Teaching without Telling: Designing Tasks for Fractions

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NCTM Annual Meeting
San Diego, CA
April 4, 2019

MEASUREMENT APPROACH TO RATIONAL NUMBERS

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National Science Foundation
WHERE DISCOVERIES BEGIN

DRL-1020154

WHAT WE DO….

We study the way children think and learn about fractions.

We develop Hypothetical Learning Trajectories (HLT)

1. A learning goal
2. A likely path for learning
3. The instructional tasks that help children move along that path

GOALS FOR THE PRESENTATION

1. Explore using learning trajectories for fraction concepts to inform instruction
2. Provide you with research-based sample tasks that allow you to address the teaching and learning of fractions in ways that are consistent with CCSSM.

OUTLINE OF THE PRESENTATION

- Background
  - Central ideas of learning fractions
- Fraction task sequences
- Discussion and questions

BACKGROUND

Despite decades of work from mathematics education researchers, two important problems in mathematics education remain unresolved.

- Results from the National Assessment of Educational Progress 2015 showed that
  - 60% of American 4th graders and 67% of 8th graders scored below the proficient level in mathematics.
- There are conceptual areas in mathematics for which instruction is often unsuccessful, especially fractions, ratio and proportion.
  - Large percentages of the student population do not develop adequate understanding of these concepts (National Mathematics Advisory Panel, 2008).

BACKGROUND

We currently lack an instructional approach that can help weak mathematics students.

- Apparently, traditional show-and-tell instructional approach has not proved to be the answer.
- Instructional designers need new approaches for engendering mathematical concepts.

The underlying approach of my research work

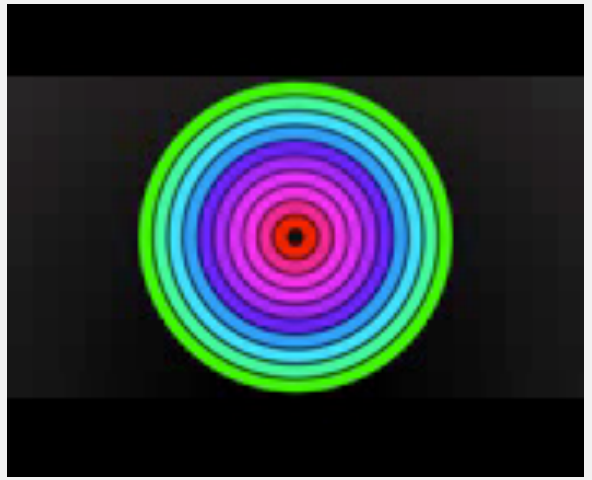
If a concept can be learned as a result of conceptual understanding, then it should be possible to engineer a sequence of tasks that promotes that understanding.

- This approach is different from *showing and telling* a mathematical concept and expecting students to grasp the concept.

MEANINGS OF FRACTIONS

Small group discussion:

- Take a few minutes to discuss possible meanings of fractions that you address in your teaching.



MEANINGS OF FRACTIONS

Meaning	Example
Part-whole	$\frac{2}{3}$: A whole is partitioned into three equal parts, and two of those parts are shaded.
Measure	$\frac{4}{5}$: Four iterations of a length ($\frac{1}{5}$) that when iterated five times would give a length of one unit
Division (Quotient)	The amount each person would get when $10 were shared amongst four people
Operator	A rule for making two-thirds of some other amount: $\frac{2}{3}$ of 30 square feet.
Ratio	Three dollars per four chocolate bars

MEANINGS OF FRACTIONS

What about $\frac{4}{3}$, how does this fraction fit into the meanings below?

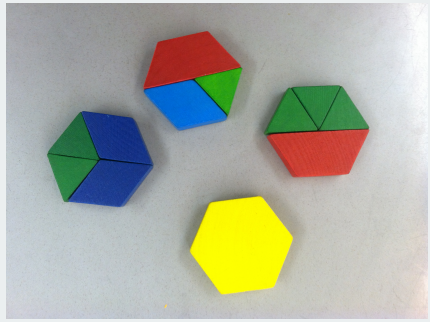
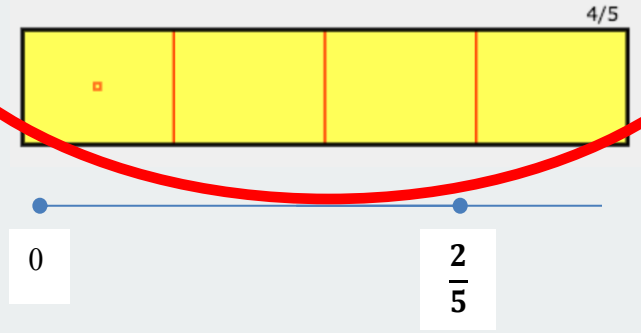
Meaning	Example	
Part-whole	$\frac{2}{3}$: A whole is partitioned into three equal parts, and two of those parts are shaded.	• Necessary to build fraction concepts in early grades. • With this meaning, students may stay in the whole number concepts without moving to the concept that fractions are numbers. • For advanced fraction concepts, this meaning is not sufficient.
Measure	$\frac{4}{5}$ Four iterations of a length ($\frac{1}{5}$) that when iterated five times would give a length of one unit	• Focuses on the length of a bar, rather than how many parts in the bar. **The tasks that will be shared during this presentation involve in fraction as a measure meaning.**

CCSS-M FRACTION PROGRESSIONS

Standard	Progression
Grade 2: Geometry—(2.G.A.3)	Partition regions halves, thirds, half of, a third of, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.
Grade 3: Number and Operations—Fractions 1 (3.NF.1)	"Understand a fraction $\frac{a}{b}$ as the quantity formed by a parts when a whole is partitioned into b equal parts."
Grade 4: Number and Operations—Fractions 4a (4.NF.4a)	"Understand a fraction $\frac{a}{b}$ as a multiple of $\frac{1}{b}$. For example, use a visual fraction model to represent $\frac{5}{4}$ as the product $5 \times (\frac{1}{4})$, recording the conclusion by the equation $\frac{5}{4} = 5 \times (\frac{1}{4})$."
Grade 5: Number and Operations—Fractions 5b (5.NF.5b)	"Interpret multiplication as scaling (resizing), by explaining why multiplying a number by a fraction greater than 1 results in a product greater than the given number."

FRACTION MODELS

The tasks that will be shared during this presentation involve in length model: length of a bar.

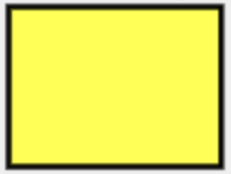
Area model	**Length model**	**Set model**
Drawing rectangles, pattern blocks 	Length model or number line 	Discrete objects 

MEASUREMENT APPROACH

Unit

Quantity

Partial unit



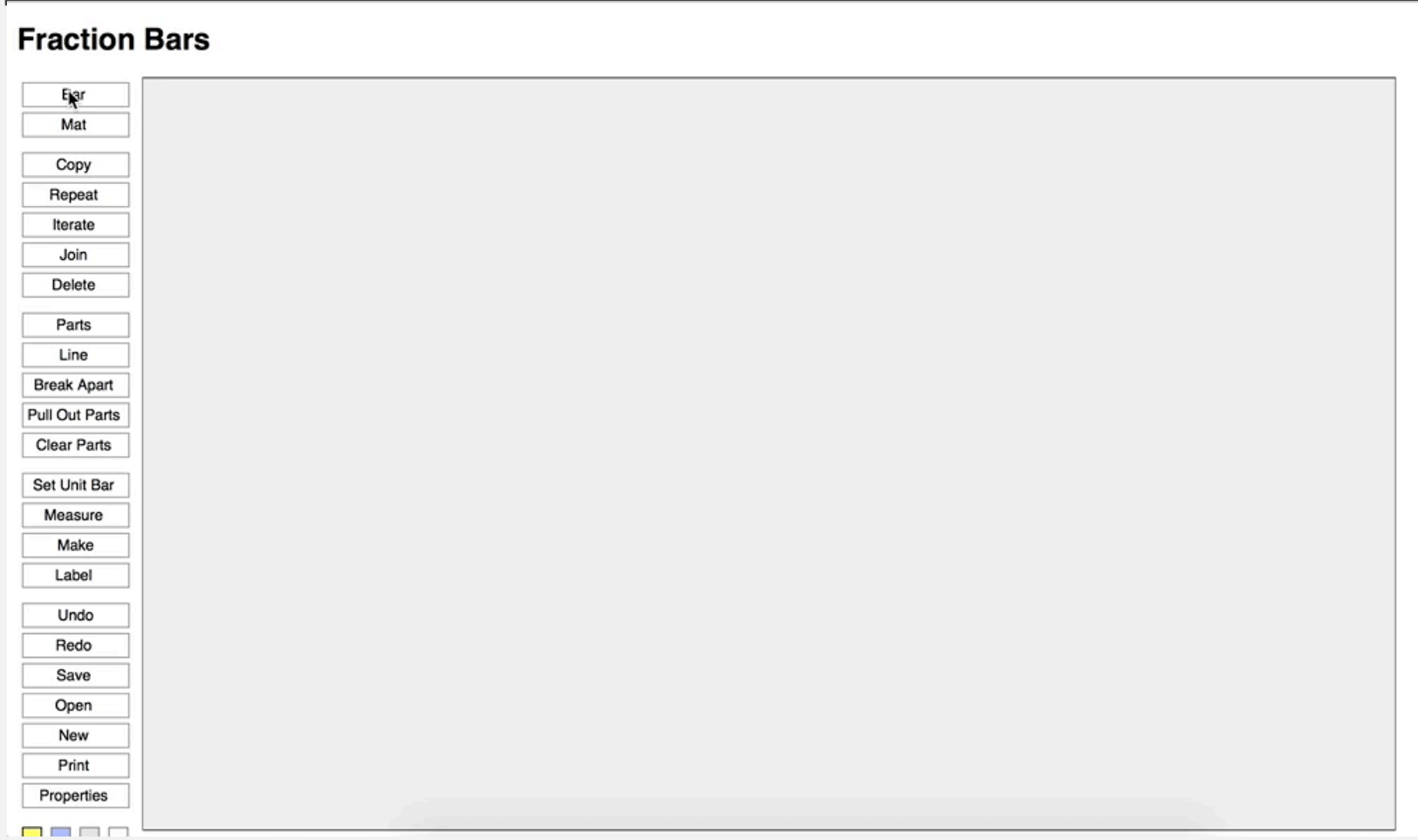
The number of times the partial unit measures the quantity

$\frac{4}{5}$ of a unit

The number of times the partial unit measures the original unit

Partitioning and iterating actions

FRACTION BARS PROGRAM



Kaput Center for Research and Innovation in STEM Education (2015). Fraction bars [software]. [Retrieved from http://www.kaputcenter.umassd.edu/products/ software/fractionbars/].

ANALYZING TASK SEQUENCES

1. Initial Concept of Fraction: Fraction-as-Measure
2. **Comparing Fractions**
3. **Improper Fraction**
4. Relation of Mixed Number and Improper Fraction
5. Fraction of a Set Fraction of a Whole Number with a Whole Number Result
6. **Recursive Partitioning:**
7. Unit Fraction of a Unit Fraction
8. **Equivalent Fractions**
9. Addition & Subtraction of Fractions
10. Meaning of Multiplication
11. Reinvention of Multiplication-of-Fractions Algorithm

Learning goal

1. Determine the prior concepts that are required for a learning goal
2. Develop an initial task that could lead to the learning goal

Tasks

3. Give possible student solutions
4. Analyze the structure of the task
5. Determine the next task that could lead to the learning goal

COMPARING FRACTIONS

CCSS.MATH.CONTENT.3.NF.A.3.D

Compare two fractions **with the same numerator** or **the same denominator** by reasoning about their **size**.

Recognize that comparisons are valid only when the two fractions refer to **the same whole**.

Record the results of comparisons with the symbols >, =, or <, and **justify** the conclusions, e.g., by using a visual fraction model.

COMPARING FRACTIONS

Comparing fractions that have the same denominator ($\frac{a}{b}$ and $\frac{c}{b}$)

Comparing unit fractions ($\frac{1}{a}$ and $\frac{1}{b}$)

Comparing fractions that have the same numerator ($\frac{a}{b}$ and $\frac{a}{c}$)

COMPARING FRACTIONS WITH COMMON DENOMINATORS

- **Determine the prior concepts required for this learning goal.**
- **What could be a task to start with?**

Comparing fractions that have the same denominator ($\frac{a}{b}$ and $\frac{c}{b}$)

The denominator specifies the partial unit. The numerator specifies the number of those partial units. If two fractions have the same denominator (i.e., same partial unit), the one with the larger numerator (i.e., more partial units) is the larger fraction.

COMPARING FRACTIONS WITH COMMON DENOMINATORS

Task 1. This bar is 1 unit long.

Create a bar that is $\frac{3}{8}$ of a unit long.

Create a bar that is $\frac{5}{8}$ of a unit long.

Which of the bars you created is longer?

Task 2. Which fraction is larger $\frac{4}{7}$ or $\frac{6}{7}$? Justify your answer. (Use the bars only if needed.)

Task 3. Which fraction is larger $\frac{17}{40}$ or $\frac{7}{40}$? Justify your answer.

- What are the possible student responses?
- Why do you think this particular task sequence was structured in that particular way?

COMPARING UNIT FRACTIONS

- **Determine the prior concepts required for this learning goal.**
- **What could be the task to start with?**

Comparing unit fractions ($\frac{1}{a}$ and $\frac{1}{b}$)

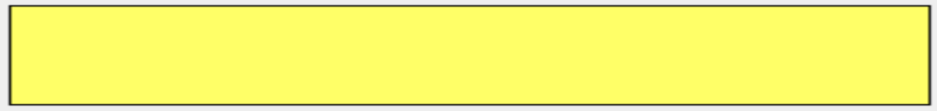
The denominator specifies how many times the unit fraction will measure the unit. A unit fraction that measures the unit more times must be the smaller. That is, to fit more equal parts in a unit, those parts have to be smaller.

COMPARING UNIT FRACTIONS

Step 1: Estimation strategies for comparing unit fractions

- **Task 1.** This bar is one unit long. Estimate where to cut the bar to create a piece that is $\frac{1}{5}$ of a unit long.

  - How can you check to see if your estimate is accurate?
    - Check your estimate. If it is too big or too small, create a new estimate and check it out. Continue until your estimate is accurate.
- **Task 2.** This bar is one unit long. Estimate where to cut the bar to create a piece that is $\frac{1}{7}$ of a unit long.
  - Explain how you decided how big to make your estimate.

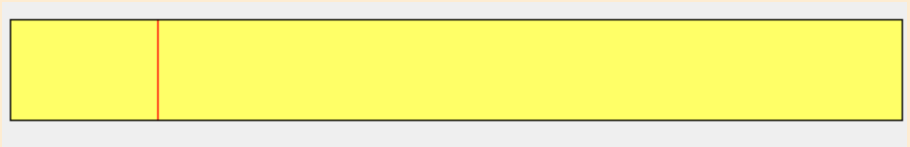


- What are the possible student responses?
- Why do you think this particular task was chosen?
- What could be the next task?

COMPARING UNIT FRACTIONS

Step 1: Estimation strategies for comparing unit fractions

Task 3. The bar below is one unit long. The line marked on the bar shows $\frac{1}{6}$ of a unit.



- Estimate where you would cut the bar to create a piece that is $\frac{1}{5}$ of a unit.
- Check to see if your estimate is accurate. If your estimate is not close, make a second estimate and see if it is closer.
- Is the part you estimated for $\frac{1}{5}$ larger or smaller than the $\frac{1}{6}$ of a unit marked on the unit bar? Explain why.

Task 4. Which is larger, $\frac{1}{13}$ or $\frac{1}{9}$? Explain.

- What are the possible student responses?
- Why do you think this particular task was structured in this way?

STUDENT REASONING: KYLIE

Kylie had memorized from her school experience that if b > d, $\frac{1}{b} < \frac{1}{d}$.

- It created a challenge to support her in having a conceptual basis for this knowledge.

After, several estimation tasks, she could reason about the size of unit fractions.

STUDENT REASONING: KYLIE

Kylie estimated $\frac{1}{5}$ of the unit and iterated it 5 times and compare it to the unit bar

After this task, the researcher ("R") had the following interaction with Kylie (the student, "S").

- R: I'm going to make...one eleventh.
- S: That is big ... That is like one fifth! That's not a good estimate.
- R: How do you know?
- S: Because, ... there wouldn't be room for eleven of those!... Because it would have to fit in the [unit].
- R: And what about if I want to make one...fiftieth?
- S: It would have to be small enough ... that it would be able to fit! But, not too small, because then there ... would be extra space.

DEVELOPING A NOTION OF IMPROPER FRACTIONS (FRACTIONS GREATER THAN 1)

Build fractions from unit fractions.

CCSS.MATH.CONTENT.4.NF.B.3

Understand a fraction $\frac{a}{b}$ **with *a* > 1 as a sum of fractions** $\frac{1}{b}$.

CCSS.MATH.CONTENT.4.NF.B.4

Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

CCSS.MATH.CONTENT.4.NF.B.4.A

Understand a fraction $\frac{a}{b}$ **as a multiple of** $\frac{1}{b}$. *For example, use a visual fraction model to represent* $\frac{5}{4}$ *as the product 5 ×* $(\frac{1}{4})$*, recording the conclusion by the equation* $\frac{5}{4} = 5 \times (\frac{1}{4})$.

CCSS.MATH.CONTENT.4.NF.B.4.B

Understand a multiple of $\frac{a}{b}$ as a multiple of 1/b, and use this understanding to multiply a fraction by a whole number. *For example, use a visual fraction model to express 3 ×* $(\frac{2}{5}$ *as 6 ×* $(\frac{1}{5})$*, recognizing this product as* $\frac{6}{5}$*. (In general, n ×* $(\frac{a}{b}) = \frac{n \times a}{b}$ *.)*

IMPROPER FRACTIONS: LEARNING GOALS

Quantities bigger than a unit can be expressed as $\frac{a}{b}$ of a unit where a > b > 0.

Consistent with the concept of (proper) fraction- as-measure, $\frac{1}{b}$ measures the unit b times, and $\frac{1}{b}$ measures the quantity a times.

An improper fraction $\frac{a}{b}$ of a unit can be created from the unit by partitioning the unit into b parts, pulling out one part, and iterating it a times (the same for proper fractions).

- **Determine the prior concepts required for this learning goal.**
- **What could be a task to start with?**

STUDENT REASONING

For a student who has a concept of fraction-as-measure, the foundational understanding is available.
However, students' concepts have been limited by the examples of fractions that they have encountered so far.

Barry:
We asked Barry to make a bar $\frac{6}{5}$ of a unit long. He created a bar that was $\frac{5}{6}$ of a unit long. After Barry realized that he had made $\frac{5}{6}$ and the researcher had asked for $\frac{6}{5}$, he asserted,
"That's impossible. You mean five sixths? The denominator, cannot be smaller than the... [numerator]."

Give the fraction name → ask them to make the fraction

Kylie:
The researcher partitioned a unit bar into 6 parts, pulled one out and iterated it 11 times.
Kylie identified the size of the new bar as "eleven sixths."

Ask to make the quantity → ask them to offer a name

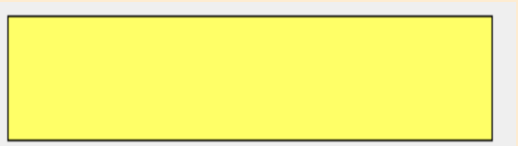
Many students would not offer an improper-fraction name, because they do not see it as an acceptable term based on their prior experience.

IMPROPER FRACTIONS

Step 1: Developing a Notion of Improper Fractions

- **Task 1.** The bar below is $\frac{1}{7}$ of a unit long. Create a new bar that is 5 times as long. How long is the new bar?

- **Task 2.** The bar below is $\frac{1}{5}$ of a unit long. Create a new bar that is 8 times as long. How long is the new bar?
- [After the student gives eight-fifths as an answer] Explain why "eight-fifths" is an appropriate way to describe the length of the bar you created.



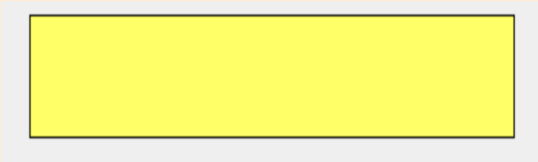
- What are the possible student responses?
- Why do you think this particular task was chosen and structured in that way?
- What could be the next task?

Next, we introduce the student the term "improper fraction/fraction greater than one" and how to write the fraction.

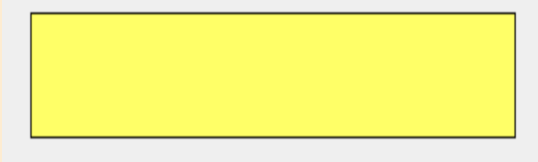
IMPROPER FRACTIONS

Step 1: Developing a Notion of Improper Fractions

- **Task 3.** The bar below is $\frac{1}{6}$ of a unit long. Create a new bar that is 7 times as long. How long is the new bar? (Write your answer as a fraction - proper or improper)



- **Task 4.** The bar below is $\frac{5}{6}$ of a unit long. Create a new bar that is twice as long. How long is the new bar? (Write your answer as a fraction - proper or improper)

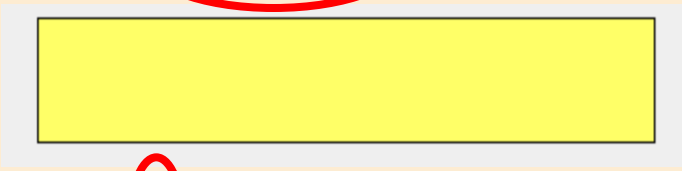


- What are the possible student responses?
- Why do you think this particular task was chosen and structured in that way?
- What could be the next task?

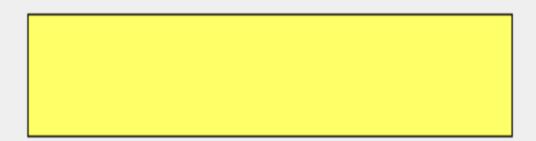
IMPROPER FRACTIONS

Step 2: Creating Improper Fractions

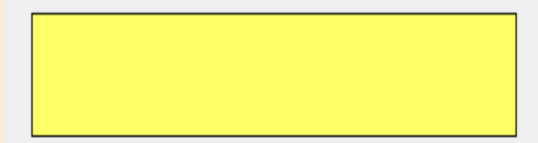
Task 5. The bar below is 1 unit long. Create a bar that is $\frac{7}{5}$ of a unit long.



Task 6. The bar below is $\frac{5}{6}$ of a unit long. Create a bar that is $\frac{15}{6}$ of a unit long.



Task 8. The bar below is $\frac{4}{3}$ of a unit long. Create a bar that is $\frac{20}{3}$ of a unit long.



- What are the possible student responses?
- Why do you think this particular task was chosen and structured in that way?

For this set of tasks, make comparisons with the Step 1 tasks (developing a notion of improper fractions)

RECURSIVE PARTITIONING LEARNING GOALS

Students will abstract that $\frac{1}{b}$ of $\frac{1}{a} = \frac{1}{ab}$.

That is, because $\frac{1}{a}$ iterates a times to the unit, and $\frac{1}{b}$ of $\frac{1}{a}$ iterates b times to $\frac{1}{a}$, $\frac{1}{b}$ of $\frac{1}{a}$ iterates ab times to the unit. Thus, $\frac{1}{b}$ of $\frac{1}{a} = \frac{1}{ab}$.

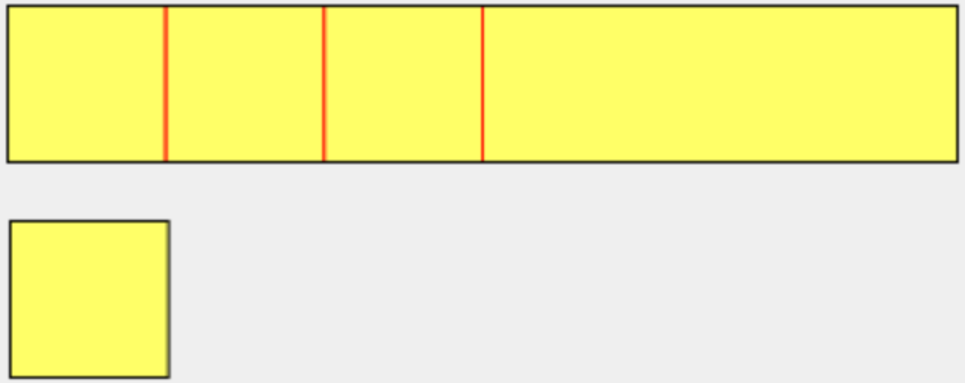
Further, they will develop the reverse concept needed to make $\frac{1}{a}$ from $\frac{1}{ab}$.

- **Determine the prior concepts required for this learning goal.**
- **What could be a task to start with?**

STUDENT REASONING

Step 1: Recursively partitioning the unit and determining the measure of subpart.

Task 1. The bar below is 1-unit long. Partition the unit bar into halves. Next, partition one of the parts into thirds. Pull out one subpart. What fraction of the unit is the subpart?



. R: Alright. Now. ... How big... is one of these little parts?
. S: A sixth! No, oh, well, of what?
R: Ah, That's a very good question – of a unit.
. S: A sixth.
R: How do you know?
S: Because three plus three is six.

What could be the next task to help Kylie reach the goal of $\frac{1}{b}$ of $\frac{1}{a}$ is $\frac{1}{ab}$?

The next task type that proved important was to give the part without the whole unit.

RECURSIVE PARTITIONING

Step 1: Recursively partitioning the unit and determining the measure of subpart.

Task 1. The bar below is 1-unit long. Partition the unit bar into halves. Next, partition one of the parts into thirds. Pull out one subpart. What fraction of the unit is the subpart?

Step 2: Taking a part of a part and determining its measure.

Task 2. The bar below is $\frac{1}{4}$ of a unit long. Pull out $\frac{1}{3}$ of the bar. What fraction of the unit is this new bar?

Task 3. [Use bar representation only if needed.] Imagine you have a bar that is $\frac{1}{3}$ of a unit long and you pulled out a subpart that was $\frac{1}{5}$ of the bar. What fraction of the unit did you pull out?

Task 4. What is $\frac{1}{4}$ of $\frac{1}{6}$?

What do you notice about this task sequence?

RECURSIVE PARTITIONING

Step 2: Obtaining $\frac{1}{mn}$ from $\frac{1}{n}$ with Fraction Bars

Task 5. If you had a bar that was $\frac{1}{5}$ of a unit long and wanted to make a bar that was $\frac{1}{10}$ of a unit long,

Would taking $\frac{1}{3}$ of the bar work?

Would taking $\frac{1}{5}$ of the bar work?

What fraction of the bar would give you a bar that is $\frac{1}{10}$ of a unit?

Task 6. If you had a bar that was $\frac{1}{6}$ of a unit long and wanted to make a bar that was $\frac{1}{18}$ of a unit, what would you need to do with the bar?

Task 7. $\frac{1}{40}$ is what fraction of $\frac{1}{8}$?

What do you notice about this task? How is this different than the previous tasks?

DISCUSSION

- How can we modify routine fraction tasks to capture some of the key characteristics of learning trajectory-based fraction tasks?
  - Accessible to students at varying levels of ability
  - Reveal student thinking with respect to partitioning and iterating
  - Consistent with CCSSM's treatment of fractions.
- How do you think you would modify our tasks for your classroom?

THANK YOU!

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