	5
The team member maintains an open mind when listening to others' ideas	1	2	3	4	5
The team member contributes equally to the project	1	2	3	4	5

Ensuring Inequities in Work are Addressed

NOTE: Each team will receive a group grade using the following rubric. To calculate your individual grade, multiply the team grade by the number of students in your group, and then, as a group, divide the points among team members.

Example: 87 x 3 group members = 261 points. The group may decide to split 75, 90, and 96, or some other split. EACH GROUP MEMBER MUST AGREE TO THE POINT DISTRIBUTION.

“What role do you typically play in your group?”

Most students are flexible

“I think the role changes depending on what group I'm in. I normally contribute to **where the group needs me to be**.”

“The roles really change depending on the topics and our workload. It also heavily **depends on the capabilities of my teammates**.”

“The **roles differ depending on what group I'm in**. Sometimes I may take the leadership role, other times I help with coding/equation formation, and I also work on presentation slides.”

Some tend towards one role

“I am **usually the coding person**, but I'd like to think that I can contribute to all aspects of group work.”

“I have consistently found myself in the position of being **very macro-level** and proposing ways that my team can go about solving a problem.”

“For some reason, I was always given the task of **divvying up the work**.”

“I tend to be the **presentation slide person**, and I particularly focus on taking on and **researching the assumptions** of our group. That said, I still do plenty of math and am **looking to diversify my roles in the future**.”

Most students report an equal division of labor

Aneesh: 25% Chris: 25% (even though he couldn't present, he still did his fair share during class workdays) Simon: 25% Me: 25%

Simon 25%, Aneesh 27%, Musa 25%, Chris 23%

Simon: 26%, Aneesh: 26%, Musa: 26%, Chris: 22%

I think each of us contributed 25% evenly.

Sometimes the group fails...

The most advanced student used techniques that no one else understood and didn’t delegate or communicate.

“I get the concept of my model, but I don't understand it fully to the point where I can comfortably modify or do calculations on my own. **The group dynamic was a challenge, since Kelly was way ahead of us. I would like to be paired with someone more on my level next time**.”

But they learn from their mistakes...

"I think this time our group cooperated really well and the work is divided evenly. I really enjoyed working with my group members, and **I hope I can continue to distribute work evenly in the future so that everyone can get the part they want to work on**."

Student Feedback

"I very much enjoyed the **collaborative** nature of everything. It was **good practice for work as a team moving forward into college and beyond.**."

"I really enjoyed the group projects as it was **fun hearing other people's ideas** and coming up with the best solution together. I hope to take away **teamwork** and the general skill of coming up with **creative ideas of solving a problem that has multiple ways of solving**."

"What I enjoyed the most is how each of my groups broke down the problems we got together. I learned a way to **find a starting point** for problems that I had no idea how to approach. I think this **problem solving skill is really important to academics and life**."

Our Favorite Modeling Problems Can Be Found At...

- Our Presentation Problems: http://tinyurl.com/ModelingNCTM2019

- HIMCM (November competition)
- M3 Challenge (March competition)
- GAIMME REPORT
- Robert Kaplinsky

Thanks!

- Lauren Shareshian, Oregon Episcopal School, shareshianl@oes.edu
- Greta Mills, Oxbridge Academy, Gmills@oapb.org
- Cheryl Gann, North Carolina School of Science and Math, gann@ncssm.edu