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Three Thought Tools for the Mathematical Modeler

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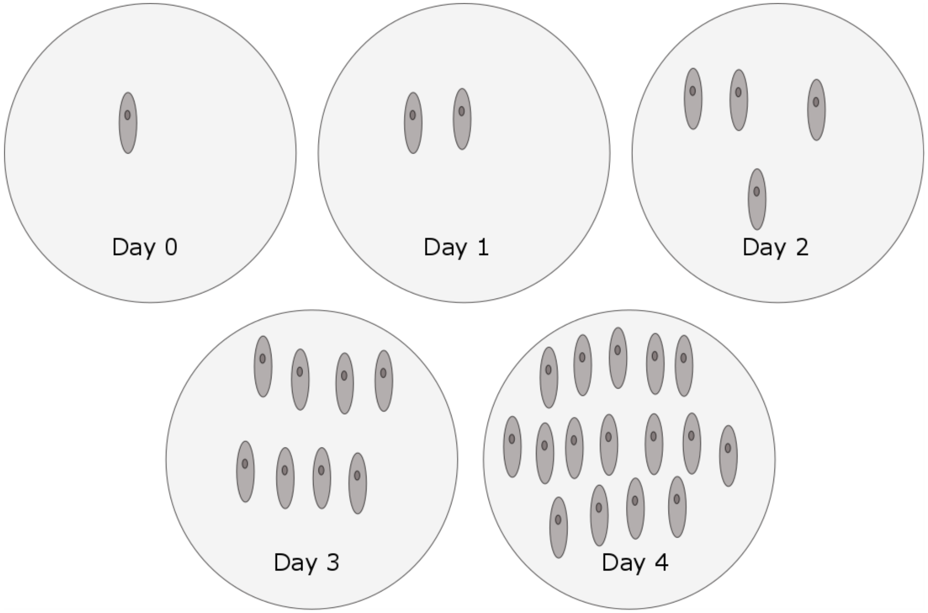
The notion of a *scientific or conceptual model*

A *scientific model or conceptual model* is a model comprised of a set of ideas that describes a process, pattern, or phenomenon in the natural world.

Conceptual Model – The raw egg contains a liquid that remains in motion even when the teacher's finger is placed upon the egg. When her finger is removed, this liquid pushes on the shell and returns the egg to motion. The hard boiled egg contains solid egg that fills the shell and when the motion of this egg is halted, the inside is halted as well.

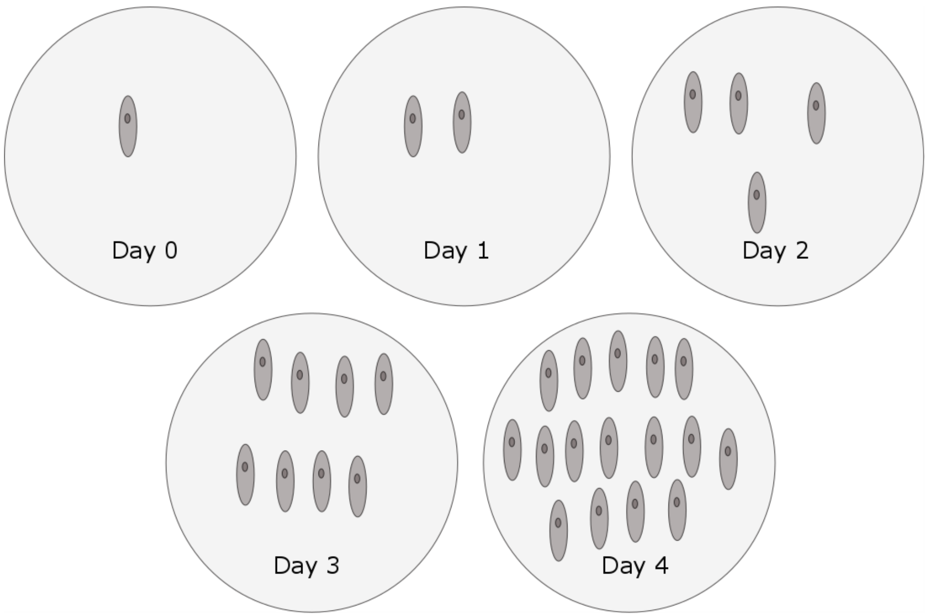
A Look at Bacteria



Scientific Model – The population of microorganisms is increasing daily through reproduction. Each microorganism present in the population reproduces once each day.

How might you translate this set of ideas into mathematics?

Express ideas using mathematics



Scientific Model – The population of microorganisms is increasing daily through reproduction. Each microorganism present in the population reproduces once each day.

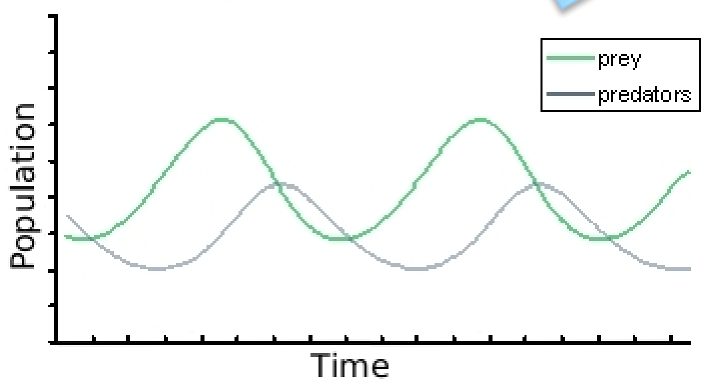
Mathematical Model -

$$P(n+1) = 2P(n)$$

$$P(0) = 1$$

$$\frac{dN}{dt} = rN - aNP$$

$$\frac{dP}{dt} = eaNP - mP$$

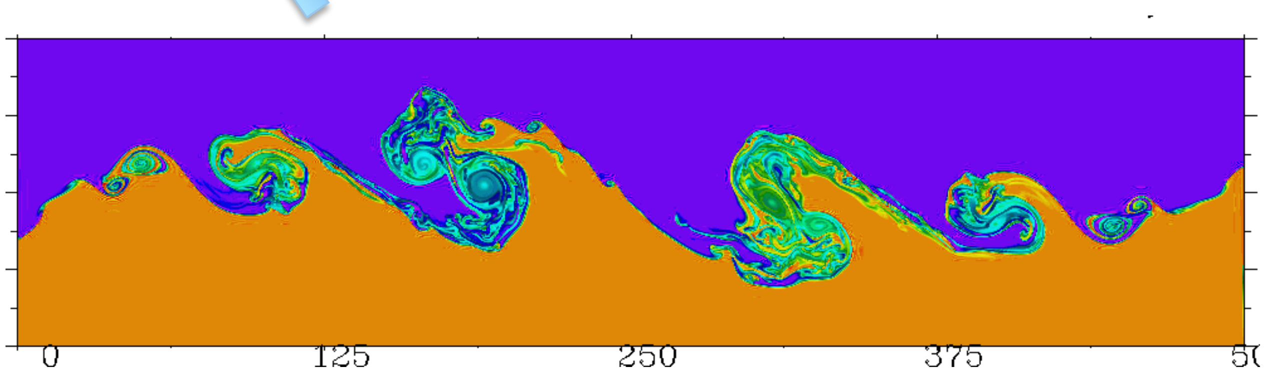


$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z}+\mu[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}]$$

$$\rho\left(\frac{\partial u_z^* U_0^2}{\partial t^* R}+u_r^* U_0\frac{\partial u_z^* U_0}{\partial r^* R}+\frac{u_\theta^* U_0}{r^* R}\frac{\partial u_z^* U_0}{\partial \theta}+u_z^* U_0\frac{\partial u_z^* U_0}{\partial z^* R}\right) = -\frac{\partial p^* U_0^2 \rho}{\partial z^* R}+\mu[\frac{1}{r^* R}\frac{\partial}{\partial r^* R}\left(r^* R\frac{\partial u_z^* U_0}{\partial r^* R}\right)+\frac{1}{(r^* R)^2}\frac{\partial^2 u_z^* U_0}{\partial \theta^2}+\frac{\partial^2 u_z^* U_0}{\partial (z^* R)^2}]$$

$$\rho\frac{U_0^2}{R}\left(\frac{\partial u_z^*}{\partial t^*}+u_r^*\frac{\partial u_z^*}{\partial r^*}+\frac{u_\theta^*}{r^*}\frac{\partial u_z^*}{\partial \theta}+u_z^*\frac{\partial u_z^*}{\partial z^*}+\frac{\partial p^*}{\partial z^*}\right) = \mu\frac{U_0}{R^2}[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_z^*}{\partial r^*}\right)+\frac{1}{r^{*2}}\frac{\partial^2 u_z^*}{\partial \theta^2}+\frac{\partial^2 u_z^*}{\partial z^{*2}}]$$

$$\frac{\partial u_z^*}{\partial t^*}+u_r^*\frac{\partial u_z^*}{\partial r^*}+\frac{u_\theta^*}{r^*}\frac{\partial u_z^*}{\partial \theta}+u_z^*\frac{\partial u_z^*}{\partial z^*}+\frac{\partial p^*}{\partial z^*} = \frac{1}{Re}[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_z^*}{\partial r^*}\right)+\frac{1}{r^{*2}}\frac{\partial^2 u_z^*}{\partial \theta^2}+\frac{\partial^2 u_z^*}{\partial z^{*2}}]$$



Rayleigh Instability

Plateau [1873] considered the capillary break-up of cylindrical liquid jets to perturbations

$$r(z,\phi) = R_0\left[1 + \epsilon\cos\frac{2\pi z}{\lambda}\cos n\phi\right] + \frac{\epsilon^2}{2}R_2$$

The volume over a wavelength is given by

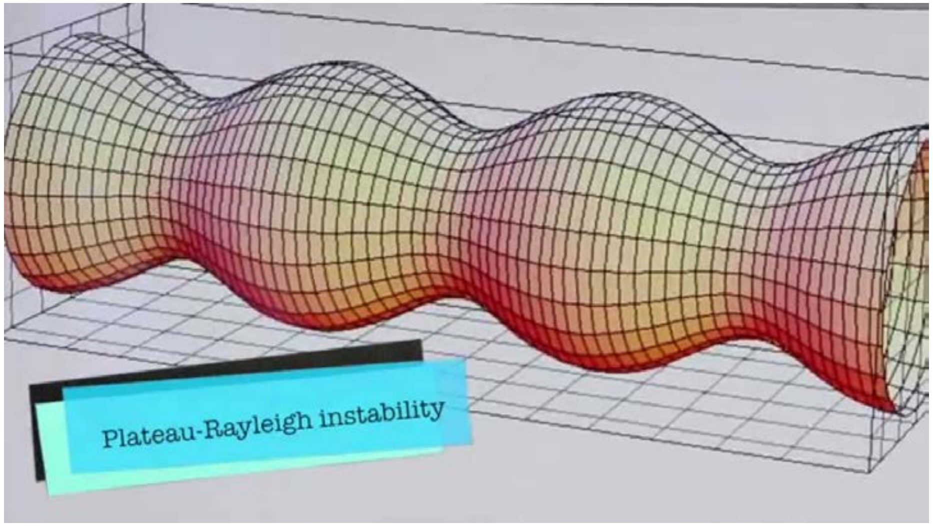
$$\text{Volume} = \pi R_0^2\,\lambda + \epsilon^2\pi\,R_0\,\lambda\left\{R_2 + \frac{R_0}{4}\right\} + O(\epsilon^3)$$

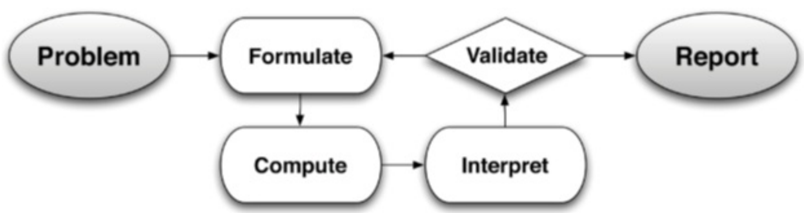
so $R_2 = -R_0/4$ preserves the volume through $O(\epsilon^2)$. Then

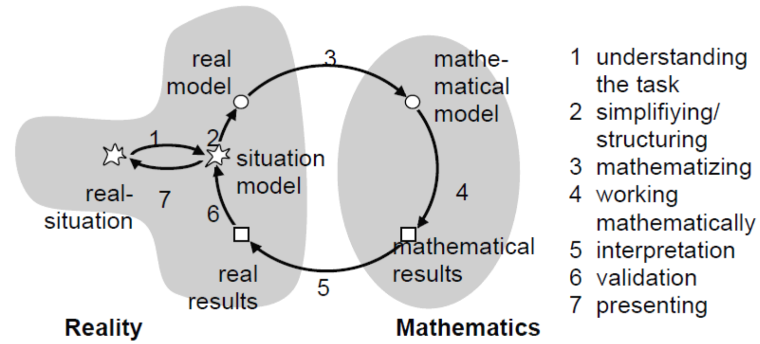
$$\text{Area} = 2\pi\,R_0\,\lambda + \frac{\epsilon^2\,\pi\,R_0\,\lambda}{4}\left[\left(\frac{2\pi R_0}{\lambda}\right)^2 + (n^2-1)\right] + O(\epsilon^3)$$

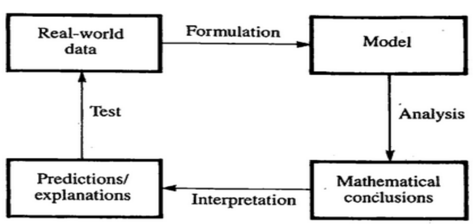
so $n \geq 1$ increases the area, but $n = 0$ with $\lambda > 2\pi\,R_0$ decreases the area.

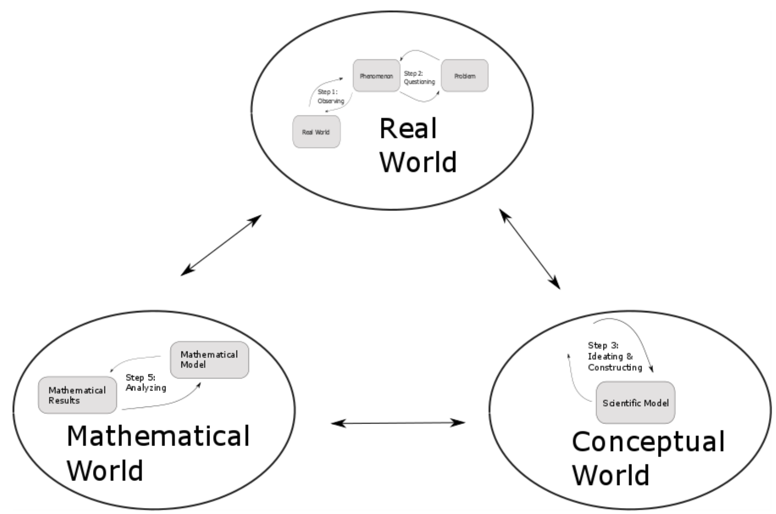
Rayleigh [1878] considered a dynamical model to get growth rates of instability.

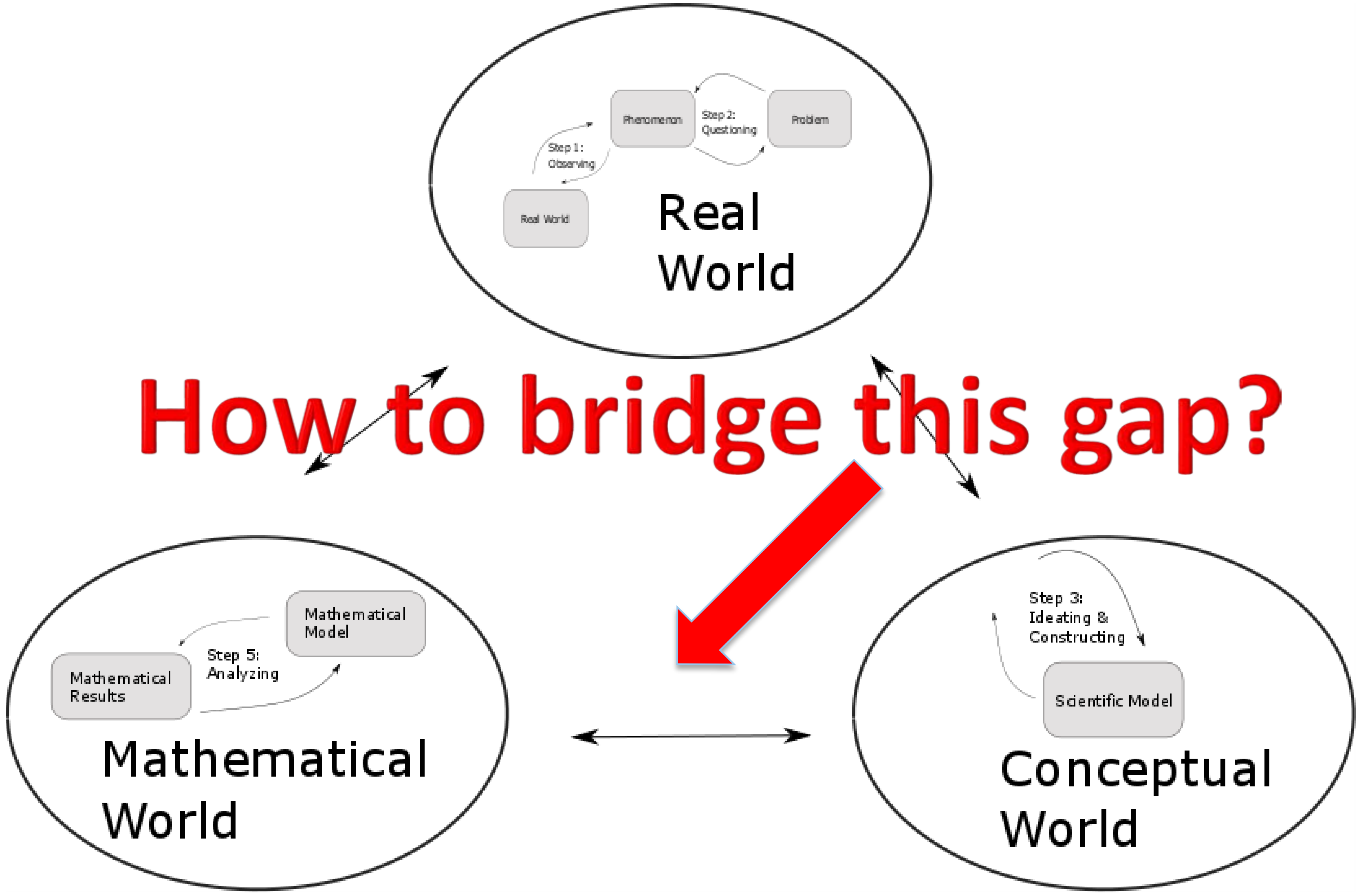



Problem
Formulate
Validate
Report
Compute
Interpret

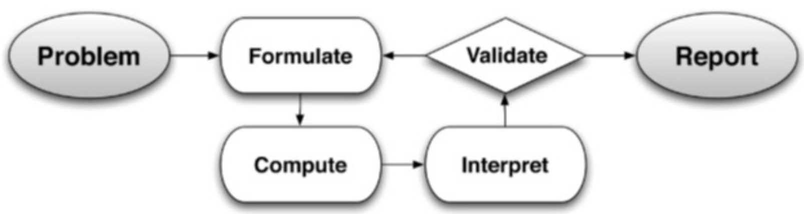

real model
mathe-matical model
situation model
real-situation
real results
mathematical results
Reality
Mathematics
1 understanding the task
2 simplifiying/ structuring
3 mathematizing
4 working mathematically
5 interpretation
6 validation
7 presenting


Real-world data
Formulation
Model
Test
Analysis
Predictions/ explanations
Interpretation
Mathematical conclusions


Real World
Mathematical Model
Step 5: Analyzing
Mathematical Results
Mathematical World
Step 3: Ideating & Constructing
Scientific Model
Conceptual World


Phenomenon
Step 2: Questioning
Problem
Step 1: Observing
Real World
Real World
How to bridge this gap?
Mathematical Model
Step 5: Analyzing
Mathematical Results
Mathematical World
Step 3: Ideating & Constructing
Scientific Model
Conceptual World

"These Standards are not intended to be new names for old ways of doing business. They are a call to take the next step." (CCSSM, p. 5)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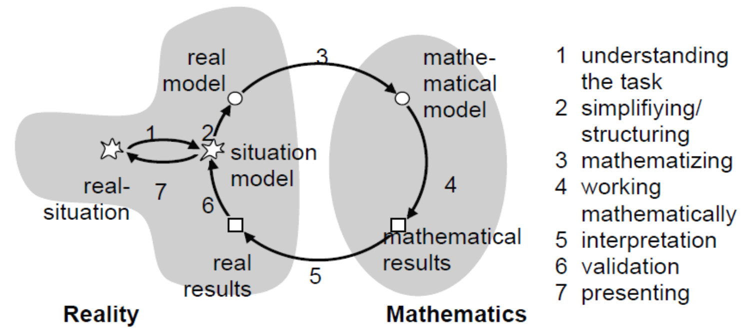
real model
mathe-matical model
real-situation
situation model
mathematical results
real results
1
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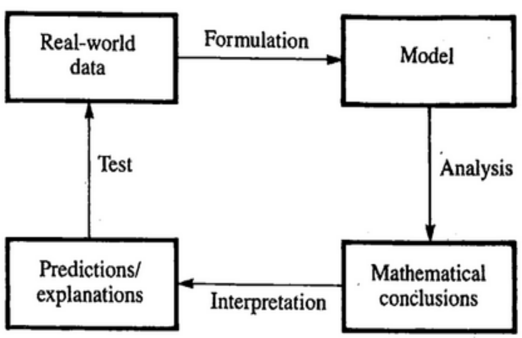
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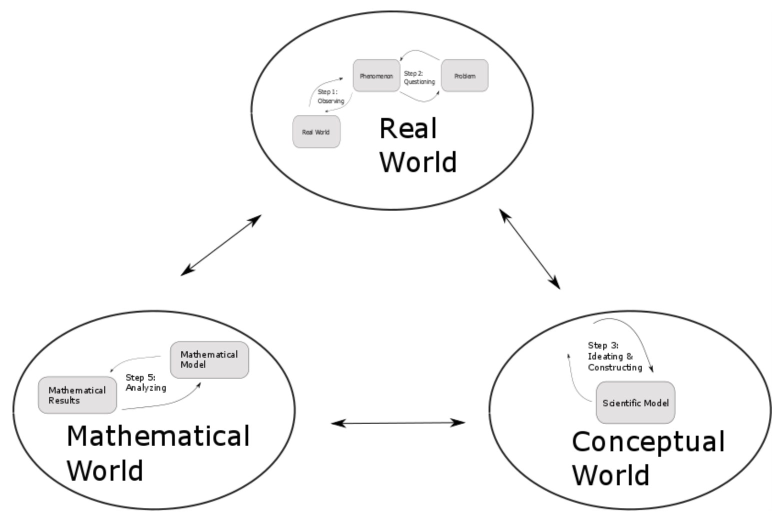
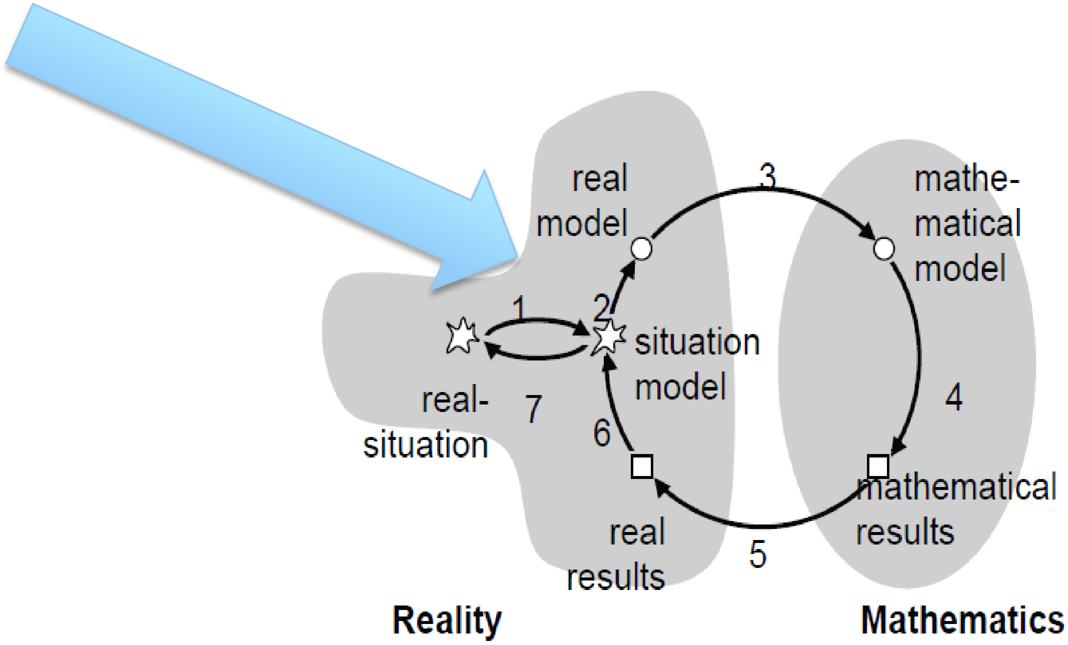

Real World
Mathematical World
Conceptual World

A German Modeling Cycle



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4 working mathematically
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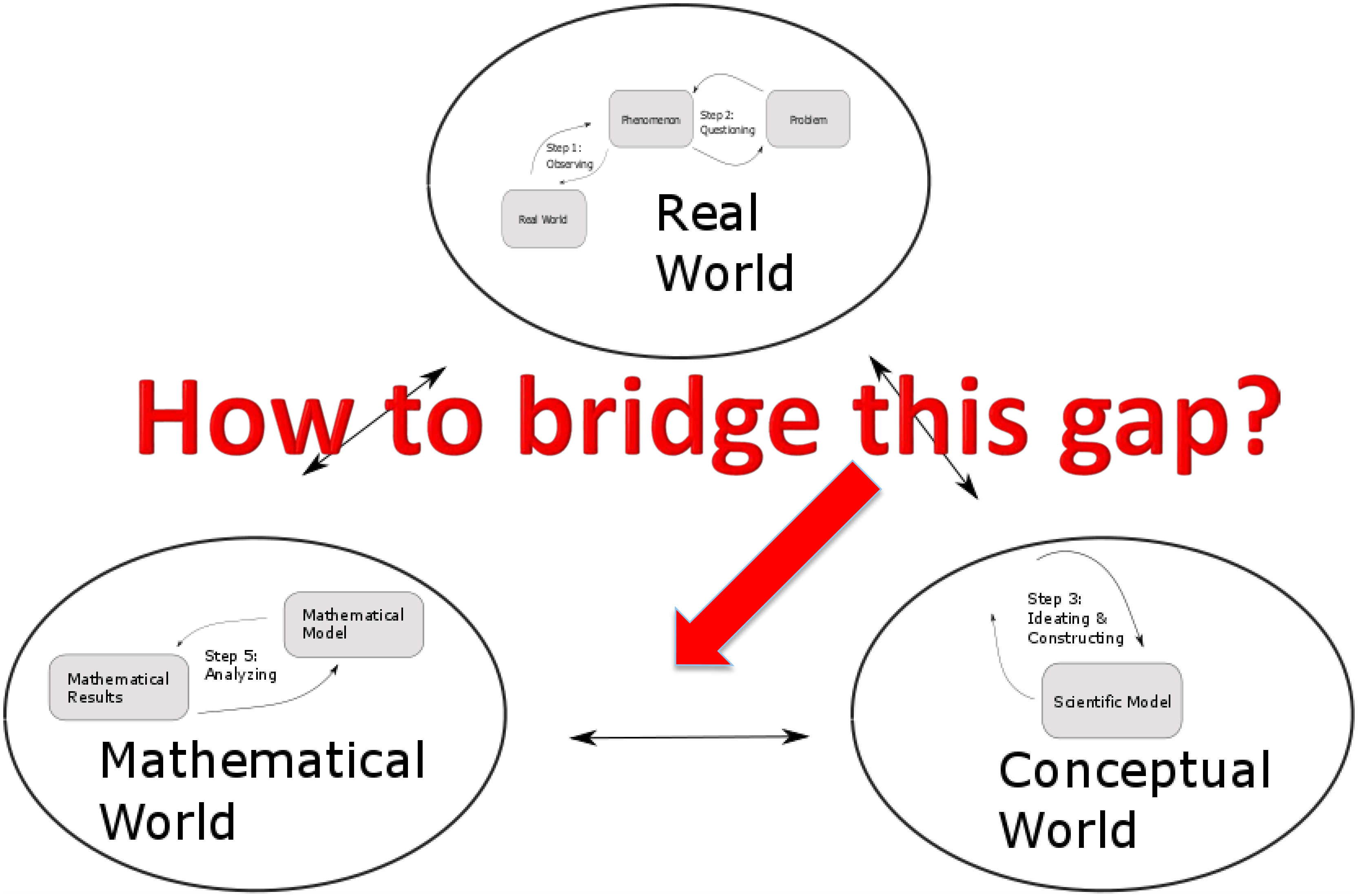
6 validation
7 presenting

Blum (2011, p. 18)

How to bridge this gap?

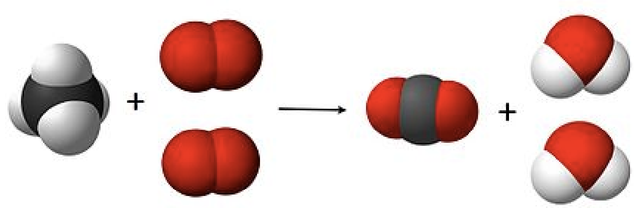


Three Thought Tools of the Mathematical Modeler

Our goal for today is to introduce you to three *thought tools*, simple, yet powerful ideas, about how a wide-range of real-world phenomena behave. These thought tools allow the modeler to readily translate ideas about the real-world into the language of mathematics.

Thought Tool #1

How do we think about populations that interact?



$$CH_4 + 2O_2 \longrightarrow CO_2 + 2H_2O$$

Experiment – Paper Tossers

Thought Tool – In a situation where you have interacting individuals, call them *reactants*, the amount of reactants that change into something else, called *products*, is simply <u>proportional</u> to how many reactants you have.

Experiment – Paper Tossers

$$P(n) = Number\ of\ paper\ tossers\ at\ step\ n$$

Our thought tool tells us…

$$P(n+1) = P(n) + bP(n) - aP(n)$$

Birth rate!

Death rate

Experiment – East or West?

East/East – Nothing happens

East/West – East sits down

West/West – Nothing happens

Experiment – East or West?

Thought Tool – In a situation where you have interacting individuals, call them *reactants*, the amount of reactants that change into something else, called *products*, is simply <u>proportional</u> to how many reactants you have. If you have more than one reactant, it's proportional to the product of those reactants. This is called the *Law of Mass Action.*

Mathematics – East or West?

$$E(n) = Number\ of\ individuals\ who\ live\ east\ of\ the\ Mississipi$$

$$W(n) = Number\ of\ individuals\ who\ live\ west\ of\ the\ Mississipi$$

Our thought tool tells us…

$$E(n+1) = E(n) - aE(n)W(n)$$

$$W(n+1) = W(n)$$

Now, you try!

$$S(n) = Number\ of\ sheep\ at\ time\ step\ n$$

$$F(n) = Number\ of\ fox\ at\ time\ step\ n$$

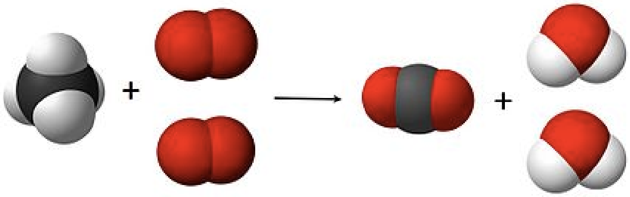
$$S(n+1) = S(n) + bS(n) - aS(n) - cS(n)F(n)$$

$$F(n+1) = F(n) + bF(n) - dF(n)$$

Use the simple thought tool (Law of Mass Action) to write down a set of mathematical equations that tell you the number of sheep and the number of fox at time step n+1. You'll need an equation for each!

Thought Tool #1

How do we think about populations that interact?



$$CH_4 + 2O_2 \longrightarrow CO_2 + 2H_2O$$

Law of Mass Action – In a system with interacting reactants and products, the amount of reactants that change into products is proportional to the product of the amount of reactants.

Whenever you have a system like this, whether it's foxes and sheep, chemicals, zombies, or a disease spreading through a population, think Mass Action and you'll be able to build a mathematical model!

Thought Tool #2

How do we keep track of stuff?

How do we know how many parking spaces are in the garage?

Thought Tool #2

Imagine the garage is empty and closed. In this case, we know that the number of parking spaces, *P*, does not change. That is, *P* is a *conserved quantity*. In a closed system, a conserved quantity is one that does not change, that is, it is *conserved*.

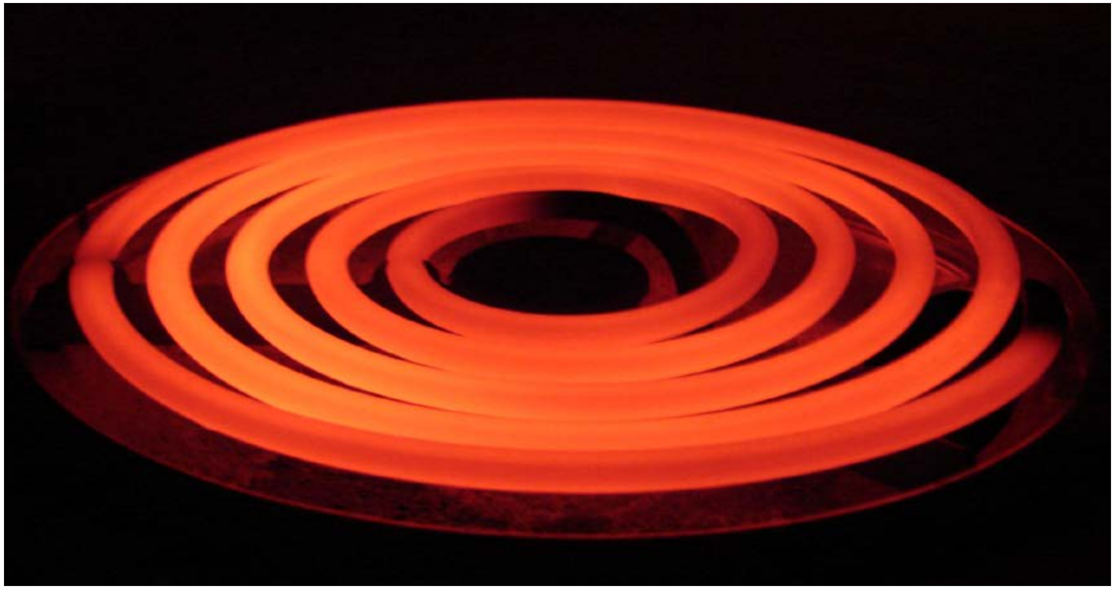
Thought Tool #2

Conservation of Parking Spaces $P = Constant$

The nice thing about conserved quantities is that when the system is not closed or isolated, we can track how the quantity changes by accounting for what flows into the system and what flows out!

$$P = P_0 - Cars\ in + Cars\ out$$

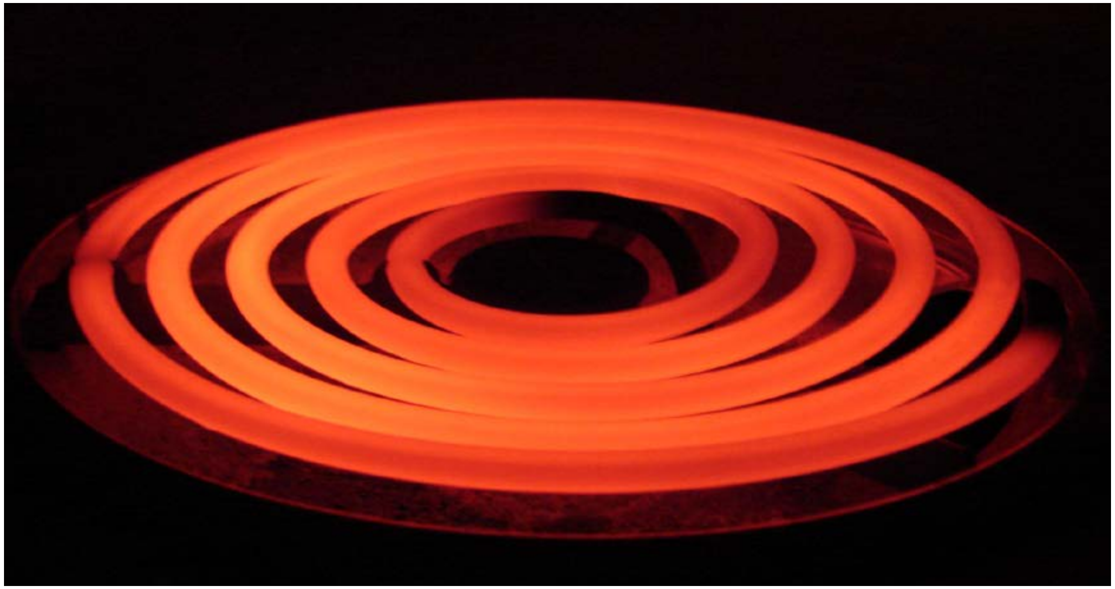
This very simple idea is at the heart of building mathematical models of how things heat up and cool down, how fluid moves, how traffic flows on the highway, and much more.



Imagine you have a hot object like a cup of coffee, or a stove ring and you wish to understand how it cools down? How do you build a mathematical model of such a situation?

Energy is a conserved quantity. To model the cooling of a heated body, we simply need to apply the conservation law idea to the energy of the system.

$$E(t + dt) = E(t) - Energy\ lost\ to\ surroundings$$



$$E(t + dt) = E(t) - \text{Energy lost to surroundings}$$

$$E(t + dt) = E(t) - q(t)dt$$

$$\frac{E(t + dt) - E(t)}{dt} = -q(t)$$

$$\frac{E(t+dt)-E(t)}{dt} = -q(t)$$

$$\frac{dE}{dt} = -q(t)$$

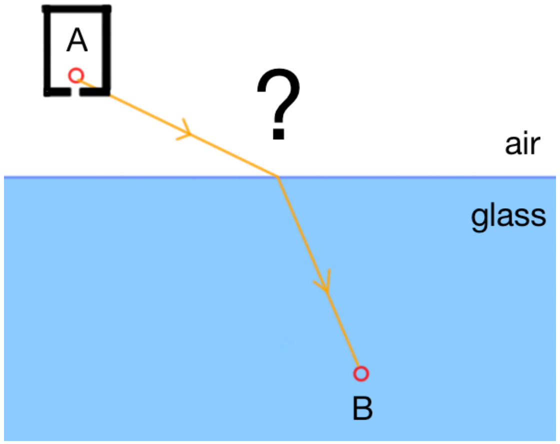
This is just our conservation law!

To obtain something useful, we rely on the idea that both the energy and the heat loss are proportional to the temperature! Newton!

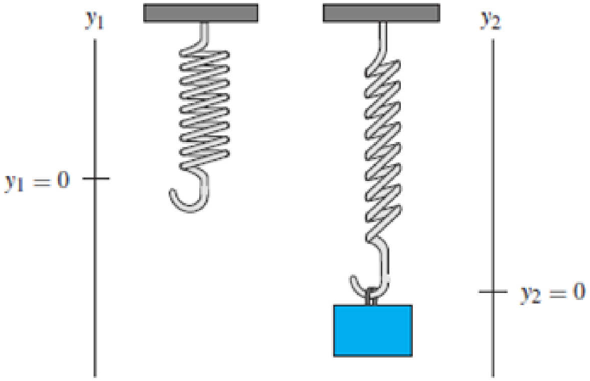
$$mc\frac{dT}{dt} = -hA(T-T_A)$$

To build this mathematical description of our cooling body, we used the idea that energy is a conserved quantity and the idea that energy is measured by temperature. This idea of conservation applies to mass, energy, momentum, cars, and many more real-world systems!

Thought Tool #3



Nature acts economically.



Thought Tool #3

The key to building a mathematical model of many real-world systems is to realize that *nature acts economically*. That is, many real-world systems behave *as if* they are trying to make some quantity as small as possible. This idea is the domain of minimization principles and applies to soap films, hanging chains, light waves, and many other real-world systems.

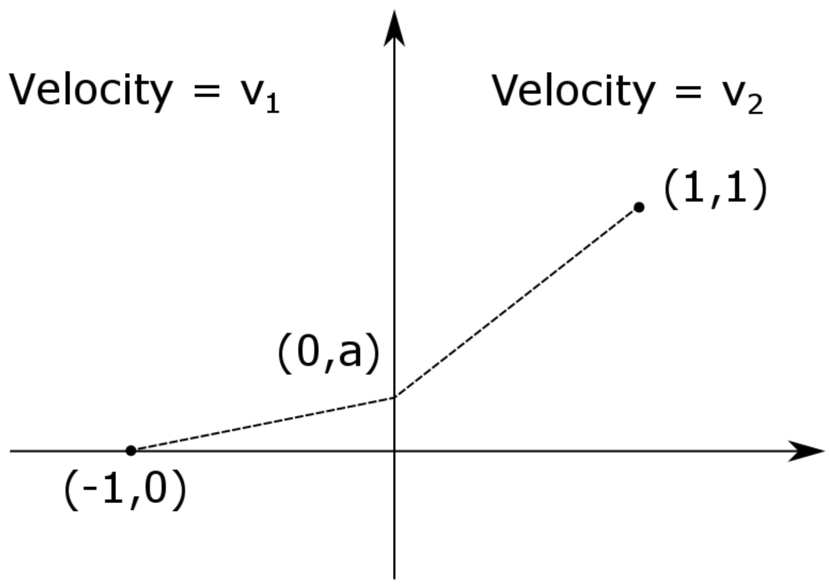
Thought Tool #3 - Example

Why are soap bubbles spherical? How do you think about that mathematically?

Key idea – A soap film tries to minimize its surface area, in this case, surrounding a fixed volume. That's a mathematical problem, whose answer is a sphere!

A Simple Example

Figure 6-8



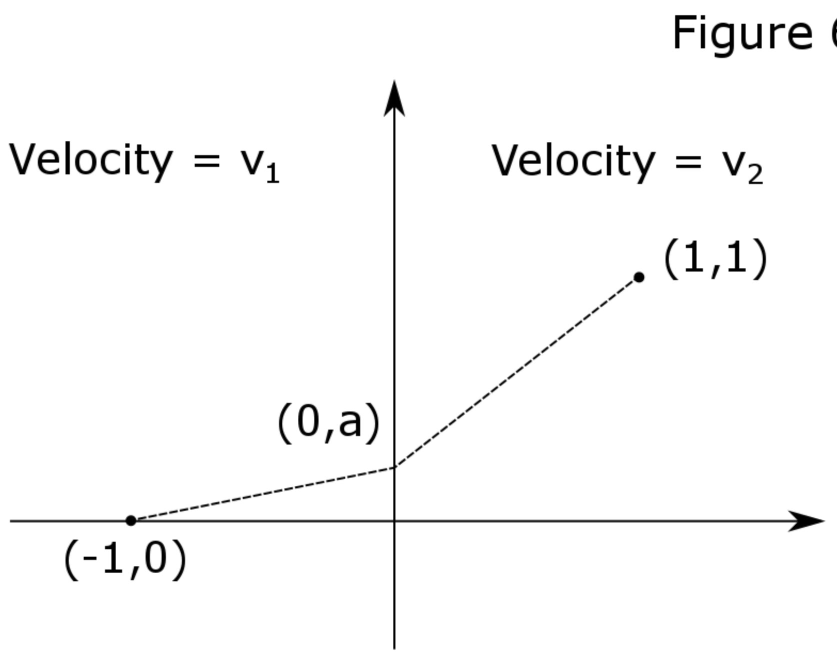
A light ray leaves the point (-1,0) and arrives at the point (1,1). In the second quadrant it travels with one fixed velocity, in the first quadrant, with another. What path does the light ray follow?

Fermat's Principles of Least Time

The light ray "chooses" the path that minimizes its total travel time.

A Simple Example

$$d = vt$$



Travel time in second quadrant $\frac{\sqrt{1+a^2}}{v_1}$

Travel time first quadrant $\frac{\sqrt{1+(1-a)^2}}{v_2}$

$$T(a) = \frac{\sqrt{1+a^2}}{v_1} + \frac{\sqrt{1+(1-a)^2}}{v_2}$$

Three Thought Tools

Thought Tool #1 – When trying to model a system with independent interacting elements, think about the Law of Mass Action.

Thought Tool #2 – When you want to model a system that changes, look for conserved quantities and then use the idea of a Conservation Law to track these changes.

Thought Tool #3 – When you want to model a system that chooses a shape or a path, look for quantities that are minimized.

Mysteries, Models, and Mathematics

- Part I: So You Want to Model? The Art of Mathematical Modeling
- Part II: Putting it Into Practice: Where do I Go from Here?

Understanding the Modeling Cycle

More Thought Tools for Teaching Modeling

Modeling Tasks

A Look Inside a Secondary Classroom

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Some final thoughts and resources

Contact us:

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Michelle Cirillo (mcirillo@udel.edu, @Udmichy)

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