The Development of Additive Reasoning

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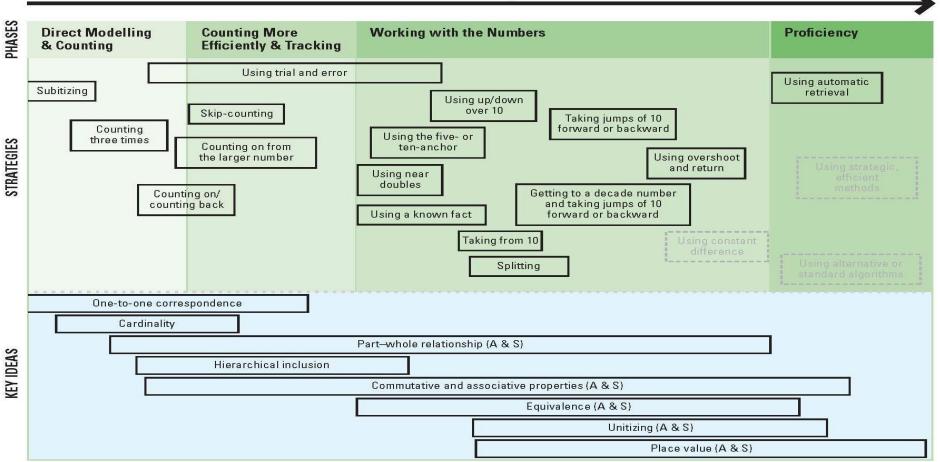
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Overview

◎ Development of Additive Reasoning

◎ Changing the script for Subtraction

◎ Building towards Algorithms with Meaning

Student Continuum of Numeracy Development: Addition and Subtraction

PHASES	Direct Modelling & Counting	Counting More Efficiently & Tracking	Working with the Numbers	Proficiency

STRATEGIES

- Subitizing
- Counting three times
- Counting on/ counting back
- Using trial and error
- Skip-counting
- Counting on from the larger number
- Using up/down over 10
- Using the five- or ten-anchor
- Using near doubles
- Using a known fact
- Taking from 10
- Splitting
- Taking jumps of 10 forward or backward
- Using overshoot and return
- Getting to a decade number and taking jumps of 10 forward or backward
- Using constant difference
- Using automatic retrieval
- Using strategic, efficient methods
- Using alternative or standard algorithms

KEY IDEAS

- One-to-one correspondence
- Cardinality
- Part–whole relationship (A & S)
- Hierarchical inclusion
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Concrete Representational Abstract

An Effective Fact Fluency Progression:

Counting → **Reasoning strategies** → **Retrieval**

Students start counting accurately and move to counting efficiently (e.g. counting on from the largest number)

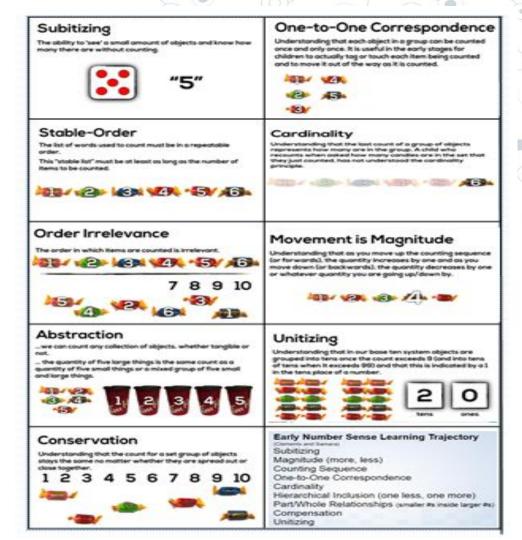Explicitly teach reasoning strategies so students make use of known facts and relationships to derive unknown facts.

Frees up working memory to allow students to work with more complex problems

Principles of Counting and Early Number Trajectory Handout
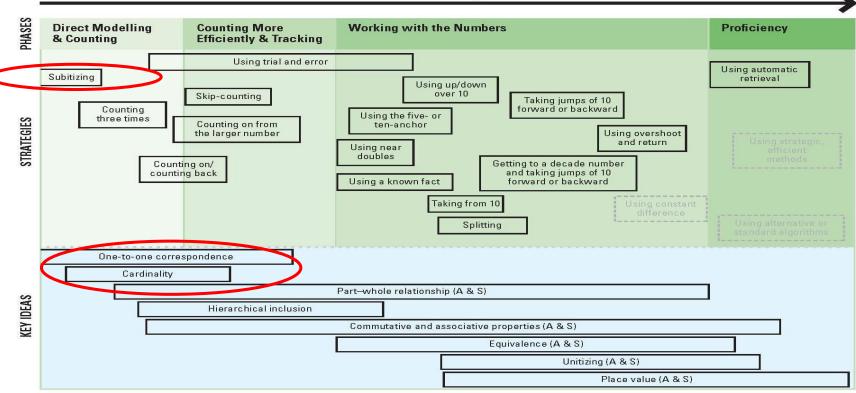
[Digital version](#)

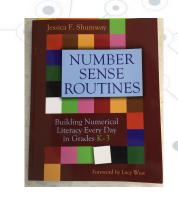Images from Kyle Pearce, Great Essex School Board

"Fact mastery relies significantly on how well students have constructed relationships of numbers and how well they understand the operations."
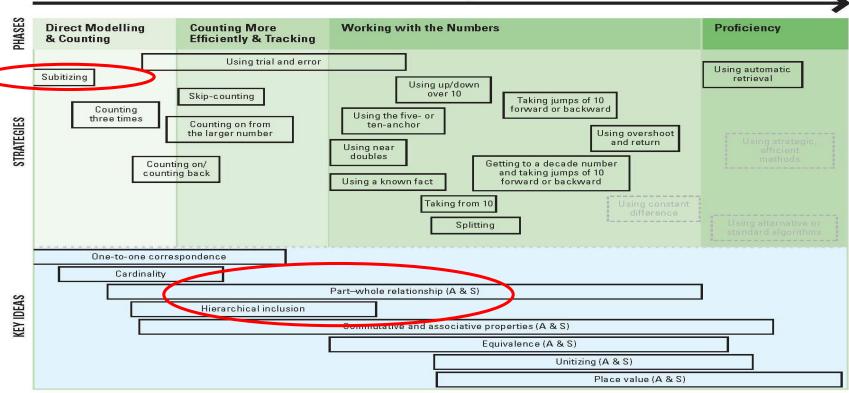
(Van de Walle, 2004, p. 156)

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Count Around the Circle

Play this game with your table group.
Count backwards by 1 starting at 12.
Whoever says "zero" gets a counter.
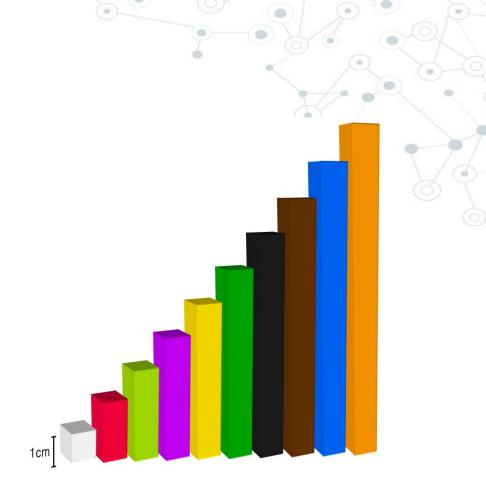First person to get 3 counters wins.

How can you modify this game to meet the needs of the students in your classroom?

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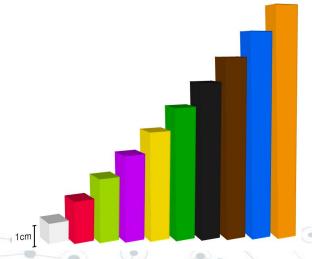
Build the Staircase

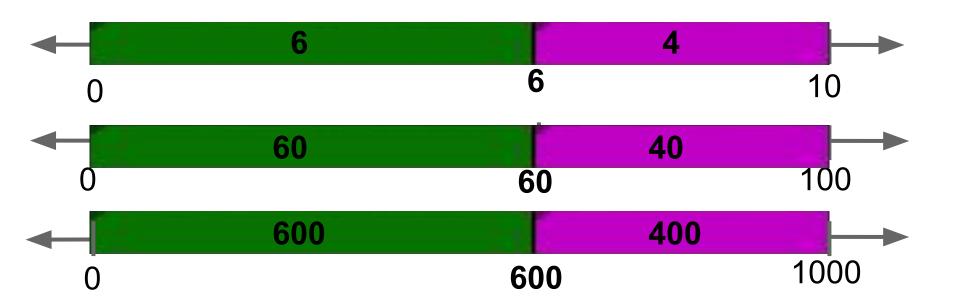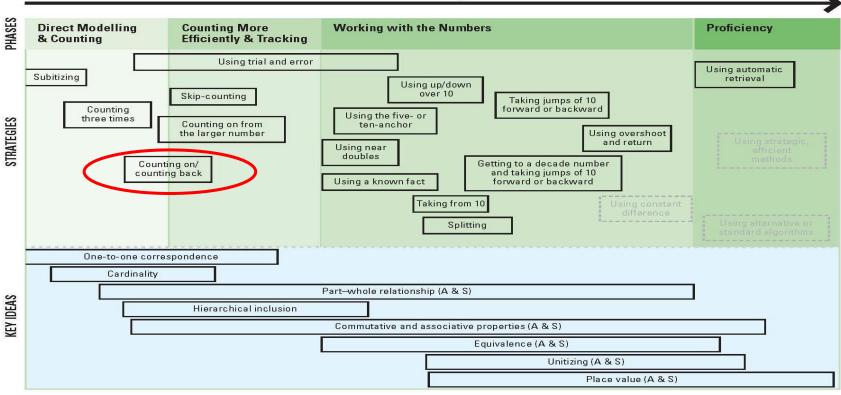## What values would you assign to each rod?

1cm

On your whiteboard, use the rods to decompose 10 in different ways.

1cm

Use those same decompositions to decompose 100 on the left side of your board.
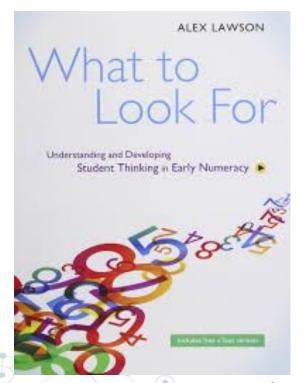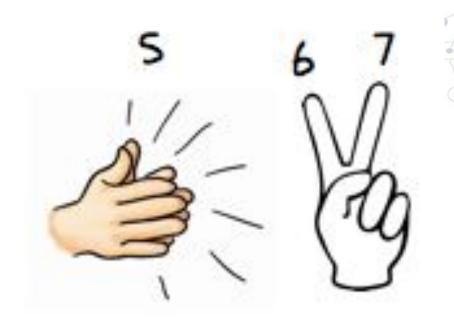For example:

4 + 6 =   40+60=

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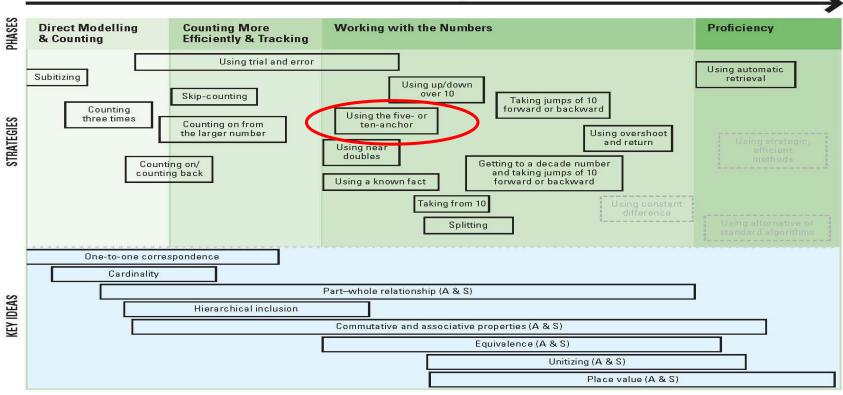- Unitizing (A & S)
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Real Counting On

(p. 163)

Real Counting On Instructions:

◎ Shuffle the number cards, place them face down in a pile.
◎ Player 1 turns over the top card and says the number aloud then puts that many counters in the cup. Player 1 places the card, face up, next to the cup as a reminder of how many counters are in the cup.
◎ Player 2 rolls the die, says the number aloud, then puts that many counters next to the cup
◎ Together, the players decide how many counters there are in all. The can verify their answer by combining the counters, then counting.

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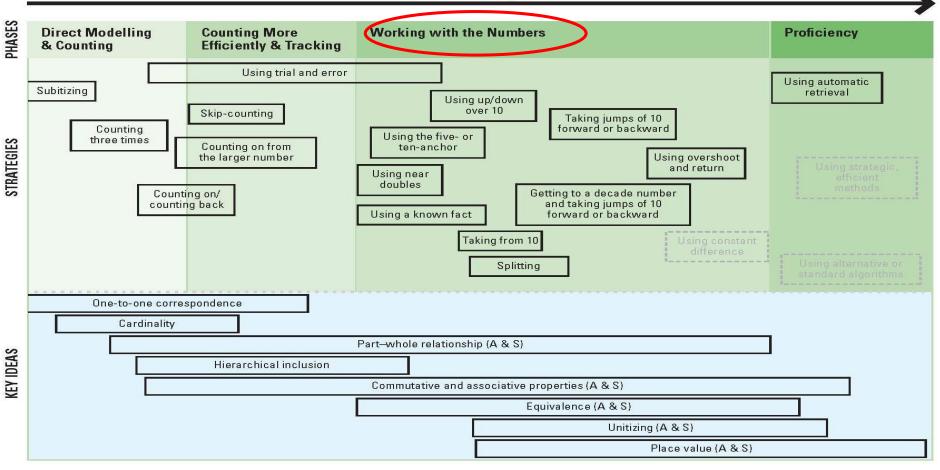
Fact Fluency Game

Materials: Deck of cards between 2 people

Each person chooses 10 cards places them face up.
Face cards are worth ten.
Estimate who has the highest total.
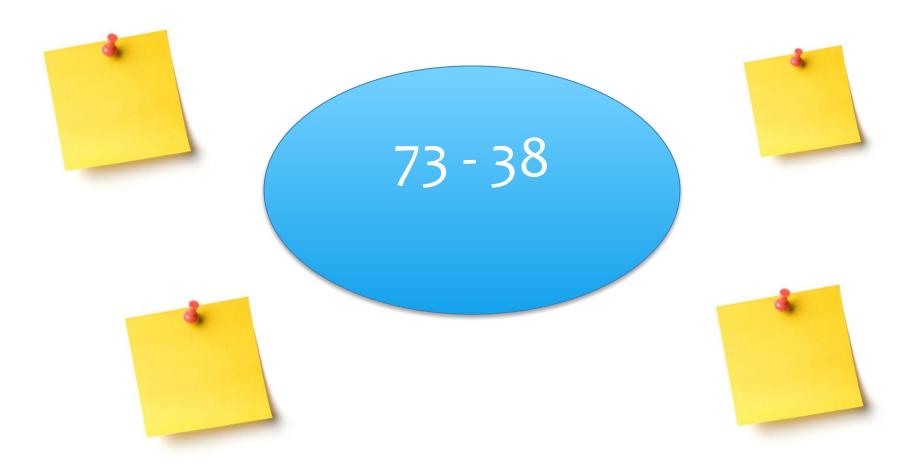Arrange cards in a manner that is easy to count, determine who has the highest amount.

Discuss:
How could you use this game to observe how well students use 5 and 10 as an anchor?

How would you modify this game to your grade?

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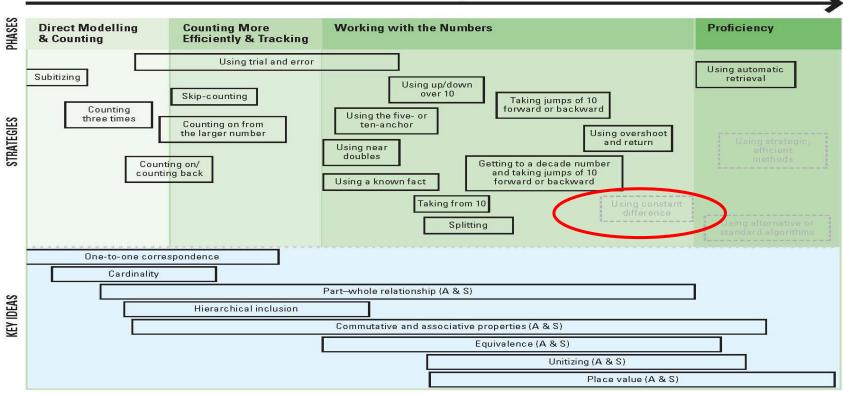**Solve in 4 different ways (at least 1 using a number line) and identify your strategies.**

73 - 38

Combining

◎ Count by 1's
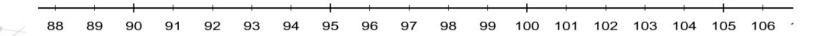◎ Add in chunks
◎ Compensate

Finding the Difference

◎ Count by 1's
◎ Add up in chunks
◎ Remove in chunks
◎ Compensate (make the question easier)

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Shifting the Difference

Using the broken ruler tool, measure your white board marker. Write 4 subtraction sentences to show how long the marker is.

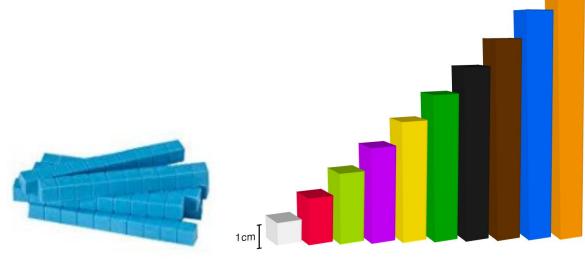**Which subtraction sentence is easiest to work with? Why?**

88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106

Shift the difference to solve:

$$81 - 69$$
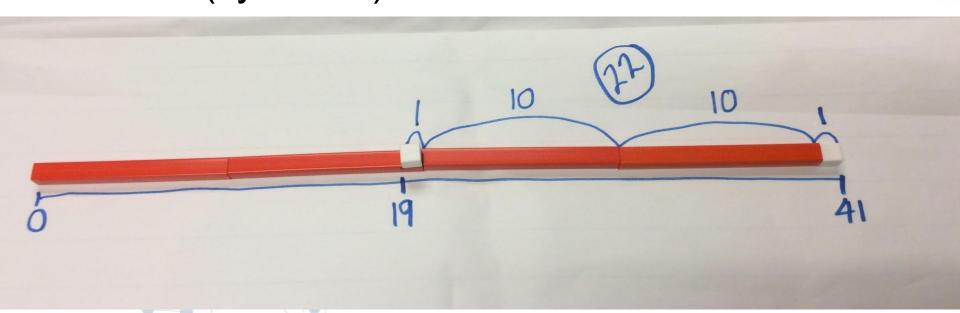$$21 - 9$$

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Write a number sentence that matches your actions.

How much greater is…

41 than 19?

1cm

Use the Relational Rods (concrete).
Trace your number line (representational).
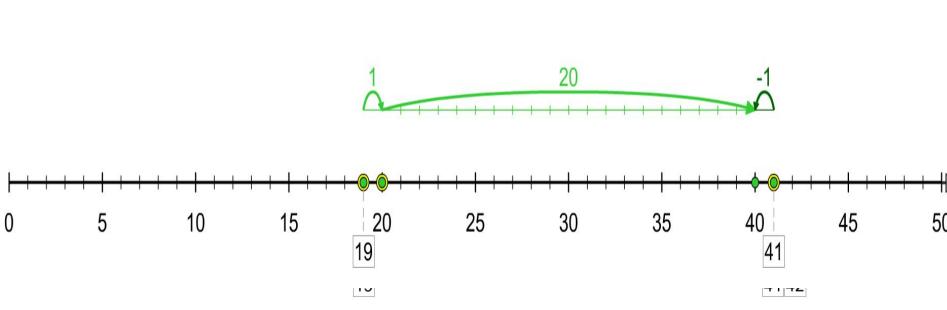Draw proportional jumps and label with
numbers (symbolic).

Subtraction is not just removal.
You can:

◎ Count by 1's
◎ Add up in chunks
◎ Remove in chunks
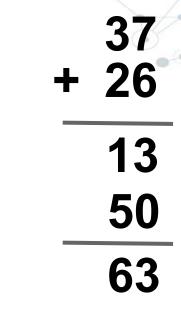◎ Compensate (make the question easier)

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Solve:
41 - 19

Addition:
Building towards Algorithms with Meaning

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$$37 \rightarrow 30 + 7$$
$$+26 \rightarrow 20 + 6$$
$$63 \quad\quad 50 + 13$$

$$37 \rightarrow 30 + 7$$
$$+26 \rightarrow 20 + 3 + 3$$
$$63 \quad\quad 50 + 10 + 3$$

Left to right or right to left: what are the advantages and disadvantages?

$$
\begin{array}{r}
37 \\
+\ 26 \\
\hline
50 \\
13 \\
\hline
63
\end{array}
$$

$$
\begin{array}{r}
37 \\
+\ 26 \\
\hline
13 \\
50 \\
\hline
63
\end{array}
$$

Left to Right or
Right to Left?
What are the
advantages and
disadvantages?

Try both with:
  189
+237

$$
\begin{array}{r}
189 \\
+237 \\
\hline
300 \\
110 \\
16 \\
\hline
426
\end{array}
$$

$$
\begin{array}{r}
189 \\
+237 \\
\hline
16 \\
110 \\
300 \\
\hline
426
\end{array}
$$

Possible Advantages of "the North American Standard Algorithm":
-you write down less
-it focuses on digit thinking

Possible Disadvantages of "the North American Standard Algorithm":
-it focuses on digit thinking
-it does not attend to magnitude
-it may hide an understanding of what happens with regrouping
-it limits number flexibility

Key Ideas

- Students do need reliable and efficient methods (algorithms) of performing operations.
- Develop algorithms based on understanding.
- As students build understanding and fluency, they can be coached to write less.

Link to Slides

http://tinyurl.com/nctmadditive