

Link to Google slides: <https://tinyurl.com/StructureTablesNCTM19>

Connecting Function Representations by Looking for and Making Use of Structure in In-Out Tables

NCTM 2019

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William, 6th grade



Link to Google slides: <https://tinyurl.com/StructureTablesNCTM19>

William, 5th grade



“We’re learning about these things called ‘**expressions.**’

I don’t like them because **they don’t seem finished.**”

$$4 + 5 + 4$$

William, 5th grade



“We’re learning about these things called ‘expressions.’

I don’t like them because they don’t seem finished.”

$$4 + 5 + 4 = 13$$

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Expressions have meaning beyond computation

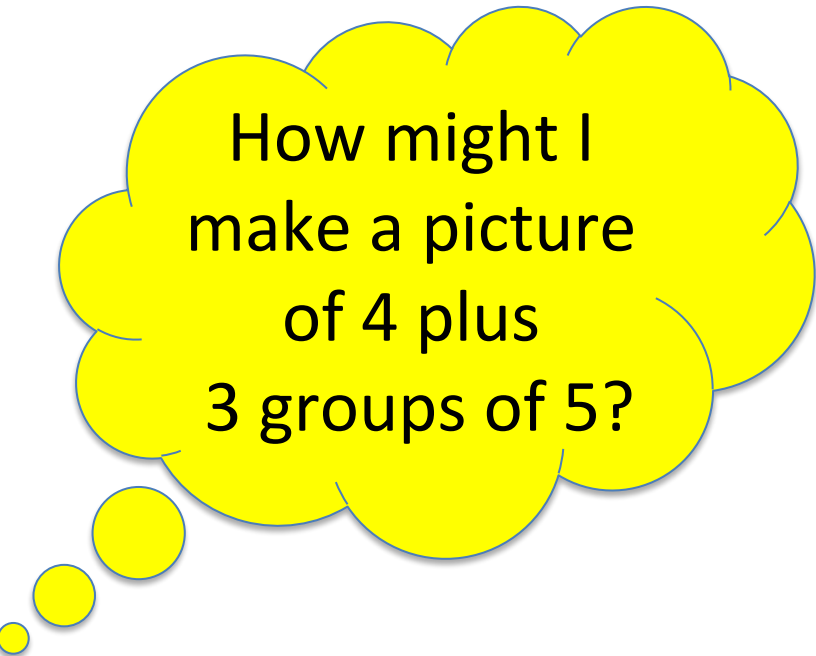
$$4 + (3 \cdot 5)$$

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Expressions have meaning beyond computation

$$4 + (3 \cdot 5)$$

 *“groups of”*

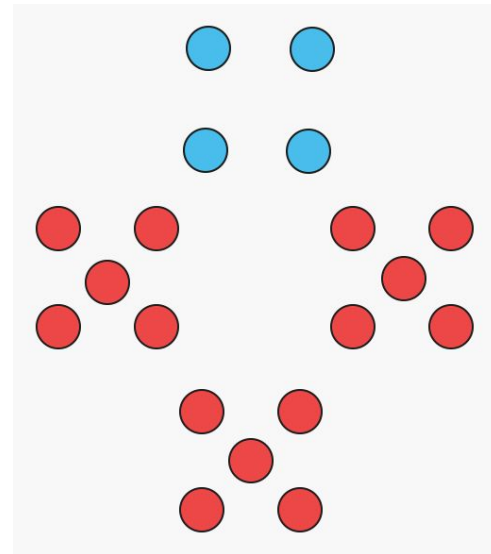


How might I
make a picture
of 4 plus
3 groups of 5?

Expressions have meaning beyond computation

$$4 + (3 \cdot 5)$$

How might I
make a picture
of 4 plus
3 groups of 5?

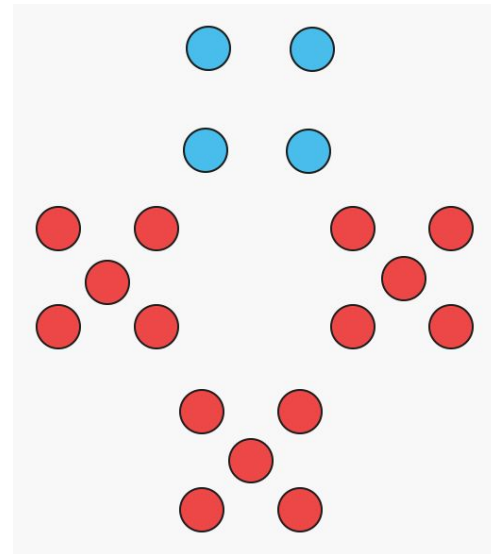


Expressions have meaning beyond computation

$$4 + (3 \cdot 5)$$

“Writing for
Structure”

The mathematical
symbols accurately
reflect the *structure*
of the dot picture

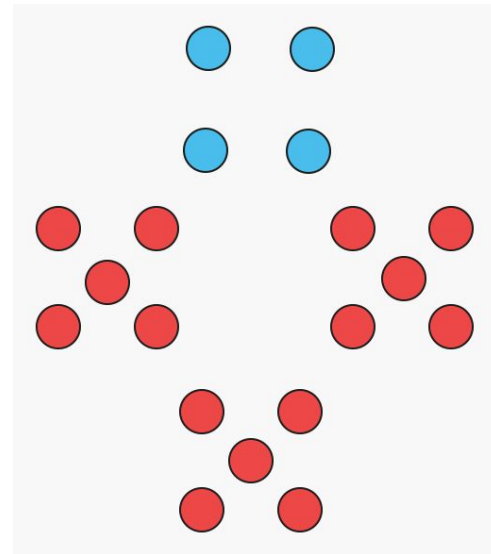


Expressions have meaning beyond computation

$$4 + (3 \cdot 5)$$

“Writing for
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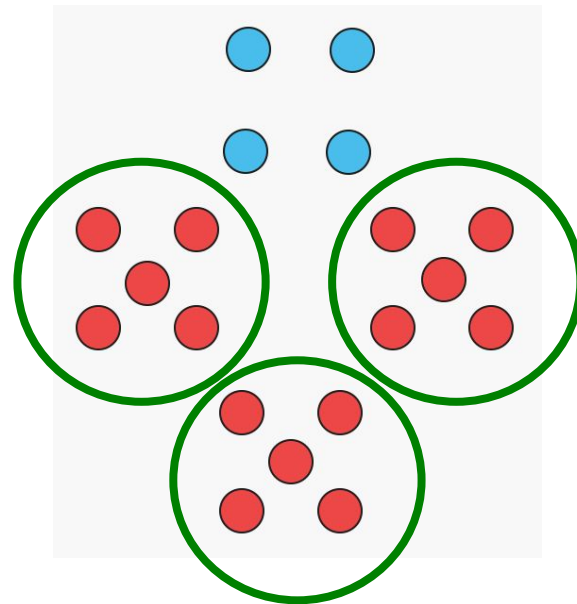


Expressions have meaning beyond computation

$$4 + (3 \cdot 5)$$

“Writing for
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The mathematical
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reflect the *structure*
of the dot picture



Expressions have meaning beyond computation

$$4 + (3 \cdot 5)$$



“Writing for
Structure”

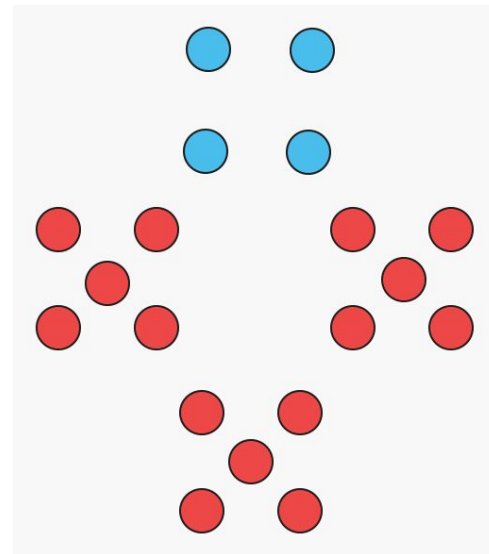
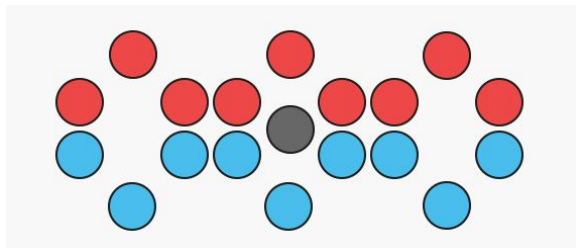
The mathematical
symbols accurately
reflect the *structure*
of the dot picture

Make a picture
that has 19 dots
but that you would
NOT describe as
 $4 + (3 \times 5)$



What expression would you write to describe
your picture?

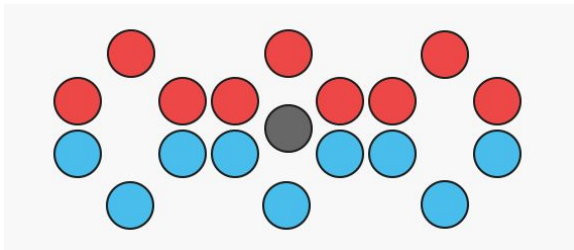
Writing for Structure

 $4 + (3 \cdot 5)$ 



Writing for Structure

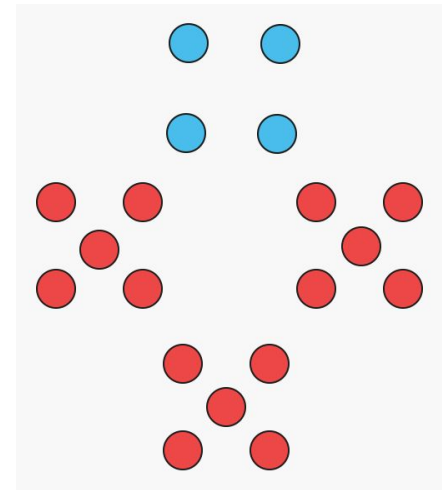
 $4 + (3 \cdot 5)$ 



$$2 \cdot 9 + 1$$

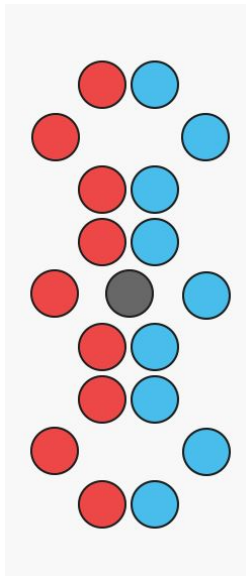
$$3 \cdot 6 + 1$$

$$3 \cdot (3 + 3) + 1$$

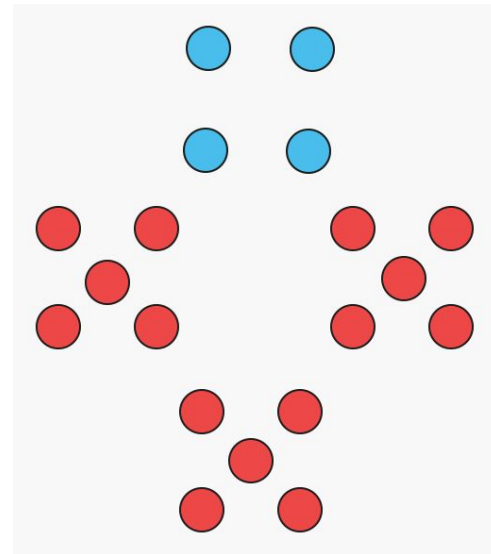


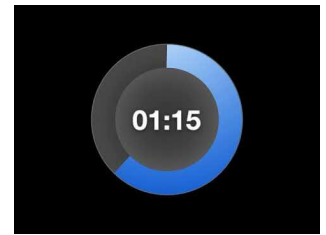
Writing for Structure

?????



$$4 + (3 \times 5)$$





Symbol Sense

Without calculating, what are some things you can say are true about these number patterns?

$$(1 \cdot 2) + 7$$

$$(2 \cdot 3) + 7$$

$$(3 \cdot 4) + 7$$

$$(4 \cdot 5) + 7$$

...

$$5\left(\frac{1}{2}\right)^1$$

$$5\left(\frac{1}{2}\right)^2$$

$$5\left(\frac{1}{2}\right)^3$$

$$5\left(\frac{1}{2}\right)^4$$

...

$$1 + 2 \cdot 3$$

$$2 + 2 \cdot 4$$

$$3 + 2 \cdot 5$$

$$4 + 2 \cdot 6$$

...

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Symbol Sense → **Arithmetic** understanding

I can *fluently* (accurately, flexibly, efficiently) **calculate the value** of expressions.

$$(1 \cdot 2) + 7 = 9$$

$$(2 \cdot 3) + 7 = 13$$

$$(3 \cdot 4) + 7 = 19$$

$$(4 \cdot 5) + 7 = 27$$

...

$$5\left(\frac{1}{2}\right)^1 = 2.5$$

$$5\left(\frac{1}{2}\right)^2 = 1.25$$

$$5\left(\frac{1}{2}\right)^3 = 0.625$$

$$5\left(\frac{1}{2}\right)^4 = 0.3125$$

...

$$1 + 2 \cdot 3 = 7$$

$$2 + 2 \cdot 4 = 10$$

$$3 + 2 \cdot 5 = 13$$

$$4 + 2 \cdot 6 = 16$$

...

Symbol Sense → **Algebraic** understanding

I can investigate and make sense of mathematical expressions *without calculating*.

The structure of the expression has meaning for me,
and I can use that structure to solve problems, make conjectures, and investigate patterns and relationships.

$$(1 \cdot 2) + 7$$

$$(2 \cdot 3) + 7$$

$$(3 \cdot 4) + 7$$

$$(4 \cdot 5) + 7$$

...

$$5\left(\frac{1}{2}\right)^1$$

$$5\left(\frac{1}{2}\right)^2$$

$$5\left(\frac{1}{2}\right)^3$$

$$5\left(\frac{1}{2}\right)^4$$

...

$$1 + 2 \cdot 3$$

$$2 + 2 \cdot 4$$

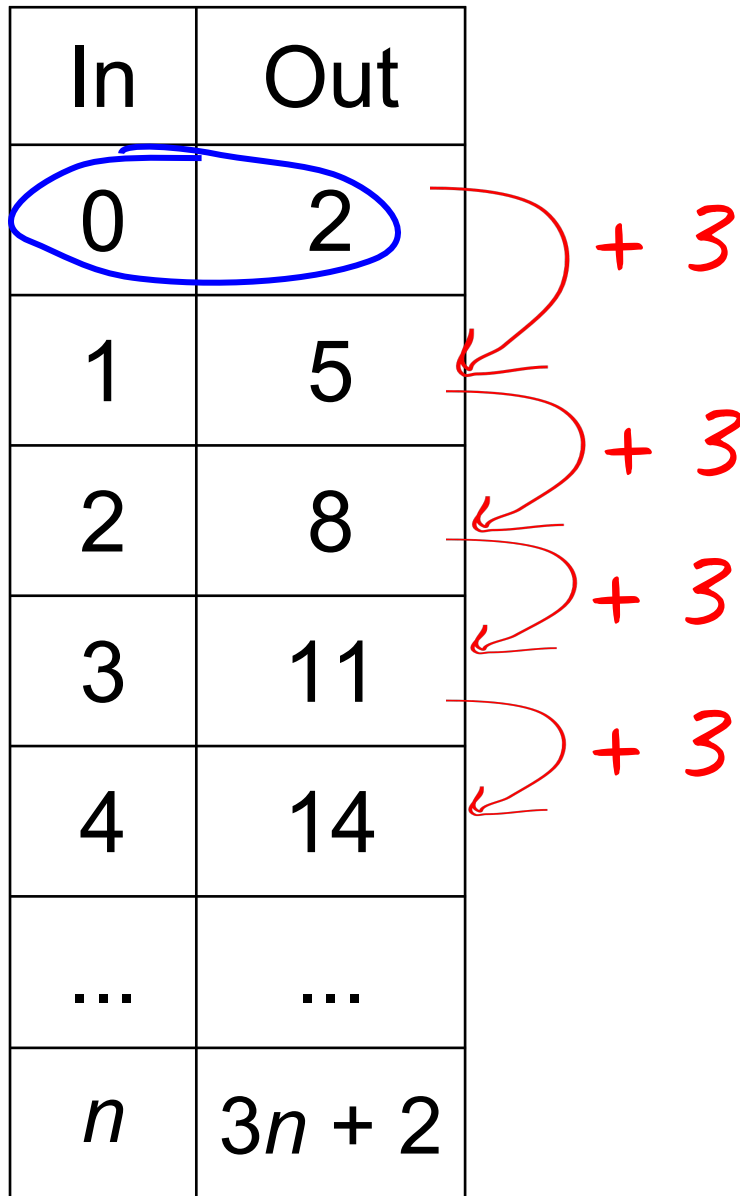
$$3 + 2 \cdot 5$$

$$4 + 2 \cdot 6$$

...

Let's Talk Tables

In	Out
0	2
1	5
2	8
3	11
4	14
...	...
n	$3n + 2$



The diagram illustrates the relationship between input and output values in a table. A blue oval highlights the first row (0, 2). Red arrows and "+ 3" labels show the constant difference between consecutive output values: 2 to 5 (+3), 5 to 8 (+3), 8 to 11 (+3), and 11 to 14 (+3).

Let's Talk Tables

In	Out
0	2
1	5
2	8
3	11
4	14
...	...
n	$3n + 2$

Diagram illustrating a linear relationship. The input values (In) are 0, 1, 2, 3, 4, ..., n . The corresponding output values (Out) are 2, 5, 8, 11, 14, ..., $3n + 2$. Red arrows indicate a constant difference of +3 between consecutive output values. A blue circle highlights the output value 2 for input 0, and another blue circle highlights the output value $3n + 2$ for input n . A blue arrow points from the first output to the last, and a red arrow points from the last output back to the first.

In	Out
0	2
1	7
2	18
3	35
4	58
...	...

Diagram illustrating a quadratic relationship. The input values (In) are 0, 1, 2, 3, 4, ..., n . The corresponding output values (Out) are 2, 7, 18, 35, 58, ..., $3n^2 + 2$. Red arrows indicate the first differences between consecutive output values: +5, +11, +17, +23, ... Purple arrows indicate the second differences, which are constant at +6. A blue circle highlights the output value 2 for input 0. A blue arrow points from the first output to the last. To the right of the table, the quadratic formula is written: $3X^2 + _ X + _ 2$. A purple arrow points from the output value 2 to the constant term in the formula, and another purple arrow points from the output value 7 to the coefficient of X in the formula.

Symbol Sense → **Arithmetic** understanding

I can *fluently* (accurately, flexibly, efficiently)
calculate the value of expressions.

In	Out
0	2
1	5
2	8
3	11
4	14
...	...
n	$3n + 2$

Diagram illustrating a sequence of numbers (Out) corresponding to input values (In). The sequence starts at 2 for input 0 and increases by 3 for each subsequent input value (1, 2, 3, 4, ...). The general formula for the output is $3n + 2$.

In	Out
0	2
1	7
2	18
3	35
4	58
...	...

Diagram illustrating a sequence of numbers (Out) corresponding to input values (In). The sequence starts at 2 for input 0 and increases by 6 for each subsequent input value (1, 2, 3, 4, ...). The general formula for the output is $3X^2 + X + 2$.

Symbol Sense → **Algebraic** understanding

I can investigate and make sense of mathematical expressions *without calculating*.

The structure of the expression has meaning for me,

and I can use that structure

to solve problems,
make conjectures,
and investigate
patterns and relationships.

In	Out
0	2
1	5 + 3
2	8
3	11
4	14
...	...

+3

Symbol Sense → **Algebraic** understanding

I can investigate and make sense of mathematical expressions *without calculating*.

The structure of the expression has meaning for me,

and I can use that structure

to solve problems,
make conjectures,
and investigate
patterns and relationships.

In	Out
0	2
1	$2 + 3$
2	$2 + 2 \cdot 3$
3	11
4	14
...	...

$+3$

$+3$

Symbol Sense → **Algebraic** understanding

I can investigate and make sense of mathematical expressions *without calculating*.

The structure of the expression has meaning for me,

and I can use that structure

to solve problems,
make conjectures,
and investigate
patterns and relationships.

In	Out
0	2
1	$2 + 3$
2	$2 + 2 \cdot 3$
3	$2 + 1 \cdot 3$
4	$2 + 4 \cdot 3$
...	...
n	$2 + n \cdot 3$

$\curvearrowright +3$
 $\curvearrowright +3$
 $\curvearrowright +3$
 $\curvearrowright +3$

Symbol Sense → **Algebraic** understanding

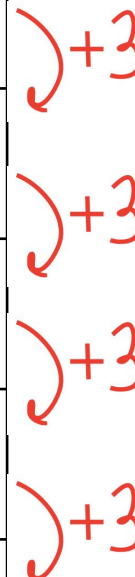
I can investigate and make sense of mathematical expressions *without calculating*.

The structure of the expression has meaning for me,

and I can use that structure

to solve problems,
make conjectures,
and investigate
patterns and relationships.

In	Out
0	$2 + 0 \cdot 3$
1	$2 + 1 \cdot 3$
2	$2 + 2 \cdot 3$
3	$2 + 3 \cdot 3$
4	$2 + 4 \cdot 3$
...	...
n	$2 + n \cdot 3$



Arithmetic Thinking

*How can I calculate
each step?*

In	Out
0	2
1	5
2	8
3	11
4	14
...	...
n	$3n + 2$

Algebraic Thinking

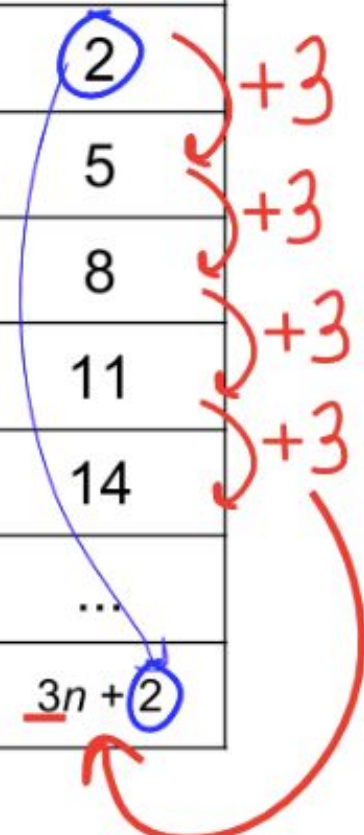
*What do all these numbers have in
common?*

In	Out
0	$2 + 0 \cdot 3$
1	$2 + 1 \cdot 3$
2	$2 + 2 \cdot 3$
3	$2 + 3 \cdot 3$
4	$2 + 4 \cdot 3$
...	...
n	$2 + n \cdot 3$

There are many ways to *write for structure*

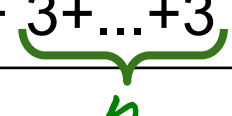
Arithmetic Thinking

In	Out
0	2
1	5
2	8
3	11
4	14
...	...
n	$3n + 2$



Algebraic Thinking

In	Out
0	2
1	2+3
2	2+3+3
3	2+3+3+3
4	2+3+3+3+3
...	...
n	$2 + \underbrace{3+\dots+3}_n$



In	Out
0	$2 + 0 \cdot 3$
1	$2 + 1 \cdot 3$
2	$2 + 2 \cdot 3$
3	$2 + 3 \cdot 3$
4	$2 + 4 \cdot 3$
...	...
n	$2 + n \cdot 3$

Writing for structure: Choices teachers make

In	Out
0	2
1	2+ <u>3</u>
2	2+3+ <u>3</u>
3	2+3+3+ <u>3</u>
4	2+3+3+3+ <u>3</u>
...	...
n	2 + <u>3+3+...+3</u>

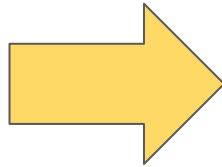
constant

+3 is the change from one step to the next

n times

Rewrite the table in a way that reveals the structure

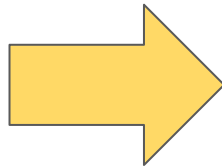
IN	OUT
0	7
1	12
2	17
3	22
4	27
...	...
n	



IN	OUT
0	
1	
2	
3	
4	
...	...
n	

Rewrite the table in a way that reveals the structure

IN	OUT
0	7
1	12
2	17
3	22
4	27
...	...
n	



IN	OUT
0	$7 + 0 \cdot 5$
1	$7 + 1 \cdot 5$
2	$7 + 2 \cdot 5$
3	$7 + 3 \cdot 5$
4	$7 + 4 \cdot 5$
...	...
n	$7 + n \cdot 5$

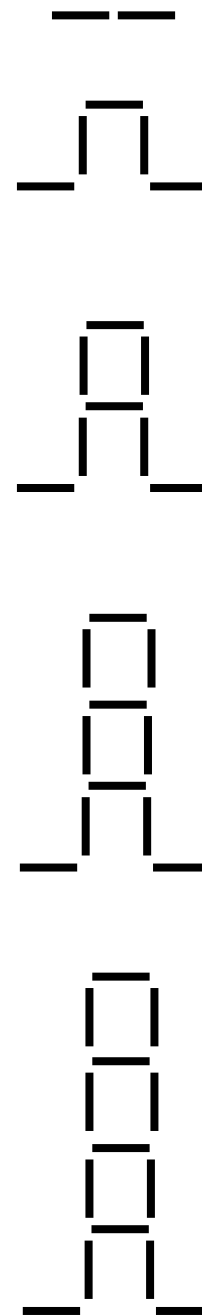
Choose one table and make a visual pattern that represents the structure



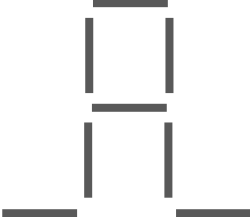
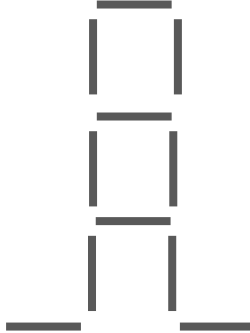
IN	OUT
0	2
1	2+3
2	2+3+3
3	2+3+3+3
4	2+3+3+3+3
...	...
n	$2 + \underbrace{3 + \dots + 3}_n$

IN	OUT
0	$7 + 0 \cdot 5$
1	$7 + 1 \cdot 5$
2	$7 + 2 \cdot 5$
3	$7 + 3 \cdot 5$
4	$7 + 4 \cdot 5$
...	...
n	$7 + n \cdot 5$

Terry's Toothpick Pattern

In	Out
0	2
1	2+3
2	2+3+3
3	2+3+3+3
4	2+3+3+3+3
...	...
n	$2 + \underbrace{3 + \dots + 3}_n$



Step	# toothpicks	Visual pattern
0	2	
1	2 + 3	
2	2 + 3 + 3	
3	2 + 3 + 3 + 3	

What can *writing for structure* in tables help you see?

x	y
0	$0+2$
1	$1+2$
2	$2+2$
3	$3+2$
4	$4+2$

x	y
0	2
1	$1+2$
2	$1+1+2$
3	$1+1+1+2$
4	$1+1+1+1+2$

x	y
0	$3 \cdot 2^0$
1	$3 \cdot 2^1$
2	$3 \cdot 2^2$
3	$3 \cdot 2^3$
4	$3 \cdot 2^4$

x	y
0	$0 \cdot 3 + 2$
1	$1 \cdot 3 + 2$
2	$2 \cdot 3 + 2$
3	$3 \cdot 3 + 2$
4	$4 \cdot 3 + 2$

x	y
0	2
1	$3+2$
2	$3+3+2$
3	$3+3+3+2$
4	$3+3+3+3+2$

x	y
0	3
1	$3 \cdot 2$
2	$3 \cdot 2 \cdot 2$
3	$3 \cdot 2 \cdot 2 \cdot 2$
4	$3 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

What can *writing for structure* in tables help you see?

x	y
0	$0+2$
1	$1+2$
2	$2+2$
3	$3+2$
4	$4+2$

x	y
0	2
1	$1+2$
2	$1+1+2$
3	$1+1+1+2$
4	$1+1+1+1+2$

x	y
0	$0 \cdot 3 + 2$
1	$1 \cdot 3 + 2$
2	$2 \cdot 3 + 2$
3	$3 \cdot 3 + 2$
4	$4 \cdot 3 + 2$

x	y
0	2
1	$3+2$
2	$3+3+2$
3	$3+3+3+2$
4	$3+3+3+3+2$

What can *writing for structure* in tables help you see?

x	y
0	$0+2$
1	$1+2$
2	$2+2$
3	$3+2$
4	$4+2$



x	y
0	2
1	$1+2$
2	$1+1+2$
3	$1+1+1+2$
4	$1+1+1+1+2$



x	y
0	$0 \cdot 3 + 2$
1	$1 \cdot 3 + 2$
2	$2 \cdot 3 + 2$
3	$3 \cdot 3 + 2$
4	$4 \cdot 3 + 2$



x	y
0	2
1	$3+2$
2	$3+3+2$
3	$3+3+3+2$
4	$3+3+3+3+2$

$\left. \begin{array}{l} \text{ } \\ \text{ } \\ \text{ } \\ \text{ } \end{array} \right\} +3$

What can *writing for structure* in tables help you see?

$$y = mx + b$$

These *look*
so different!

$$y = a \cdot r^x$$

x	y
0	$0 \cdot 3 + 2$
1	$1 \cdot 3 + 2$
2	$2 \cdot 3 + 2$
3	$3 \cdot 3 + 2$
4	$4 \cdot 3 + 2$

x	y
0	2
1	$3 + 2$
2	$3 + 3 + 2$
3	$3 + 3 + 3 + 2$
4	$3 + 3 + 3 + 3 + 2$

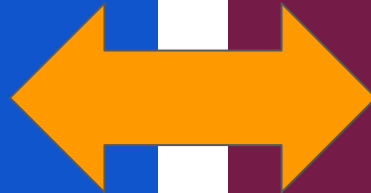
x	y
0	$3 \cdot 2^0$
1	$3 \cdot 2^1$
2	$3 \cdot 2^2$
3	$3 \cdot 2^3$
4	$3 \cdot 2^4$

x	y
0	3
1	$3 \cdot 2$
2	$3 \cdot 2 \cdot 2$
3	$3 \cdot 2 \cdot 2 \cdot 2$
4	$3 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

The commonalities
between arithmetic
and geometric
sequences is
revealed when
writing for structure

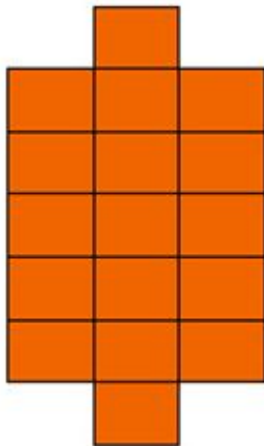
A “structure-focused” introduction to In-Out tables...

Dot Talks

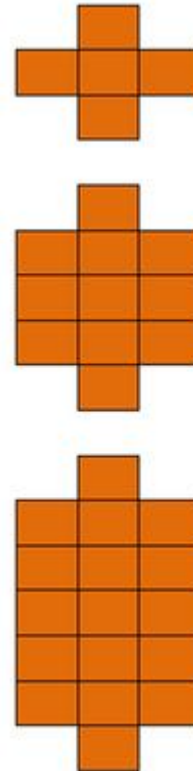


Pattern Talks

How did you determine the number of squares?



<http://www.visualpatterns.org>



Dot Talks

Pattern Talks

1. What can you know without calculating?

$$(0 \cdot 3) + 1$$

$$(1 \cdot 3) + 1$$

$$(2 \cdot 3) + 1$$

$$(3 \cdot 3) + 1$$

...

Dot Talks

Pattern Talks

1. What can you know without calculating?

$$(0 \cdot 3) + 1$$

$$(1 \cdot 3) + 1$$

$$(2 \cdot 3) + 1$$

$$(3 \cdot 3) + 1$$

...

2. Make an In-Out table and find the rule

In	Out
0	$(0 \cdot 3) + 1$
1	$(1 \cdot 3) + 1$
2	$(2 \cdot 3) + 1$
3	$(3 \cdot 3) + 1$
...	...
n	$(n \cdot 3) + 1$

Dot Talks

Pattern Talks

1. What can you know without calculating?

$$(0 \cdot 3) + 1$$

$$(1 \cdot 3) + 1$$

$$(2 \cdot 3) + 1$$

$$(3 \cdot 3) + 1$$

...

2. Make an In-Out table and find the rule

In	Out
0	$(0 \cdot 3) + 1$
1	$(1 \cdot 3) + 1$
2	$(2 \cdot 3) + 1$
3	$(3 \cdot 3) + 1$
...	...
n	$(n \cdot 3) + 1$

3. Can you rewrite to show the structure?

In	Out
0	7
1	12
2	17
3	22
...	...
n	

Rewrite the table in a way that reveals the structure

x	y
0	2
1	5
2	8
3	11
4	14



x	y
0	2
1	$2+3$
2	$2+3+3$
3	$2+3+3+3$
4	$2+3+3+3+3$

x	y
0	7
1	12
2	17
3	22
4	27



x	y
0	
1	
2	
3	
4	

Connecting Representations--A Structured View

Table

In	Out
0	2
1	2+3
2	2+3+3
3	2+3+3+3
4	2+3+3+3+3
...	...
n	$2+3+3+\dots+3$ <i>n times</i>

Graph

Try it!

Use the structure to
graph these points
without calculating.

Try to *show the
structure*
on the graph.

Connecting Representations--A Structured View

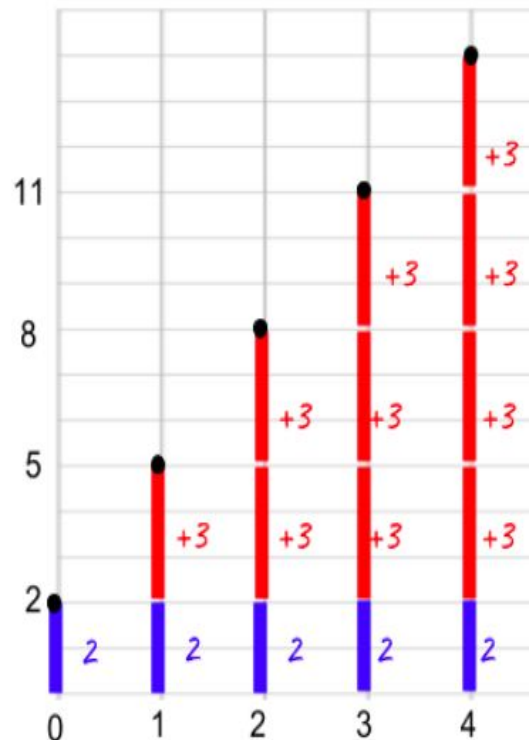
Table

In	Out
0	2
1	2+3
2	2+3+3
3	2+3+3+3
4	2+3+3+3+3
...	...
n	$2+3+3+\dots+3$

constant

n times

Graph



Connecting Representations--A Structured View

Table

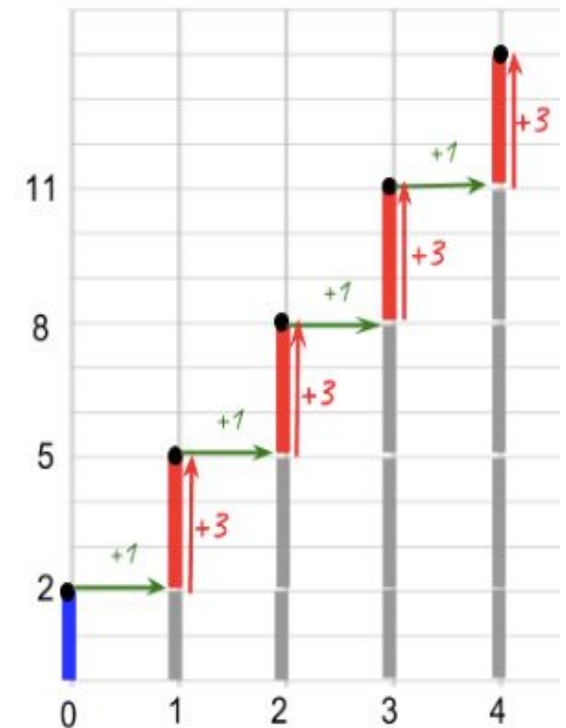
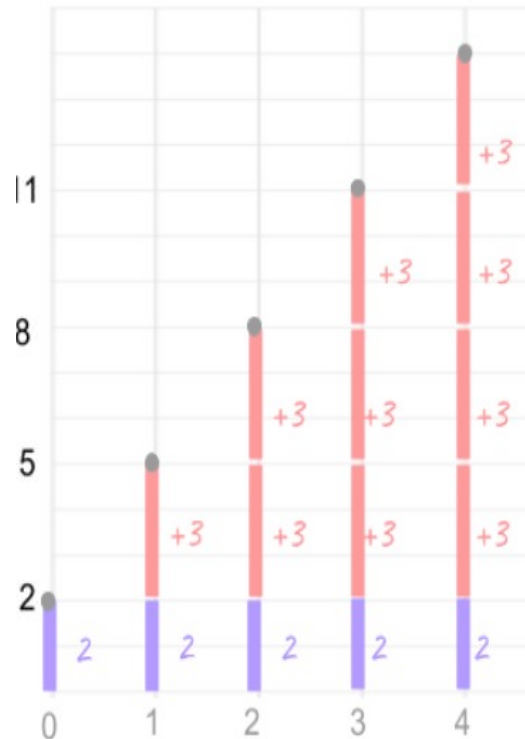
In	Out
0	2
1	2+3
2	2+3+3
3	2+3+3+3
4	2+3+3+3+3
...	...
n	2+3+3+...+3

constant (pointing to the 2 in the Out column)

+1 (curved arrows between rows 0-1, 1-2, 2-3, 3-4)

n times (under the ellipsis in the Out column for n)

Graph



Connecting Representations--A Structured View

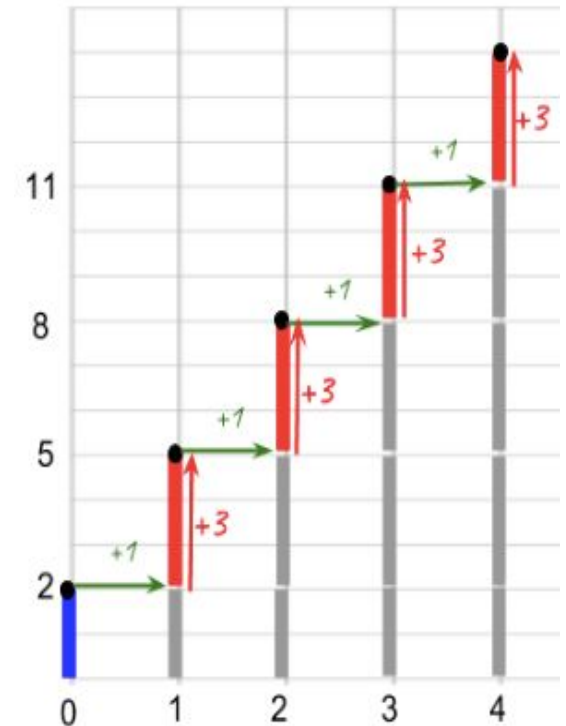
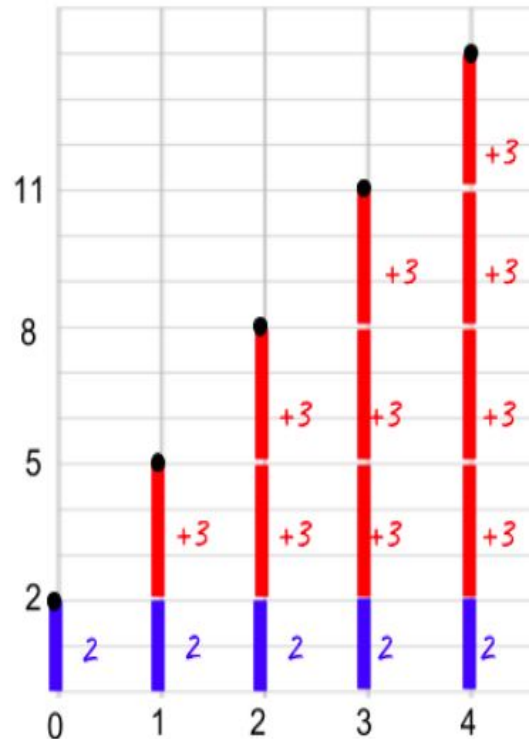
Table

In	Out
0	2
1	2+3
2	2+3+3
3	2+3+3+3
4	2+3+3+3+3
...	...
n	$2+3+3+\dots+3$

constant (pointing to the 2 in the first row)

n times (under the repeated 3s in the last row)

Graph



Connecting Representations--A structured view

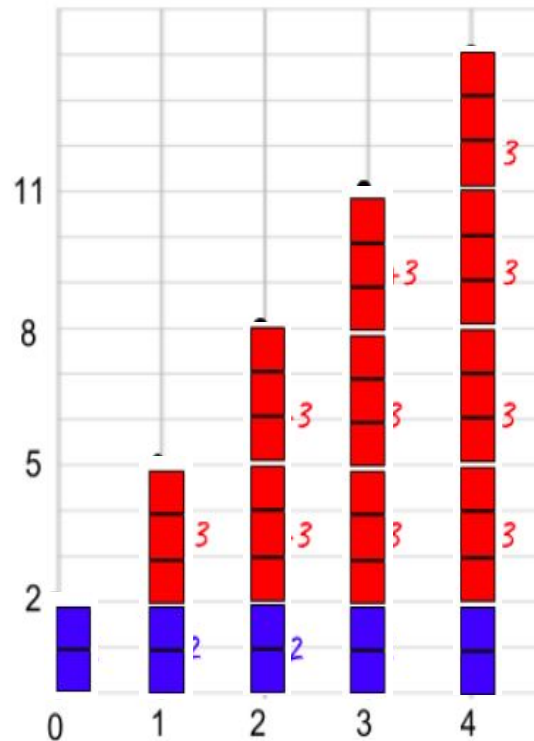
Table

In	Out
0	2
1	2+3
2	2+3+3
3	2+3+3+3
4	2+3+3+3+3
...	...
n	$2+3+3+\dots+3$

constant

n times

Graph



Visual Pattern

Connecting Representations--A structured view

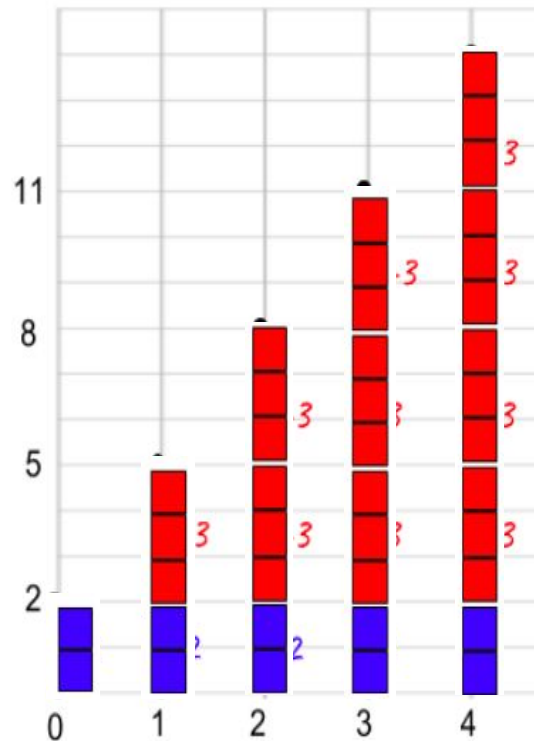
Table

In	Out
0	2
1	2+3
2	2+3+3
3	2+3+3+3
4	2+3+3+3+3
...	...
n	$2+3+3+\dots+3$

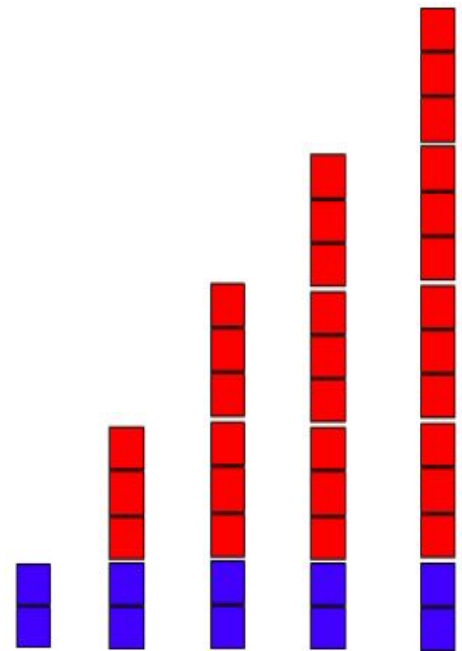
constant

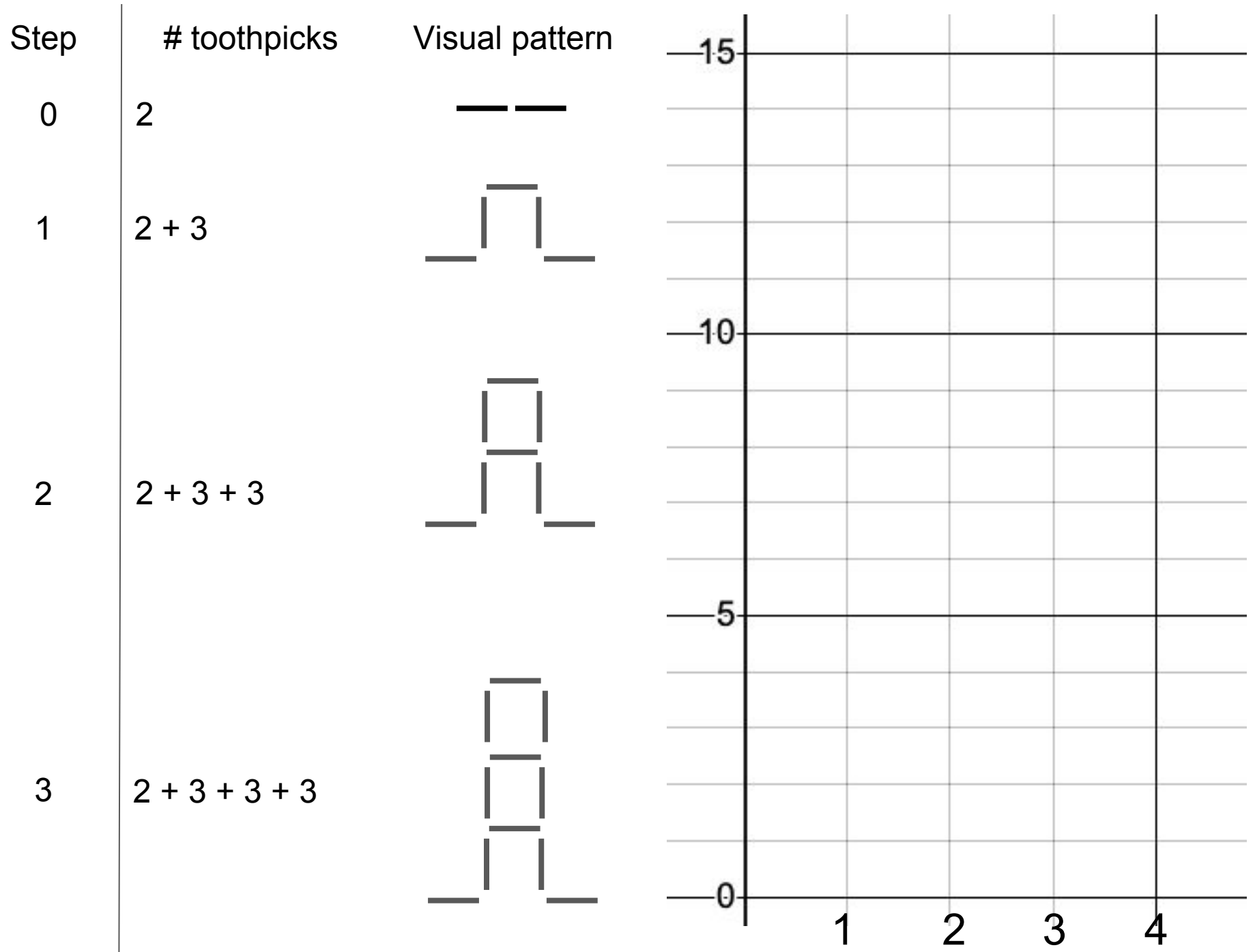
n times

Graph



Visual Pattern





Connecting Representations--A structured view

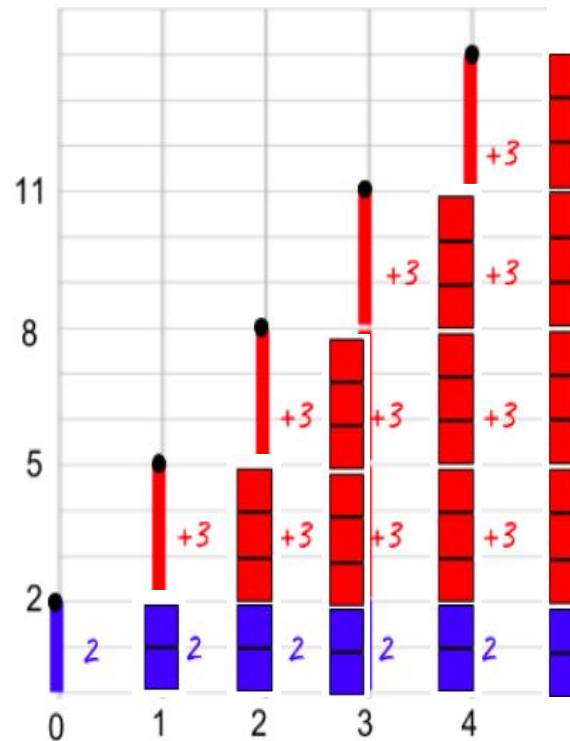
Table

In	Out
0	2
1	2+3
2	2+3+3
3	2+3+3+3
4	2+3+3+3+3
...	...
n	$2+3+3+\dots+3$

constant (pointing to the blue '2' in the first row)

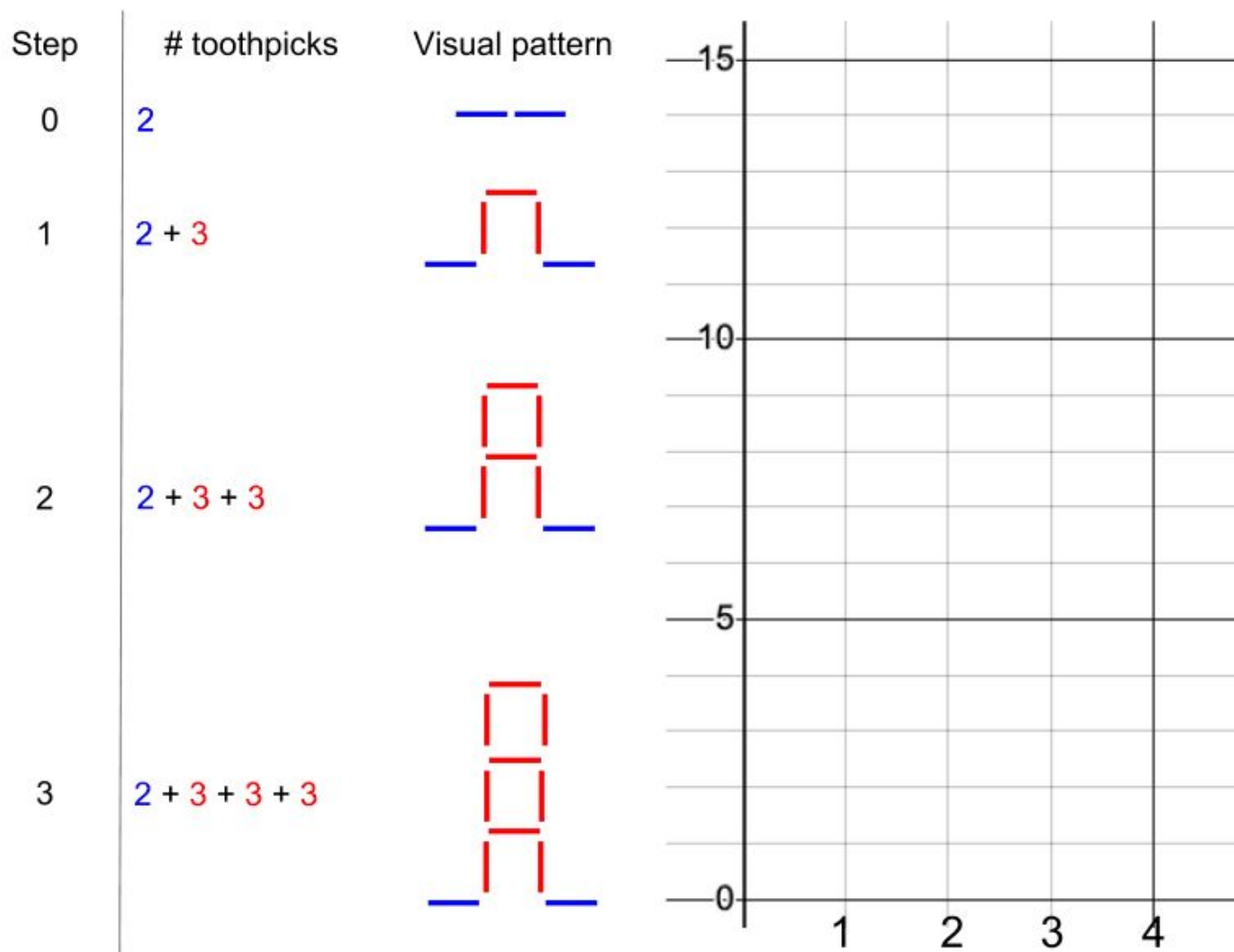
n times (under the ellipsis in the last row)

Graph



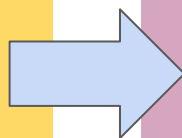
Visual Pattern

Connecting Representations using Structure

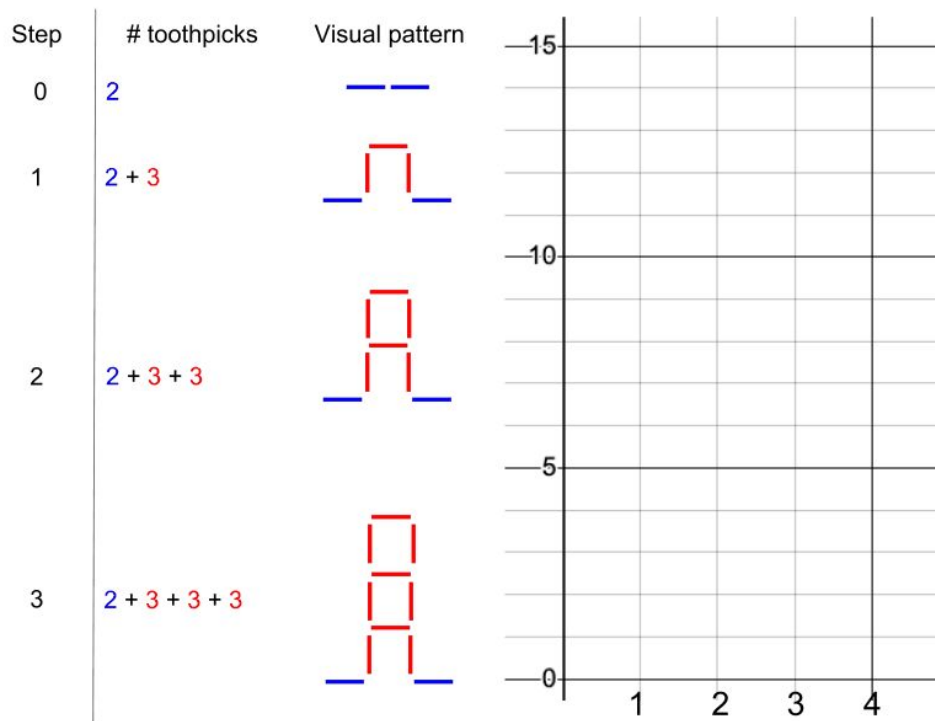


Stop Motion Animation

Google Slides:
1 per move
(26 for this animation)



[Tall Tweets](#) to turn
Google Slides into gif



Same or Different?

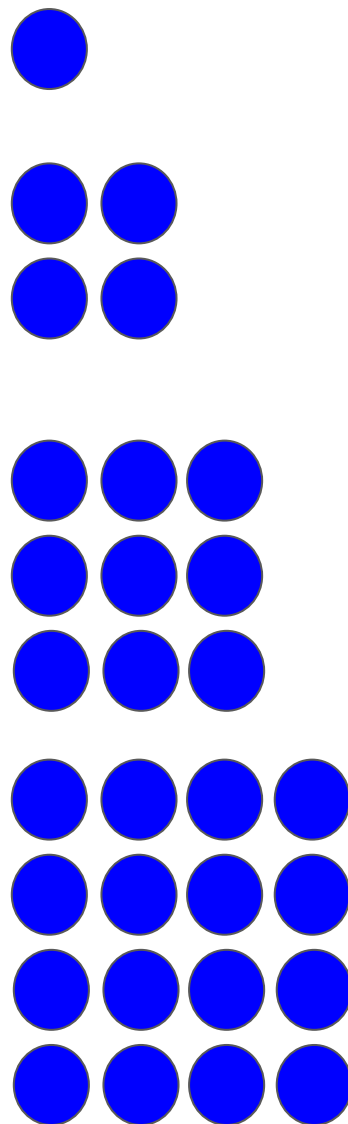
In	Out
0	2
1	$2 + 3$
2	$2 + 3 + 3$
3	$2 + 3 + 3 + 3$
4	$2 + 3 + 3 + 3 + 3$
...	...
n	

In	Out
0	2
1	$2 \cdot 3$
2	$2 \cdot 3 \cdot 3$
3	$2 \cdot 3 \cdot 3 \cdot 3$
4	$2 \cdot 3 \cdot 3 \cdot 3 \cdot 3$
...	...
n	

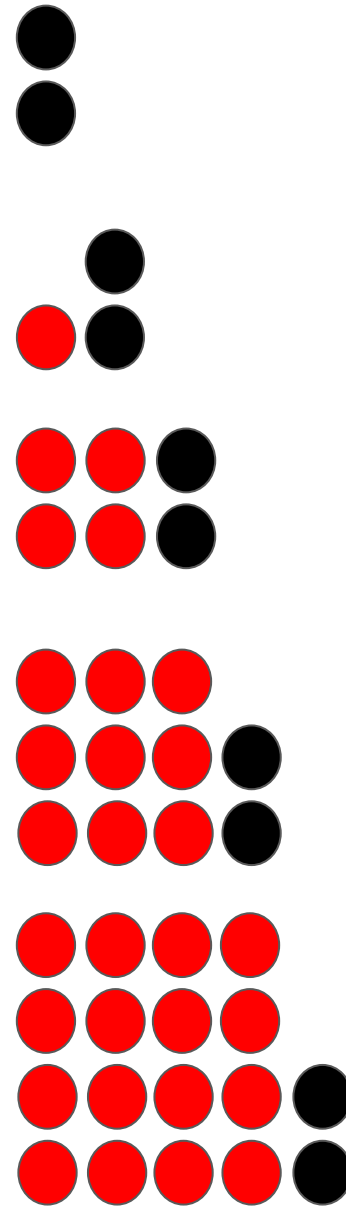
In	Out
0	$3 + 0^2$
1	$3 + 1^2$
2	$3 + 2^2$
3	$3 + 3^2$
4	$3 + 4^2$
...	...
n	$3 + n^2$

In	Out
0	$3 + 2 \cdot 0^2$
1	$3 + 2 \cdot 1^2$
2	$3 + 2 \cdot 2^2$
3	$3 + 2 \cdot 3^2$
4	$3 + 2 \cdot 4^2$
...	...
n	$3 + 2 \cdot n^2$

In	Out
0	0^2
1	1^2
2	2^2
3	3^2
4	4^2
...	...
n	n^2



In	Out
0	$0^2 + 2$
1	$1^2 + 2$
2	$2^2 + 2$
3	$3^2 + 2$
4	$4^2 + 2$
...	...
n	$n^2 + 2$



In	Out
0	0^2
1	1^2
2	2^2
3	3^2
4	4^2
...	...
n	n^2

In	Out
0	$2 \cdot 0^2 + 4 \cdot 0 + 3$
1	$2 \cdot 1^2 + 4 \cdot 1 + 3$
2	$2 \cdot 2^2 + 4 \cdot 2 + 3$
3	$2 \cdot 3^2 + 4 \cdot 3 + 3$
4	$2 \cdot 4^2 + 4 \cdot 4 + 3$
...	...
n	$2 \cdot n^2 + 4 \cdot n + 3$

Link to Google slides: <https://tinyurl.com/StructureTablesNCTM19>

Thank you for coming!

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Do you love multiplication and integers?!

3-4 pm TODAY
Sapphire KL
Minus times minus is
plus: The reason for
this we will discuss

