Integral Defined Functions:
Discovering the Fundamental Theorem of Calculus with Technology

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I. Approximations with Rectangles
(Finding the Area Under Curves by Approximating with Rectangles)

The area under a curve $y = f(x)$ can be approximated through the use of Riemann sums: $Area = \sum_{k=1}^{n} f(x_k)\Delta x_k$

$LRAM_n$ = Sum of *n* rectangles using the left-hand *x*-coordinate of each interval to find the height of the rectangle.
$RRAM_n$ = Sum of *n* rectangles using the right-hand *x*-coordinate of each interval to find the height of the rectangle.
$MRAM_n$ = Sum of *n* rectangles using the midpoint *x*-coordinate of each interval to find the height of the rectangle.

Let $f(x) = x^2 + 1$ on $[0,3]$.

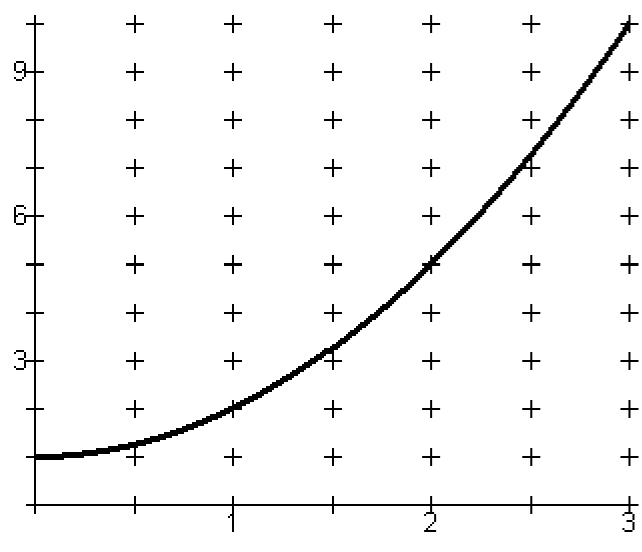
If there are 3 intervals, what is the value of Δx_k ? ___________

If there are 6 intervals, what is the value of Δx_k ? ___________

If there are *n* intervals and all *n* intervals are the same width, what is the value of Δx_k ? _______________

For each of the following, draw the indicated rectangles and find the area of the rectangles.

On the first figure, draw three rectangles using the left-hand rule. ($LRAM_3$)
Find the area of each rectangle, and find the sum of the areas.



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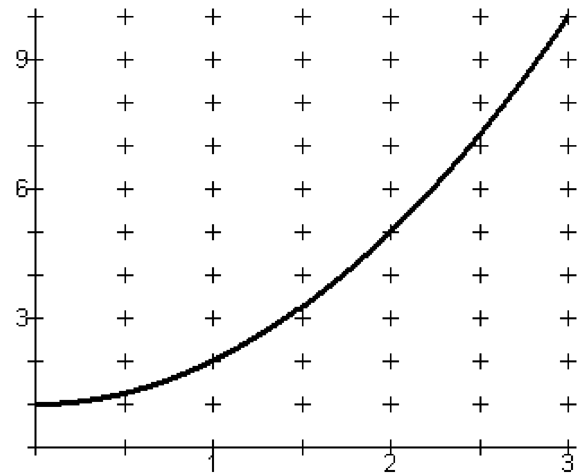
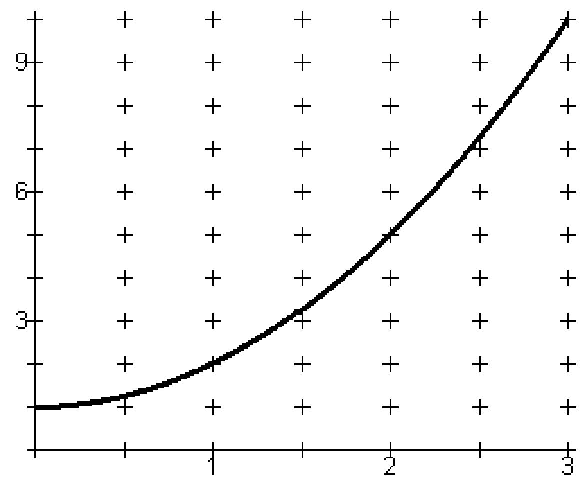
Draw six rectangles using the left-hand rule. ($LRAM_6$)

Find the area of each rectangle, and find the sum of the areas.

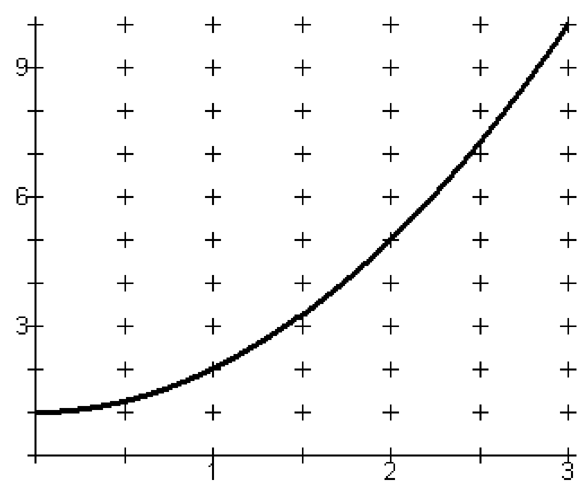


Draw and compute $RRAM_6$



Draw three rectangles using the mid-point rule. ($MRAM_3$)

Find the area of each rectangle, and find the sum of the areas.



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Use the **DRAWRECT** program on the TI-84+ to complete the table on the right for the function $f(x) = x^2 + 1$. Use this program up to $n = 50$, and then use the **RAM** program (Rectangular Approximation Method).

Discuss whether these will under estimate or over estimate the actual area.

n	LRAM	MRAM	RRAM
6			
10			
50			
100			
500			

Let $f(x) = 3^x$ on $[-1,3]$.

Use the **DRAWRECT** and **RAM** programs to complete the following table. Work in groups of three, with each person taking one of the three types of approximation.

Discuss whether these will under estimate or over estimate the actual area.

n	LRAM	MRAM	RRAM
8			
10			
50			
100			
500			

The **definite integral** is the limit of a Riemann sum as the number of intervals approach infinity (width of the interval approaches zero) as indicated in the following formula:

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x_k = \int_a^b f(x)\,dx.$$

The **fnInt** (MATH **9:fnInt(**) option on the calculator produces a numerical integral of the function. It computes an approximation of the total area from $x = a$ to $x = b$. The TI-84+ with current operating system and the TI-84 CE will show an integral symbol unless the calculator is in Classic Mode. The syntax on a TI-83+ or TI-84+ in classic mode is **fnInt(** *function, variable of integration, left endpoint, right endpoint*).

Sketch a graph and shade the indicated region of each of the following functions. Use fnInt to approximate the value of each definite integral.

1. $\int_0^2 \left(x^2+1\right)dx$

2. $\int_{-2}^0 \left(x^2+1\right)dx$

3. $\int_0^{-2} \left(x^2+1\right)dx$

Why is the answer to problem 3 a negative number?

4. $\int_0^1 \left(x^2-1\right)dx$

Why is the answer to problem 4 a negative number?

5. $\int_{-2}^2 \sin(x)dx$ Explain the answer to problem 5?

Draw a sketch for each problem. Use fnInt to determine the area under the curve.

1. $\int_0^2 x^3\,dx$
2. $\int_{-2}^0 x^3\,dx$
3. $\int_{-2}^2 x^3\,dx$
4. $\int_{-1}^2 x^3\,dx$
5. $\int_{-2}^1 x^3\,dx$
6. $\int_0^{-2} x^3\,dx$
7. Give a geometric explanation why some of the answers are negative and some are positive. If any of the answers equal zero, give a reason why.

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II. Integral Defined Functions

What is the relationship between the area function $A(x)$ and the original function $f(t)$?

Problem 1: Given $f(t) = 2$ on $[0, x]$. Find the values of an area function, $A(x)$ which represents the area under the graph of f from $t = 0$ to $t = x$ for different values of x.

Fill in the chart below by evaluating $A(x) = \int_0^x 2\,dt$.

x	0	1	2	3	-1	-2
$A(x)$						

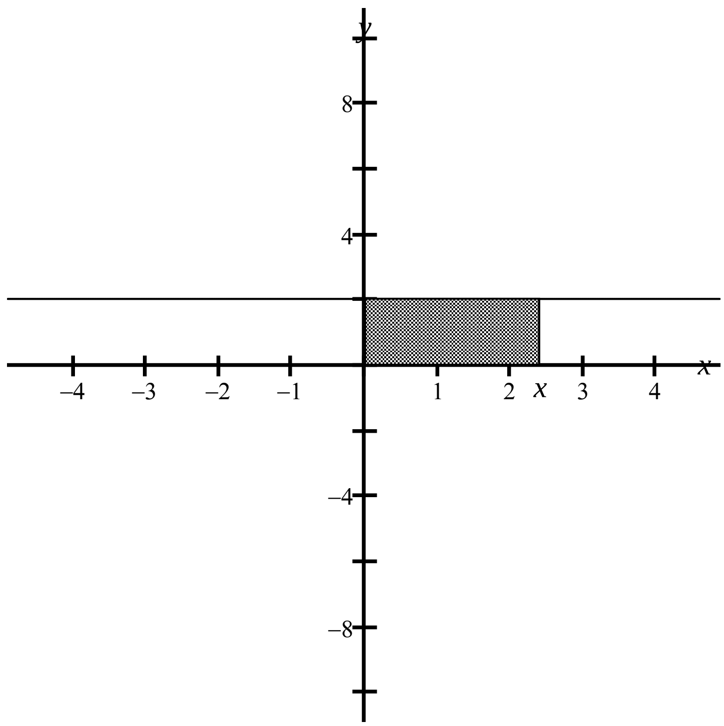
To the right there is a sketch of the function $y = f(t) = 2$ on $[0, x]$ and the region from 0 to x shaded.

1. Plot the values of $A(x)$ on the axis.

2. Sketch the graph of $A(x)$.

3. Find a function $A(x)$ that fits the data points.

4. $A(x) =$ ____________________.



Complete the steps 1 to 4 above for the following functions. Use the same axis to sketch the graphs.

$f(t) = 2$ on $[1, x]$ $\int_1^x 2\,dt$

x	0	1	2	3	-1	-2
$B(x)$						

$f(t) = 2$ on $[-1, x]$ $\int_{-1}^x 2\,dt$

x	0	1	2	3	-1	-2
$C(x)$						

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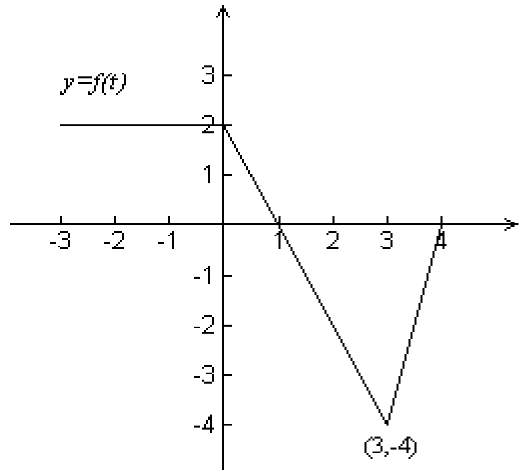
Problem 2.

Let $f(t)$ be defined by the graph shown on the right.



Let $g(x) = \int_0^x f(t)dt$.

Rewrite the following integrals in terms of g .

$\int_0^1 f(t)dt =$

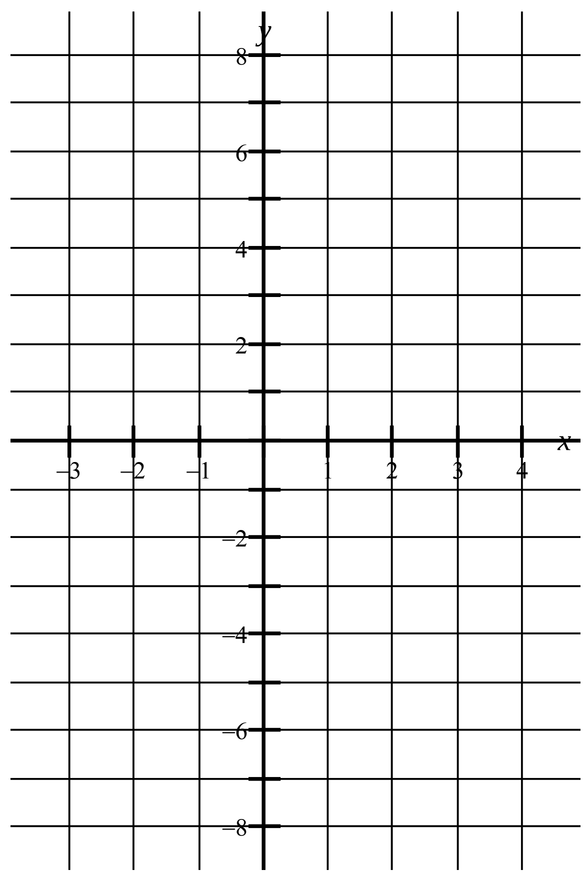
$\int_0^{-2} f(t)dt =$

$\int_1^4 f(t)dt =$

Let $h(x) = \int_{-3}^x f(t)dt$ and $k(x) = \int_1^x f(t)dt$. Fill in the chart below.

x	-3	-1	0	1	3	4
$g(x)$						
$h(x)$						
$k(x)$						

Plot the data from the table on the graph to the right.



What do you notice about the three graphs?

Express the functions $h(x)$ and $k(x)$ in terms of $g(x)$.

Earlier in the course, we found the antiderivatives of functions. We also learned that a function has infinitely many antiderivatives, each differing by a constant.

Does this relate to the graphs drawn on the right? Explain.

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Problem 3: Let $f(t) = 2t - 2$. Sketch the graph of f on $[-2, 4]$ on the graph below.

Let $A(x) = \int_0^x (2t-2)\,dt$. Remember, the function A represents the area between the graph of f and the x-axis.

Use the graph of f to answer the following questions.

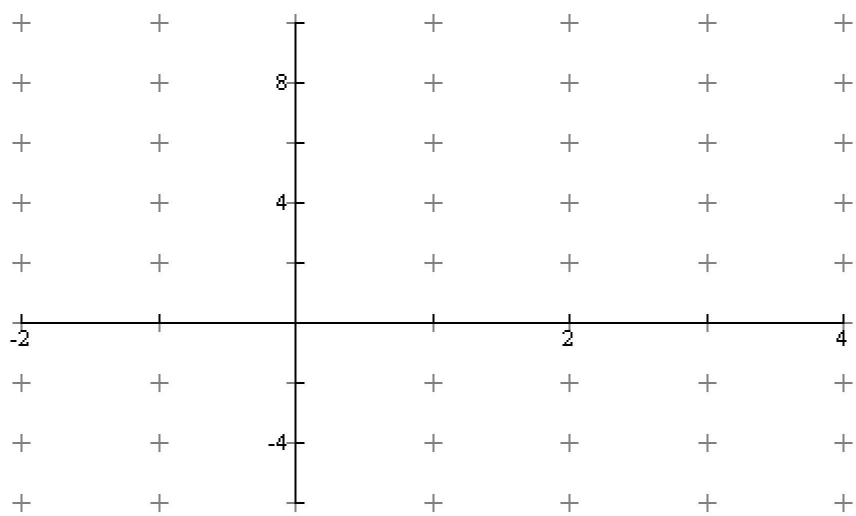
$A(0) =$ _______ $A(1) =$ _______ $A(2) =$ ______

$A(3) =$ _______ $A(4) =$ _______

$A(-1) =$ _______ $A(-2) =$ _______

Plot the coordinates of the points found above in the same window as the sketch of the graph of $f(t)$.

Sketch what you think $A(x)$ might look like.



Verify your sketch by graphing $\int_0^x (2t - t)dt$ on the calculator.

Find a function $A(x)$ that fits the data points. $A(x) =$ ______________________

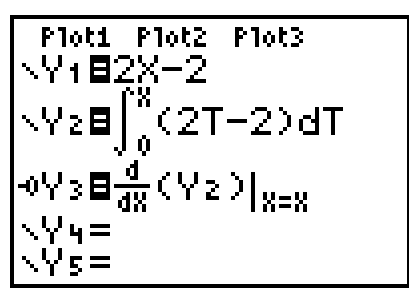
The function $A(x)$ has a minimum at x =? The function $f(x)$ has at zero at x =?

The minimum of the function $A(x)$ occurs at a _____________ of $f(x)$.

Enter the functions shown on the screen to the right.
Note the graph type for Y3.

Press WINDOW and change Xres to 4.

Graph in a decimal window. (ZOOM 4)



What can you conclude about the function $A(x)$ and the function $f(x)$?

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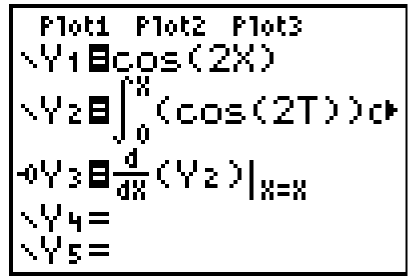
Problem 4: How do other functions behave?

Investigate the function $g(x)=\int_0^x \cos(2t)dt$.

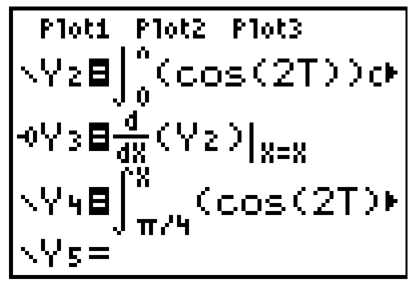


Enter the functions shown on the screen to the right. Note the graph type for Y3.

Press WINDOW and change to $\left[0,2\pi\right]$ by $\left[-1.5,1.5\right]$ and change Xres to 4.

What can you conclude about the derivative of the function $g(x)$?

Add the function $h(x)=\int_{\pi/4}^x \cos(2t)dt$ by graphing in Y4 as shown on the right.



This changes the lower limit of the integral.

How is this graph of Y4 different from Y2?

How does this relate to the graph of Y1/Y3?

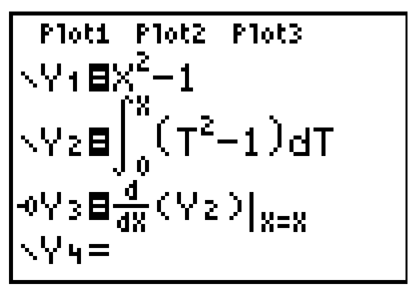
Discuss any relationships between the function $\cos(2t)$ and the graphs of $g(x)$ and $h(x)$?

Where do the graphs of $g(x)$ and $h(x)$ appear to have maximums and minimums?

How does this relate to the function $\cos(2t)$?

Problem 5:

Investigate the function $k(x)=\int_0^x \left(t^2-1\right)dt$.



Enter the functions shown on the screen to the right. Note the graph type for Y3.

Press WINDOW and change Xres to 4.

Graph in a ZOOM 4 window.

What can you conclude about the derivative of the function $k(x)$?

Find a new function with a new lower limit of integration such that the graph of Y4 will be at least one unit higher than the graph of Y2.

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III. Fundamental Theorem of Calculus (Part 1)

$$\text{Define } A(x) = \int_a^x f(t)\,dt$$

$$\frac{d}{dx}\left(A(x)\right) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

What is $\frac{d}{dx}\left(\int_1^x t^2 dt\right)$?

$$\frac{d}{dx}\left(\int_1^x t^2 dt\right) = \frac{d}{dx}\left(\tfrac{1}{3}t^3\Big|_1^x\right)$$
$$= \frac{d}{dx}\left(\tfrac{1}{3}x^3 - \tfrac{1}{3}\right)$$
$$= \tfrac{1}{3}(3x^2) - 0$$
$$= x^2$$

What is $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right)$?

a. Let F be the antiderivative of $f(t)$.

b. Write $\int_a^x f(t)\,dt$ in symbolic form using FTC P2.

$F(x) - F(a)$

c. Take the derivative of the answer to (b) with respect to *x*.

Example: $\frac{d}{dx}\left(\int_2^x \left(\sin^2 t\right) dt\right)$

Let F be the antiderivative of $f(t)$ (i.e. $F'(t) = \sin^2(t)$).

Write $\frac{d}{dx}\left(\int_2^x \left(\sin^2 t\right) dt\right)$ in symbolic form using FTC P2: $F(t)\Big|_2^x = F(x) - F(2)$

Take the derivative with respect to *x*: $\frac{d}{dx}\left(F(x) - F(2)\right) = F'(x) - 0 = \sin^2(x) = f(x)$

Find the following derivatives:

1. $\frac{d}{dx}\left(\int_{-2}^x \sqrt{1+e^{5t}}\,dt\right)$

2. $\frac{d}{dx}\left(\int_x^6 \ln(1+t^2)\,dt\right)$

3. $\frac{d}{dx}\left(\int_2^{x^2} \sqrt{5+t^3}\,dt\right)$

4. $\frac{d}{dx}\left(\int_{x^2}^{x^3} \cos(2t)dt\right)$

5. $\frac{d}{dx}\left(\int_{x^2+x}^{\sin x} \left(t+t^4\right)dt\right)$

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