NCTM Annual Meeting San Diego 2019

To Infinity …err … Infinities and Beyond!

Presented By:

Chuck Emenaker & Gene Kramer

University of Cincinnati Blue Ash

April 4, 2019: 8:00 - 9:00 AM

San Diego Convention Center

Room: 30 C

Session 1836

UNIVERSITY OF Cincinnati

Some History of Infinity

Really … beyond infinity!?
The concept of infinity has not always been taken seriously in the mathematical community.

Some History of Infinity

While the concept of infinity dates back to the time of the Greeks, the Ancient Greeks did not define infinity formally but as a philosophical concept.

Some History of Infinity

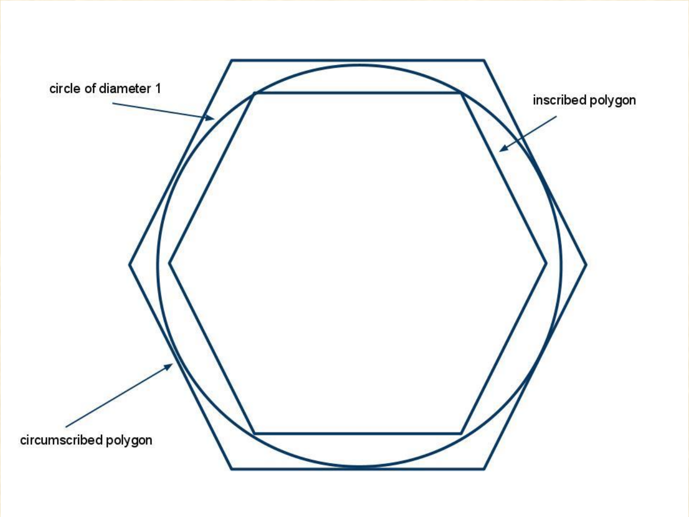
An early concept that has it's roots in infinity is the paradox of The Tortoise and Achilles. This is ascribed to Zeno of Elea, a Greek Philosopher.

Some History of Infinity

How does this paradox work?

Some History of Infinity

In 250 BCE Archimedes method of calculating the area of a circle used the idea of limits and infinity. Archimedes inscribed and circumscribed polygons about a circle. This approach is also known as the *method of exhaustions*.



Some History of Infinity

Using this idea, "Archimedes encountered two concepts which would become hugely popular later – that of limits and that of infinity, for the perfect area would be given by a polygon with infinitely many sides, so the two polygons would converge at that point. As the number of sides tends towards infinity, the difference between the area of the polygons and the area of the circle tends towards zero and the limits coincide"

Rooney, A. (2014). The Story of Mathematics. London: Arcturus Publishing Limited

Some History of Infinity

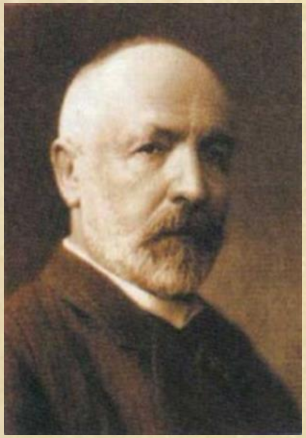
In his book, *De sectionibus conicis* (Of Conical Sections, 1655), John Wallis was the first to use the symbol ∞ which is associated with infinity.

Some History of Infinity

In the late 1600s Leibniz and Newton invented calculus. Leibniz used the idea of infinitesimals. He treated them as though they were discrete, technically a flaw in logic that others overlooked. Newton used power series, which are infinite, to express infinite sums.

Some History of Infinity

In 1874 Georg Cantor discovered uncountable sets. It was one of the most unexpected events in the history of mathematics.



(*Roads to Infinity* p. 10)

Some History of Infinity

Uncountable sets? Does this mean some infinite sets are countable????

Actually, yes!

Some History of Infinity

In 1873 Georg Cantor demonstrated the rational numbers are countable even though they are infinite. The key was the way he went about counting to allow for continued counting in both directions.

Some History of Infinity

How did he count in both directions at once when they were both going to infinity?

ENTER

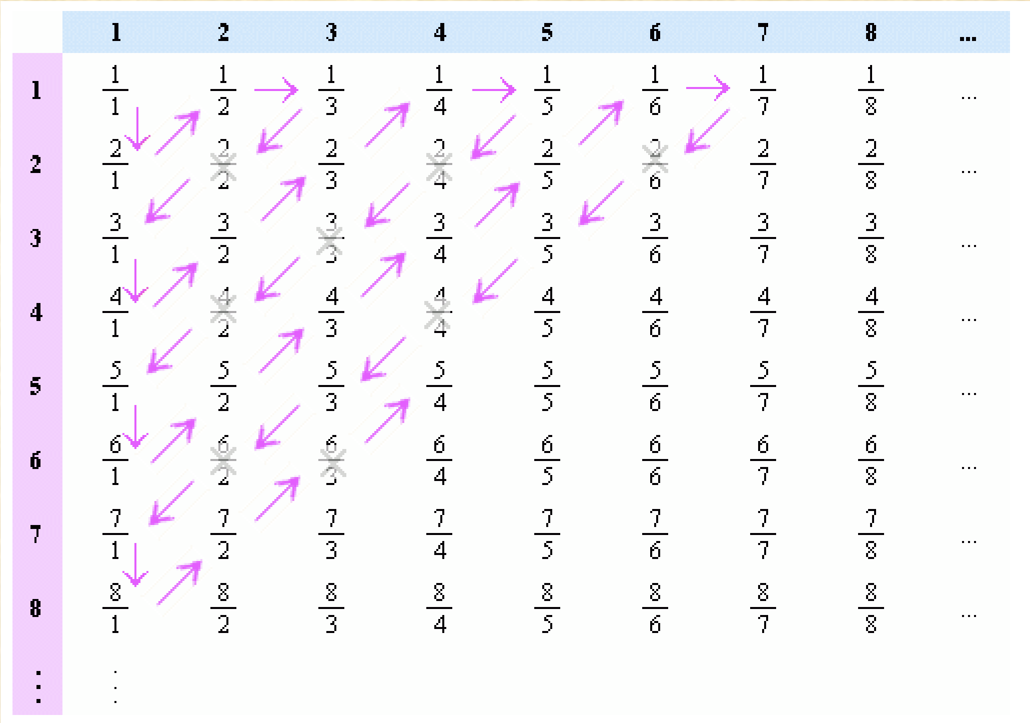
$$
\begin{array}{cccccccccc}
\ldots & 7 & 5 & 3 & 1 & 2 & 4 & 6 & \ldots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\ldots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \ldots
\end{array}
$$

Some History of Infinity

In 1891 he also showed the rational numbers have the same cardinality as the natural numbers using the now famous diagonal argument.

Some History of Infinity

The Diagonal Argument of Cantor



Some History of Infinity

OK, so some sets of numbers are countable, why are others uncountable?

Infinity Up Close & Personal

A First Encounter … Countable Sets

Have students work in small groups. Students in turn create a larger integer.

Have students discuss if there truly is a largest integer.

Perhaps one way to look at it is, infinity is the smallest number you cannot count to!

Infinity Up Close & Personal

A First Encounter … Countable Sets

Perhaps one way to look at it is, infinity is the smallest number you cannot count to!

Infinity Up Close & Personal

A Second Encounter … An Uncountable Set

Have students work in small groups. Students in turn create a larger decimal or fractional number that is less than one.

Have students discuss if there truly is a largest number less than one.

Infinity Up Close & Personal

Here is a challenge to jumpstart the idea of the size of infinity.

Have students generate a list of items that are finite and that are infinite.

Give it a try!

Infinity Up Close & Personal

Here area a couple of common answers for infinite items.

Total grains of sand on earth, not quite infinite … in fact not really that big … about 7.5 X 10^{18} grains of sand.

Total atoms in the universe …. A bit bigger. This count is estimated to be between 10^{78} and 10^{82} atoms.

Total stars in the universe … This count is estimated to be about 10^{21} stars.

Infinity Up Close & Personal

Why does scientific notation only go to 10^{99} on a calculator?

Infinity Up Close & Personal

Let's look at some examples of the implications of infinity.

Consider the following:
Does 0.999… = 1?

First what does it really mean to write 0.999…?

Infinity Up Close & Personal

Does 0.999… = 1?

While not particularly mathematical in nature, complete the following pattern

$$\frac{2}{9}=0.222\ldots,\ \frac{3}{9}=\frac{1}{3}=0.333\ldots,\ \frac{4}{9}=0.444\ldots,\ \frac{5}{9}=0.555\ldots,$$

$$\frac{6}{9}=\frac{2}{3}=0.666,\ \frac{7}{9}=0.777\ldots,\ \frac{8}{9}=0.888\ldots$$

Infinity Up Close & Personal

Is there a more mathematical way to verify 0.999… = 1?

Consider the following:

Letting $x = 0.999\ldots$

It follows that $10x = 9.999\ldots$

$10x = 9 + 0.999\ldots$

$10x = 9 + x$

$9x = 9$

$x = 1$

By the Transitive Property of Equality it follows that

$0.999\ldots = 1$

Infinity Up Close & Personal

Where did *e* come from?

There is quite a bit of history regarding the development of *e*, dating back to Napier's work with logarithms in 1618. However the first formal definition of *e* took place in 1683 while Jacob Bernoulli was considering continuously compounded interest. He needed to evaluate the limit $(1 + {}^{1}/n)^n$ as n tends to infinity, $\lim_{n\to\infty}\left(1+{}^{1}\!/_{n}\right)^{n}$.

Infinity Up Close & Personal

Where did *e* come from?

This limit, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, can also be expressed using the infinite power series

$$e^x = \sum_{n=0}^{\infty}\left(\frac{1}{n!}x^n\right) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots$$

Infinity Up Close & Personal

Where does *e* come from?

Remembering

To remember the value of ***e*** (to 10 places) just remember this saying (count the letters!):

- To
- express
- e
- remember
- to
- memorize
- a
- sentence
- to
- memorize
- this

Or you can remember the curious pattern that after the "2.7" the number "1828" appears TWICE:

2.7 1828 1828

Following the first nine digits of *e* are the digits of the angles 45°, 90°, 45° in a Right-Angle Isosceles Triangle (no real reason, just how it is):

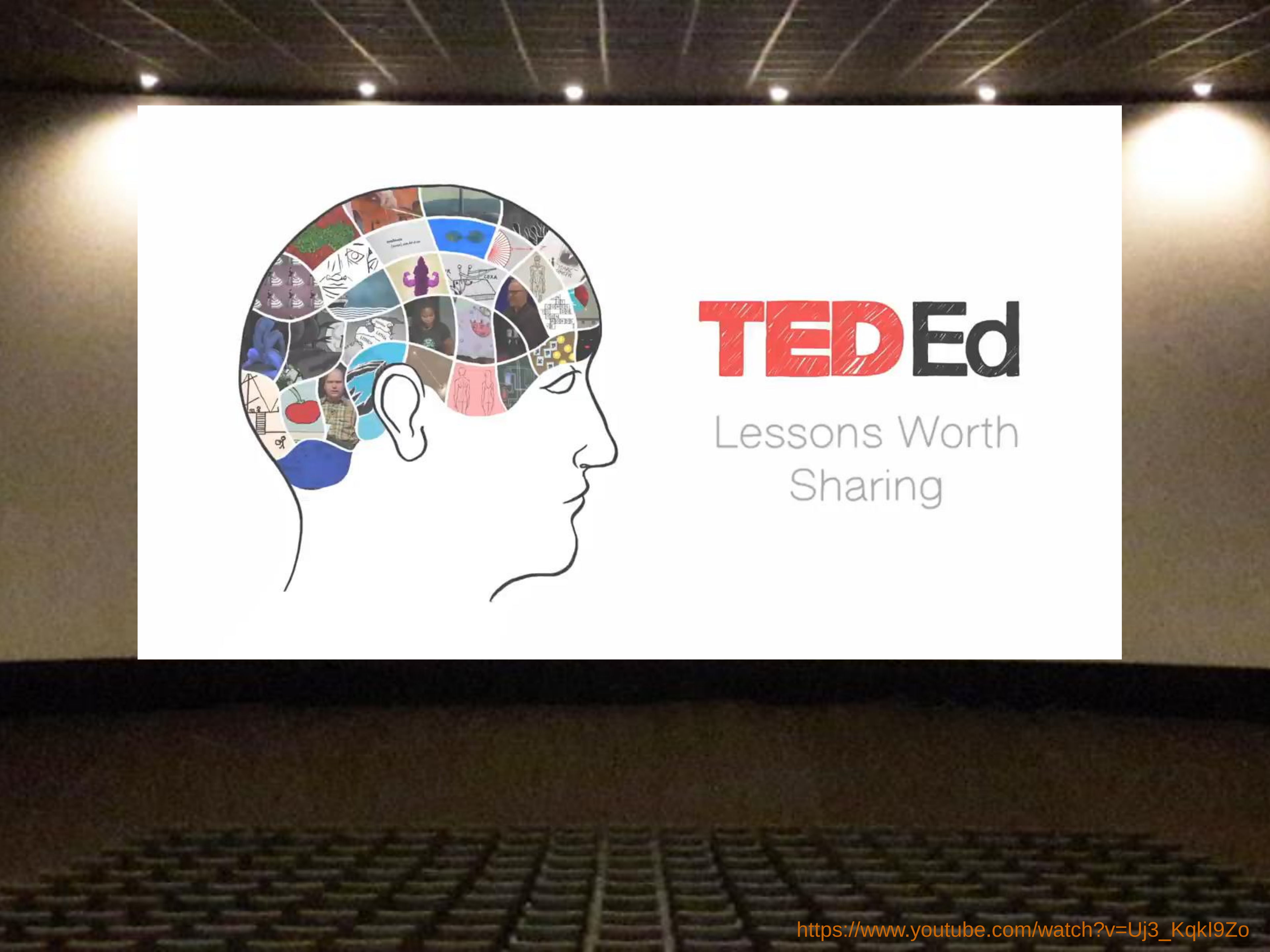
2.7 1828 1828 45 90 45

UC Blue Ash College University of Cincinnati

Infinity Up Close & Personal

Hilbert's Hotel

This idea was presented by David Hilbert in 1924. The idea was to illustrate through a thought experiment a counter-intuitive property of infinite sets. Hmmm …. perhaps there is a better way to explain this one!



https://www.youtube.com/watch?v=Uj3_KqkI9Zo

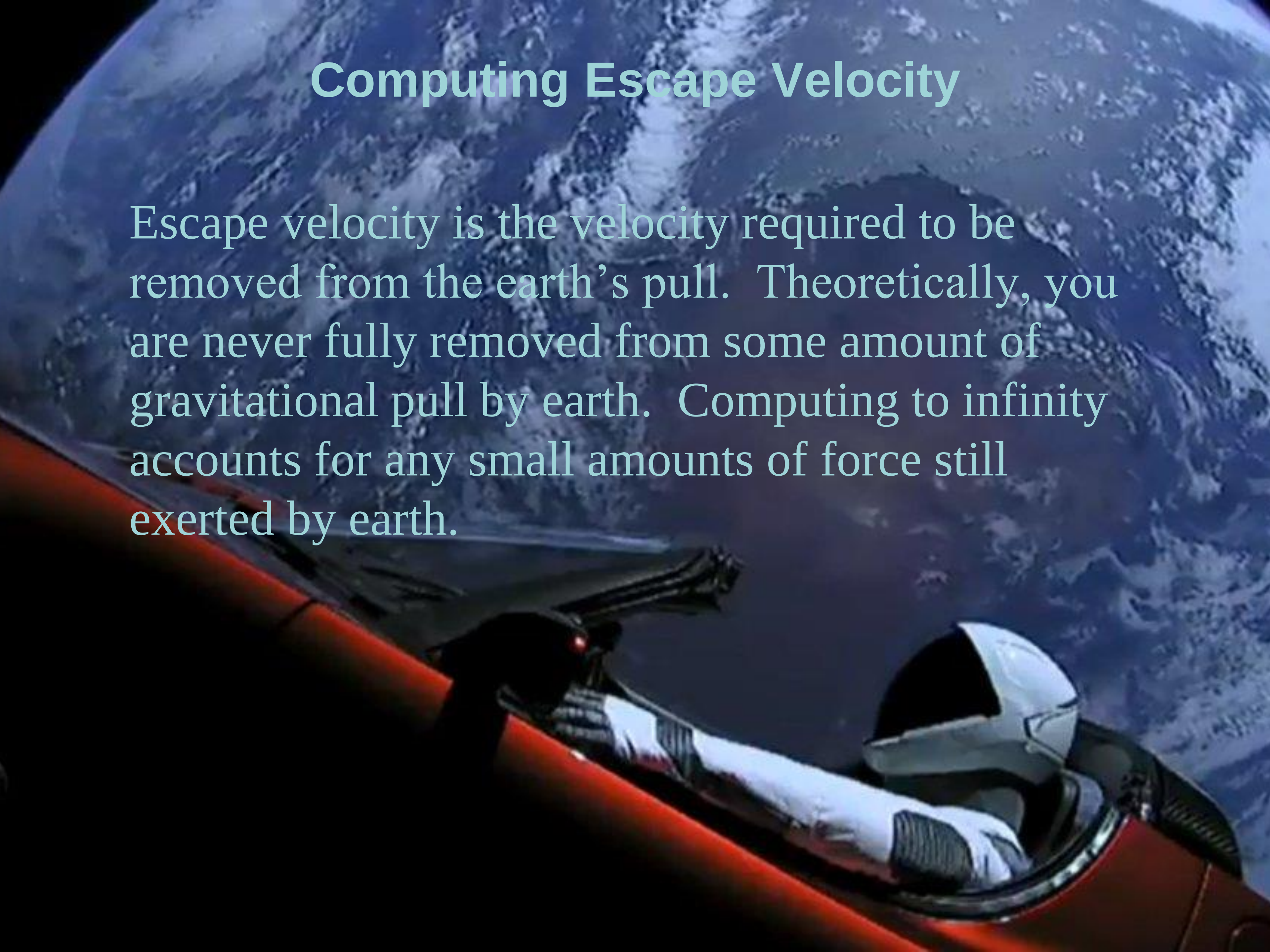
OK that's cute but what about now? You know, something that can really happen.

Elon Musk, owner of SpaceX, decided to put his Tesla Roadster into orbit of earth.

Computing Escape Velocity

How fast does a rocket need to be going to escape the earth’s gravitational pull?

This can be computed using improper integrals, which if you will recall depend on the use of infinity.



Computing Escape Velocity

Escape velocity is the velocity required to be removed from the earth’s pull. Theoretically, you are never fully removed from some amount of gravitational pull by earth. Computing to infinity accounts for any small amounts of force still exerted by earth.

Computing Escape Velocity

Two equations are needed from physics:

$$F = \frac{G m_1 m_2}{r^2}$$

and

$$E = \frac{1}{2} m v^2$$

Computing Escape Velocity

There are several values in the first equation that we can readily obtain

$G = 6.67\times10^{-11}$ N m²/kg²

m1 = 1360 kg (the mass of the car)

m2 = 6×10^{24} kg (mass of the earth)

r = 6.3×10^{6} m (radius of the earth)

yielding

$$F = \frac{Gm_1m_2}{r^2} = \frac{(6.67\times10^{-11})(1360)(6\times10^{24})}{\left(6.3\times10^{6}\right)^2}$$

Computing Escape Velocity

The work required to move the car out of the earth's gravitational pull can be computed by:

$$\int_{r_0}^{\infty} \frac{Gm_1m_2}{r^2}dr = (6.67\times10^{-11})(1360)(6\times10^{24})\int_{r_0}^{\infty}\left(\frac{1}{r^2}\right)dr$$

$$=\left(5.44\times10^{17}\right)\int_{r_0}^{\infty}\left(\frac{1}{r^2}\right)dr$$

$$=\left(5.44\times10^{17}\right)\int_{6.3\times10^{6}}^{\infty}\left(\frac{1}{r^2}\right)dr$$

Computing Escape Velocity

Evaluating the improper integral yields:

$$= \left(5.44 \times 10^{17}\right)\left(1.59 \times 10^{-7}\right) = 8.63 \times 10^{10} \text{ Joules}$$

Using the formula for energy and the mass of the car we can solve for *v*.

$$E = \frac{1}{2} m v^2 = \frac{1}{2}(1360 kg) v^2 = 8.63 \times 10^{10} \text{ Joules}$$

Computing Escape Velocity

Solving for *v*:

$$v^2 = 126,984,127 m^2/\sec^2$$

or

$$v = 11,268 m/\sec$$

This velocity can be converted to miles per hour yielding:

$$v \cong 25,206 mph$$

Thank You Very Much For Coming!

Dr. Gene Kramer
Krameree@ucmail.uc.edu

Dr. Chuck Emenaker
Charles.Emenaker@uc.edu

University of Cincinnati Blue Ash
9555 Plainfield Rd.
Cincinnati OH, 45236

https://drive.google.com/drive/folders/1c9WtKhQkRnDs-oZ5QKn4LzWKsmye88k7

UNIVERSITY OF Cincinnati