Teaching Problem Solving to ELLs Using Cognitively Guided Math Instruction and Graphic Organizers

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Learning Objectives

- Participants will be able to identify cognitive strategies for improving the mathematical problem-solving abilities of English language students.
  - Learn to teach students how to utilize cognitive strategies and self-regulated strategy development for engaging in problem solving.
    - Identify struggles encountered by English language students.
    - Learn how to use vocabulary/concept diagram for teaching key concepts.
    - Learn how implement cognitively guided math instruction with graphic organizers and learning strategies.
  - Learn to assist students in monitoring and reflecting on the problem-solving process.
    - Provide a list of prompts for self-monitoring.
    - Model how to monitor and reflect on the problem solving process.
    - Use student thinking about a problem to develop students' ability to monitor and reflect.
  - Learn to teach students how to use visual representations.
    - Select visual representations that are appropriate for students and the problems they are solving.
    - Use think-alouds and discussions to teach students how to represent problems visually.
  - Expose students to multiple problem-solving strategies.
    - Provide instruction for multiple strategies.
    - Opportunities to compare multiple strategies.
    - Ask students to generate and share multiple strategies.
  - Articulate math concepts
    - Describe relevant math concepts
    - Reveal a step by step process for solving problems.

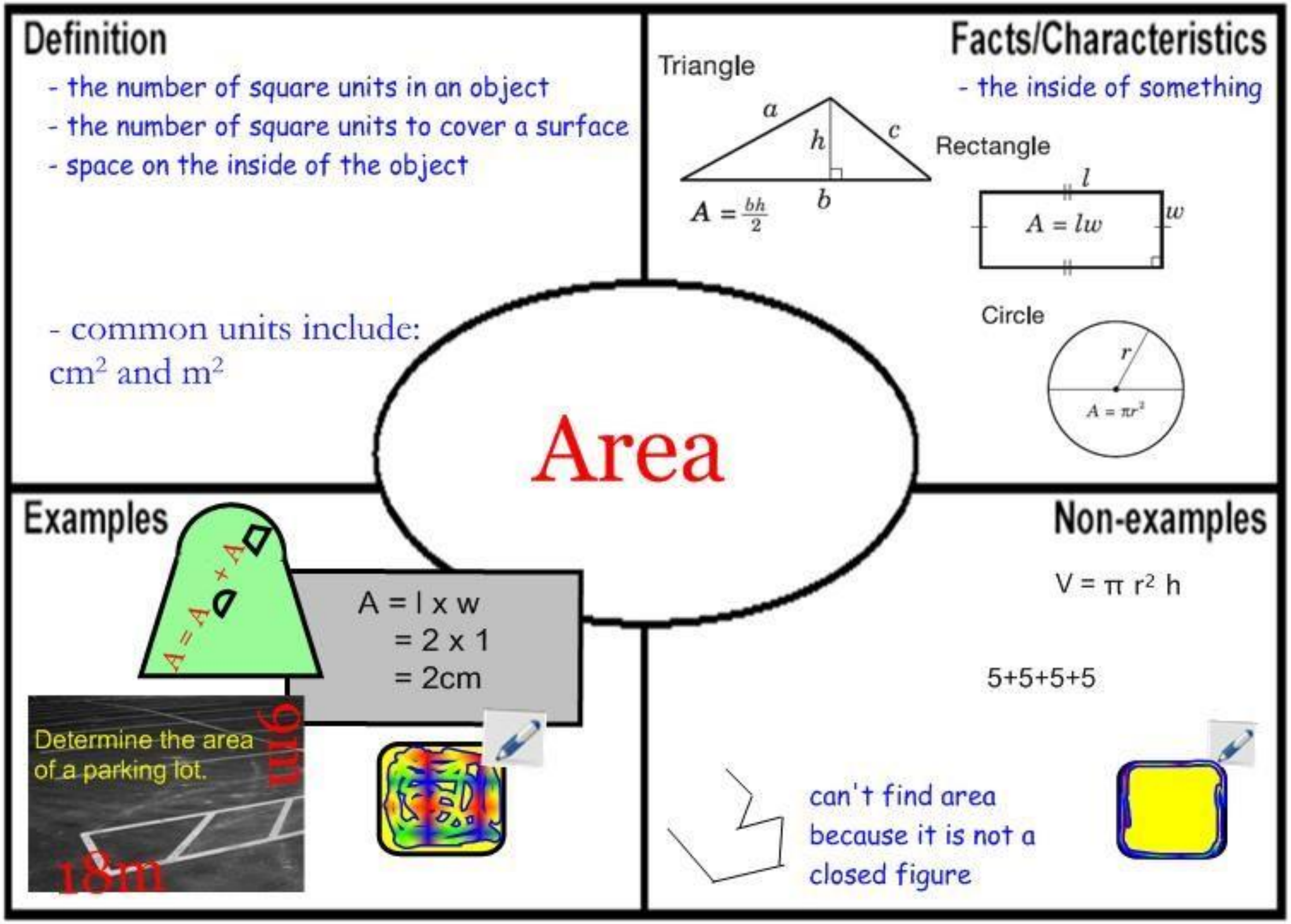
Who Do ELLs Struggle With Mathematics

- Vocabulary Issues
  - ELLs are challenged with learning the language in which the math problems and directions are written, as well as learning new math skills and a new language at the same time.
  - Some math terms have a mathematic meaning AND an everyday meaning which can confuse students.
    - Examples: Mean, Operation, Power, Root, Table
  - Teachers should provide a wide variety of opportunities for ELLs to speak, write, read, and listen. *(Students should SWRL everyday!)*
  - *https://steinhardt.nyu.edu/metrocenter/resources/glossaries* Content Glossaries in Different Languages

Who Do ELLs Struggle With Mathematics

- Symbols and Systems
  - Some symbols serve different functions in different cultures.
    - Commas and periods vary from culture to culture
    - Metric System v. the US Customary System
    - Fahrenheit v. Celsius
- Individualized Learning Systems
  - Mismatched text levels
  - Background knowledge in math
  - Self-efficacy
  - Speaking in class

Graphic Organizer for Math Vocabulary



Algebra I Assessment Problem

- A bulldozer can move 25 tons of dirt per hour, while an end loader can only move 18 tons per hour. If a bulldozer costs \$75 per hour to rent and the end loader costs \$50 per hour, how many bulldozers and end loaders were being used if 204 tons of dirt were moved in 1 hour and the cost was \$600?

Word Problem Challenges for ELLs

AREAS OF CONCERN

- Context of word problem
- Lessening focus on extraneous language, emphasizing essential math language
- Lack of support for visualization of context
- Overall, not very culturally responsive

WAYS TO MODIFY

- Change context
- Reiteration of information the student must provide in answer
- Highlight/emphasize the key math vocab within the problem
- Provide picture to help students visualize context
- Frontload necessary math vocabulary

Assessment Problem Revised

A taxi can transport 25 people **per** *hour*, while an Uber can only transport 18 people **per** *hour*. If a taxi *costs* \$75 **per** hour to operate and an Uber *costs* \$50 **per** hour to operate, how many taxis and Ubers were being used if 204 people were transported in 1 hour and the total cost was \$600?

t = taxis
u = Ubers

Number of taxis = ______________

Number of Ubers = ______________

Assessment Problem Revised & Simplified

A taxi can transport 25 people **per** *hour*, while an Uber can only transport 18 people **per** *hour*. If a taxi *costs* $75 **per** hour to operate and an Uber *costs* $50 **per** hour to operate. If 204 people were transported in 1 hour and the total cost was $600, which system best describes the situation?

t = taxis
u = Ubers

(A) $25t + 18u = 204$
$75t + 50u = 600$

(B) $75t + 50u = 204$
$25t + 18u = 600$

Conceptual Understanding

- The comprehension of mathematical concepts, operations, and relations.
- Refers to an integrated and functional grasp of mathematical ideas.
    - Students understand why a mathematical idea is important and the context in which is it useful.
    - They organize their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know.

Self-Regulation Strategy Development

- A *strategy* is a series of steps that we use to more quickly or effectively perform a specific task.
- *Self-regulation* refers to self-generated thoughts, feelings, and behaviors that are oriented to attaining goals.
- *Self-regulated learning strategies* are actions and processes such as organizing and transforming information and rehearsing or using memory aids.

Self-Regulated Strategy Development (SRSD)

- The SRSD model involves six stages:
  - Develop Background Knowledge
  - Discuss It
  - Model It
  - Memorize It
  - Support It
  - Establish Independent Practice

Concept Mastery Routine

- Concept Diagram
  - Used to define, summarize, and explain a major concept and where it fits within a larger body of knowledge.
  - Designed for teaching a complex, abstract concept in a manner to enhance the student's understanding and his ability to apply the concept.
  - Steps
    a) Identify a target concept
    b) Place that concept in a larger framework
    c) explore students' prior knowledge of the concept
    d) Specify characteristics of the concept
    e) Analyze example and non-examples
    f) Construct a definition

CONCEPT DIAGRAM

③ **Key Words**

Numerator

Denominator (divisor)

Line

Mixed

Improper

Ratio

Equal parts

½

¼

① CONVEY CONCEPT

② OFFER OVERALL CONCEPT

③ NOTE KEY WORDS

④ CLASSIFY CHARACTERISTICS

Fraction ①	Number ②

Always Present	Sometimes Present	Never Present
Numerator representing parts you have	Numerator less than denominator	Denominator that is zero
Denominator representing the number of parts in a whole	Numerator greater than denominator	
A line separating the numerator and denominator	Numerator equivalent to denominator	

⑤ EXPLORE EXAMPLES

Examples:		Non-examples:
1/2		1/0
1/4	1/1000	
Numerator/denominator		

⑥ PRACTICE WITH NEW EXAMPLES

⑦ TIE DOWN A DEFINITION

A fraction is a number that has a numerator representing parts you have, a denominator representing the number of parts in a whole and a line called a divisor separating the two.

Sample Problem

- Solve the following problem:
  - "At the County Fair, there are 36 children in line to ride the roller coaster. The roller coaster has 10 cars. Each car holds 4 children. How many children can sit 3 to a car, and how many have to sit 4 to a car?"
  - Solve the problem.

Cognitively Guided Math Instruction

- *Cognitively Guided Instruction* (CGI) is a problem-solving mathematics program designed to improve number sense and computation for students in K – 3 grades.
- The K-6 instructional strategy focuses on student knowledge and encourages teachers to pose story problems that can be solved by any means chosen by the child.
  - Problem-posing and problem-solving become the focus of the mathematics class, rather than the traditional emphasis on memorization of facts and algorithms.

Problem Type Chart

PROBLEM-SOLVING SITUATIONS

JOINING PROBLEMS		
Join: Result Unknown (JRU)	**Join: Change Unknown (JCU)**	**Join: Start Unknown (JSU)**
♦ Grandmother had 5 strawberries. Grandfather gave her 8 more strawberries. How many strawberries does Grandmother have now? 5 + 8 = □	♥ Grandmother had 5 strawberries. Grandfather gave her some more. Then Grandmother had 13 strawberries. How many strawberries did Grandfather give Grandmother? 5 + □ = 13	♠ Grandmother had some strawberries, Grandfather gave her 8 more. Then she had 13 strawberries. How many strawberries did Grandmother have before Grandfather gave her any? □ + 8 = 13
SEPARATING PROBLEMS		
Separate: Result Unknown (SRU)	**Separate: Change Unknown (SCU)**	**Separate: Start Unknown (SSU)**
♦ Grandfather had 13 strawberries. He gave 5 strawberries to Grandmother. How many strawberries does Grandfather have left? 13 - 5 = □	♥,Grandfather had 13 strawberries. He gave some to Grandmother. Now he has 5 strawberries left. How many strawberries did Grandfather give Grandmother? 13 - □ = 5	♠ Grandfather had some strawberries. He gave 5 to Grandmother. Now he has 8 strawberries left. How many strawberries did Grandfather have before he gave any to Grandmother □ - 5 = 8
PART -PART -WHOLE PROBLEMS		
Part-Part-Whole: Whole Unknown (PPW:WU)	**Part-Part-Whole: Part Unknow (PPW:PU)**	
♦ Grandmother has 5 big strawberries and 8 small strawberries. How many strawberries does Grandmother have altogether? 5 + 8 = □	♥,Grandmother has 13 strawberries. Five are big and the rest are small. How many small strawberries does Grandmother have? 13 - 5 = □ or 5 + □ = 13	
COMPARE PROBLEMS		
Comp. Difference Unknown	**Comp. Quantity Unknown**	**Comp. Referent Unknown**
♦ ♥,Grandfather has 8 strawberries. Grandmother has 5 strawberries. How many more berries does Grandfather have than Grandmother? 8 - 5 = □ or 5 + □ = 8	♠ Grandmother has 5 strawberries. Grandfather has 3 more strawberries than Grandmother. How many strawberries does Grandfather have? 5 + 3 = □	♠ Grandfather has 8 strawberries. He has 3 more strawberries than Grandmother. How many strawberries does Grandmother have? 8- 3 = □ or □ + 3 = 8
MULTIPLICATION & DIVISION PROBLEMS		
Multiplication	**Measurement Division**	**Partitive Division**
♦ Grandmother has 4 piles of strawberries. There are 3 strawberries in each pile. How many strawberries does Grandmother have? 4 x 3 = □	♦ Grandmother had 12 strawberries. She gave them to some children. She gave each child 3 strawberries. How many children were given strawberries? 12 ÷3 = □	♦ ♥,Grandfather has 12 strawberries. He wants to give them to 3 children. If he gives the same number of strawberries to each child, how many strawberries will each child get? 12 ÷ 3 = □

Problem chart based on Cognitively Guided Instruction Problem Types (Carpenter et al., 1996)

Components of Cognitively Guided Math Instruction

- Direct Modeling (concrete-representational-abstract)
  - Act out the problem.
  - Follow the sequence or steps for completing the problem.
  - Use manipulatives.
  - Use composition or decomposition of numbers.
- Counting Strategies
  - Use the following strategies: counting-all & counting-on.
- Flexible Base-Ten Strategies
  - Use invented or alternative algorithms.
- Derived Facts/Number Facts
  - Recall basic facts for addition, subtraction, multiplication and division.
  - Use derived facts, doubles or near doubles.

Direct Modeling Strategies

- **Direct Modeling** strategy represents each number in the problem with concrete objects.

Child's Solution to JRU

$6 + 5 = 11$

"Grandfather had six strawberries. One, two, three, four, five, six." (*The child sets out six counters*.) "Grandmother gave him five more. *One, two, three, four, five." (Child sets out five counters and then pushes both sets together and counts all of the counters*.) "Now, he has eleven strawberries."

Child's Solution to SRU

$11 – 5 = 6$

"Grandmother had eleven strawberries. One, two three, four, five, six, seven, eight, nine, ten, eleven, twelve." (*Child sets out eleven counters*.) "She gave five to Grandfather. One, two, three, four, five. (*Child counts out and removes five counters from the group of eleven and counts the remaining counters*.) "Now she has 'one, two, three, four, five, six. She has six.

Counting Strategies

- A child using a **Counting On/Back** strategy is able to hold a number in her/his mind and count on or back from that number while keeping track of the quantity that is added or subtracted using fingers, tally marks, or counters.

Child's Solution to JRU

$6 + 5 = 11$

"I don't have to count the six again. I just have to add five to it. I say, 'Seven, eight, nine, ten, eleven.' (*Child holds up a finger with each count*.) I have eleven."

Child's Solution to SRU

$11 - 5 = 6$

"I know Grandmother had eleven strawberries. I know she gave five away. So, I count five down. 'Eleven, ten, nine, eight, seven.' I have six left." (*Child folds a finger down with each count*.)

Flexible Base-Ten Strategies

- **Alternative or Invented Strategies** *r*efers to any strategy other than the traditional algorithm or does not involve the use of physical materials.

Child's Solution to JRU

19 + 3 = 22

"Nine ones and three ones equal twelve ones. One ten joined with twelve ones equals two tens and two ones." The sum is 22.
(Partial Sums)

Child's Solution to SRU

22 – 19 = 3

"Two ones take away nine ones equal (negative) seven. Two tens take away one ten equal one ten. One ten take away (negative) seven equals three ones. The sum is 3."
(Partial Differences)

Recalled or Derived Fact Strategies

- A child possessing good number sense is able to solve problems in flexible ways, often breaking numbers down and recombining them by using known facts, which is referred to as **deriving**.

Child's Solution to JRU

5 + 5 + 1 = 11

"I know that five and five is ten. I took one from the six to make five. But I must add one back on. It is eleven."

Child's Solution to SRU

10 – 5 = 5 or 11 – 5 = 6

"I know that ten take away five is five, but I started with eleven. The answer must be one more. It is six."

CGI Template

Cognitively-Guided Instruction Anticipatory Framework

School:	Teacher:	Grade:	Date:

Problem Type (see back of this sheet for more information)	JRU	JCU	JSU	Rationale/Goals (Why are you asking this problem type with these particular numbers?):
	SRU	SCU	SSU	
	PPW-WU		PPW-PU	
	CDU	CCDU	CRU	
	Multi.	Meas. Div.	Part. Div.	
Problem				

	No Base-Ten Evidence	Base-Ten Evidence
Direct Modeling Strategies		
Counting Strategies		

	Combining Tens and Ones Invented Algorithm	Compensating Invented Algorithm	Incrementing Invented Algorithm
Flexible/Base-Ten Strategies (Invented Algorithms)			
Recalled or Derived-Fact Strategies			

Created by Dr. Stephanie Smith, GSU with John Ballard, LPES

CGI Example

Cognitively Guided Instruction - Anticipatory Framework

School:	Teacher:	Grade: Second	Date:

Problem Type	JRU SRU / JCU SCU / JSU SSU / PPW-WU / PPW-PU / CDU M / CQU MD / CRU PD	<u>Rationale/Goals</u> (Why are you asking this problem type with these particular numbers?): Earlier in the year, we focused on adding and subtracting 10 from any number. Students practiced 10 more and 10 less from any given number using base ten blocks, solving procedurally, as well as mentally. The end goal was for students to be able compute these particular problems mentally using a variety of strategies. I wanted to extend their thinking as well as challenge their thinking by allowing them use these strategies to solve a complex problem involving reasoning and supporting their argument. I focused on students establishing logic and reason behind their answer instead of just finding an answer. I used these numbers because they were appropriate for my class as well as appropriate for students to model and compute mentally.
Problem	Heather has 12 glass sea rocks in her collection. Her brother Dave has 22 glass sea rocks in his collection. While walking on the beach, Heather found 10 more glass sea rocks, but Dave did not find any. Dave was sad. He said "You have more sea rocks than me!" Heather said, "No, I don't. You have more!" Who is right? Does Heather or Dave have more sea rocks? Show your work and how you know.	

	No Base-Ten Evidence	Base-Ten Evidence
Direct Modeling Strategies	Models 12 in base ten blocks, then counts out ten units, then counts all. 11 12 13 14 15 16 17 18 19 20 21 22	10+10+2; 12→22. Models 12 then adds 1 base ten rod, counts by tens to determine answer.
Counting Strategies	The student draws 10 sea rocks and begins counting by ones starting at 12 to determine answer. 12.... 13 14 15 16 17 18 19 20 21 22	The student begins counting at 12 then skip counts by tens to get to 22. 12, 22. Student mentally adds 10 more to number.

Flexible/Base-Ten Strategies (Invented Algorithms)	Combining Tens and Ones Invented Algorithm	Compensating Invented Algorithm	Incrementing Invented Algorithm
	10+10=20 2 20+2=22	15+10=25 25-3=22	10+10=20 20+2=22

Graphic Organizers

- A graphic organizer is an instructional tool used to illustrate a student or class's knowledge about a problem.
- A series of visual charts and tools used to represent and organize a student's knowledge or ideas.
- Graphic organizers create a connection between different ideas and allowing students to grasp how large concepts are interrelated.

Overview of Four Square

- Word problems require the employment of logic and reasoning different from the usual compute and solve drills.
- These problems require that students develop their own equation, and then perform an unnamed operation to find a solution.
- The four step process includes the following:
  1. Question
  2. Process
  3. Information
  4. Compute
  5. Solution

Creating a Four Square Graphic Organizer

- Fold a sheet of manila construction paper into four equal squares.
- In 1st box write the problem in the middle of the paper.
- In the 2nd box write the first algorithmic step required to solve the problem.
- In the 3rd box write following algorithmic step for computing the problem.
- In the 4th box write the last algorithmic step for completing the problem.
- In the 5th box write the solution to the problem.
  - Note: You can provide additional boxes for problems requiring more than 4 steps.

Four Square Template

Box 2

Box 3

Box 1

Box 4

Box 5

Single-Digit Addition with Regrouping

1st box **5 + 6 =**

2nd box

Addition

3rd box

5 ones + 6 ones = 11 ones

4th box

11 ones equal 1 ten and 1 one or 11.

5th box

The answer is 11.

Multi-Digit Addition with Regrouping

1st box **17 + 46 =**

2nd box

7 + 6 = 13 which is represented by 1 ten and 3 ones.

*Remember to carry one ten.

3rd box

10 + 40 = 50 which is represented by 5 tens.

4th box

10 + 50 + 3 = 63 which is 6 tens and 3 ones.

5th box

The answer is 63.

Multi-Digit Subtraction with Regrouping
* Practice Problem

1st box **82 - 37 =**

2nd box

I can't subtract 7 from 2 so I must borrow one ten from the tens place, which makes it 12 – 7 = 5 or 5 ones.

3rd box

I borrowed a ten from 8 (80 or 8 tens), which makes it 7 (70 or 7 tens). So 70 – 30 = 40 or 4 tens.

4th box

I have 5 ones and 4 tens, which looks like 40 + 5 = 45

5th box

The answer is 45.

Multiplication
* Practice Problem

1st box **12 x 5=**

2nd box

The first step is to multiply 5 x 10, which looks like 10 + 10 + 10 + 10 + 10 = 50.

3rd box

The next step is 2 x 5, which looks like 5 + 5 = 10.

4th box

I have 50 (5 tens) and a ten (1 ten or 10 ones), which looks like 50 + 10 = 60.

5th box

The answer is 60.

Division
* Practice Problem

1^{st} box **54 ÷ 3 =**

2^{nd} box

The first step is to divide 5 by 3, which looks like 3 $\overline{)5}$

3^{rd} box

The next step is to subtract 3 from 5, which equals 2. Put the 1 above the 5 and bring the 4 down.

4^{th} box

Now, I have 24 divided by 3 which equals 8 because 8 + 8 + 8 = 24. I place 8 next to the 1 above the 4, which looks like

$$\begin{array}{r} 18 \\ 3\overline{)54} \end{array}$$

5^{th} box

The answer is 18.

Four Square with Math

Process

Information

Question

Compute

Solution

Word Problem and Addition

Addition

"+"

4 boys

3 girls

Four boys and three girls went to the movies. How many altogether went to the movies?

$4 + 3 = \_\_\_\_$

7 Students

Four Square

Key Words	Diagram
Number Sentence	Explanation

What do you need to find?

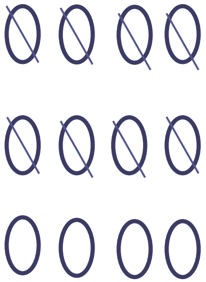
Subtraction

Key Words

Subtraction

Borrowing

Diagram



Draw 12 objects and cross out 8. There are 4 objects left.

12 - 8 =

Number Sentence

There were 12 boys in the classroom and 8 left. How many boys were left in the classroom?

Explanation

12 represents 1 ten and 2 ones. Since you can't subtract 8 ones from 2 ones, you must borrow from the tens and then subtract.

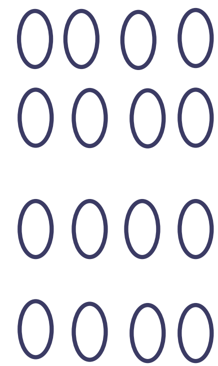
Multiplying
* Practice Problem

Key Words

Multiplication

Array

Repeated Addition



Diagram

Draw 4 rows with 4 objects in each row to represent an array.

4 x 4=

Number Sentence

There were four baskets with four bananas in each basket. How may bananas were there altogether?

Explanation

4 bananas in each of 4 baskets would look like 4 + 4 + 4 + 4 = 16, which is repeated addition or counting by 4's.

Division
* Practice Problem

Key Words
Division
Repeated Subtraction
Dividend
Divisor & Quotient

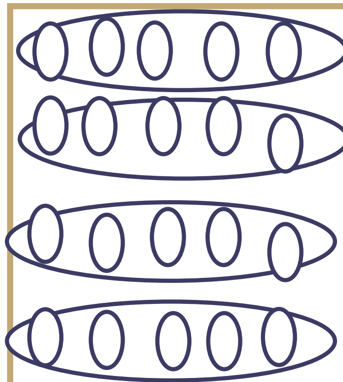
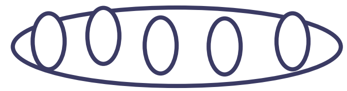


Diagram
Draw 25 objects and group them by 5’s.



$25 \div 5 =$

Number Sentence
There were 25 pennies and 5 students in the class. How many pennies did each student receive?

Explanation
Begin with 25 pennies and give each student 5 pennies. It looks like this: 25 – **5** = 20 – **5** = 15 – **5**
10 – **5** = 5 – **5** = 0

Word Problem Map

- What type of problem is this?
- The problem asked for what information?
- What cue words were used?
- These cue words suggest what operations?
- Is there a particular order in which I must perform these operations?
- *Note*: Student attempts the problem.
- Did I get an answer that seems correct?
- Have I rechecked the problem to make certain I understand it and is there anything I missed?

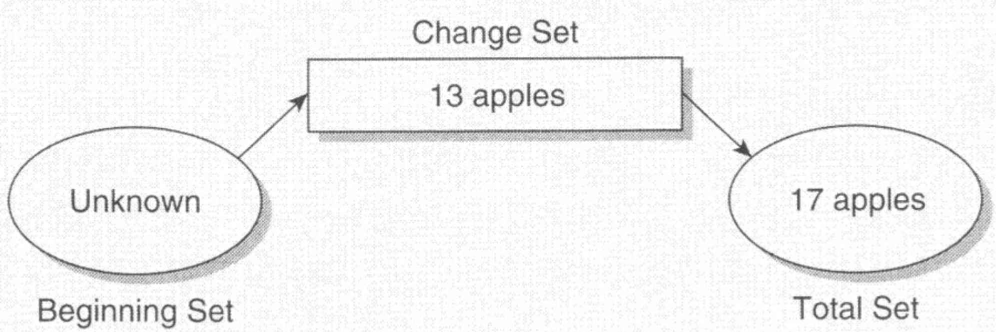
Schematic Diagrams for Word Problems

The Change Schema

One type of problem requires a change schema. These problems include a set of information that indicates change in other information in the problem.

John had some apples. Paul gave him 13 more apples. Now John has 17 apples. How many did John have in the beginning?

The change information (13 apples) must be subtracted from the total resultant set of information (17) in order to determine the start set. This change problem may be represented as follows.



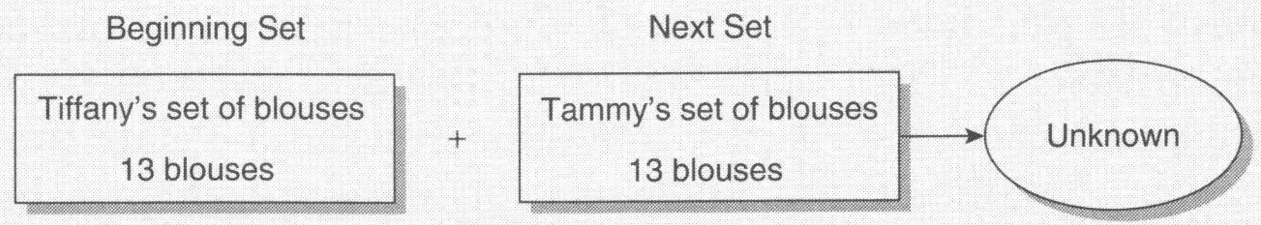
Schematic Diagrams for Word Problems

The Group Schema

In a group schema, items are grouped together from various sets. Consider the following problem:

> Tiffany owns 13 blouses that she wears to school. Her twin sister Tammy owns 13 blouses. When these girls swap clothes for school, how many blouses can they choose from?

A group schema would be represented as follows.



Schematic Diagrams for Word Problems

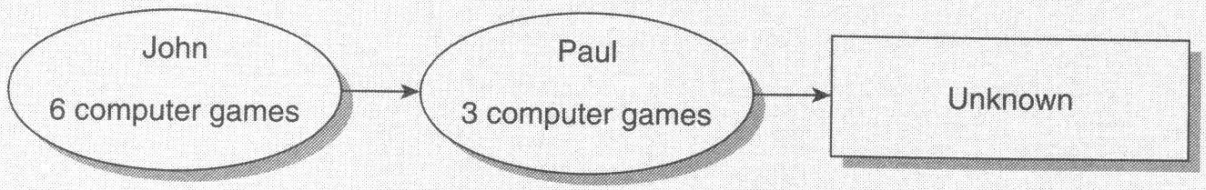
The Comparison Schema

Some word problems present "comparison" problems, which require the student to determine and subsequently compare values.

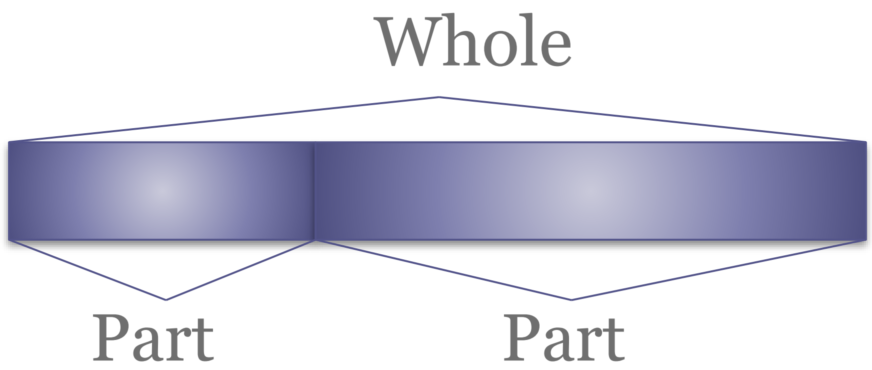
John has 6 computer games. He has 3 more than Paul. How many games does Paul have?

In order to solve this problem, the child must have a comparison schema or mental concept that includes three pieces of information: two reference quantities (the number of computer games that John has, and the difference) and a derived piece of information involving the comparison answer. Note also that the cue word *more* usually means that a child should add, but in this example it indicates subtraction.

A comparison schema would be represented as follows:



Part-Whole Model

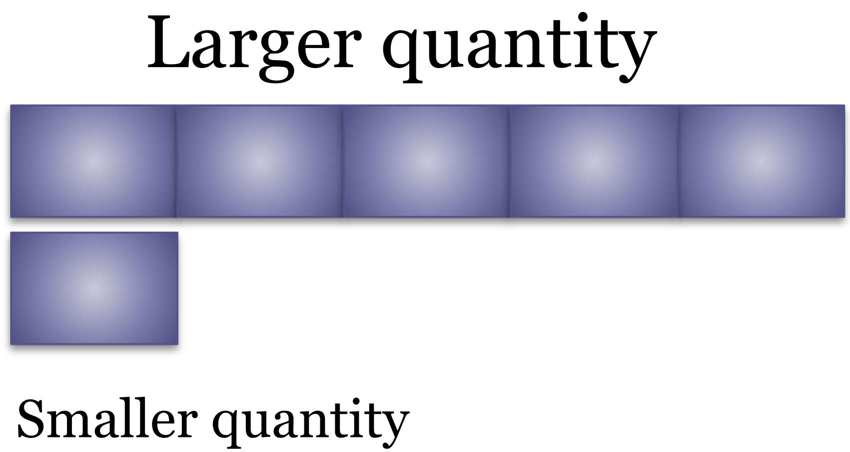


- Addition and Subtraction
  - Part + Part = Whole
  - Whole – Part = Part
- Multiplication and Division
  - One Part x Number of Parts = Whole
  - Whole / Number of Parts = One Part
  - Whole / One Part = Number of Parts

Comparison Model

- Addition and Subtraction
  - Larger Quantity - Smaller Quantity = Difference
  - Smaller Quantity + Difference = Larger Quantity
  - Larger Quantity – Difference = Smaller Quantity

Comparison Model



- Multiplication and Division
  - Larger Quantity / Smaller Quantity = Multiple
  - Smaller Quantity x Multiple = Larger Quantity
  - Larger Quantity / Multiple = Smaller Quantity

Self-regulated Strategy Development

- Develop Background Knowledge
  - Help students develop the necessary skills (e.g., vocabulary) they may need to learn the academic and self-regulation strategy.
- Discuss It
  - Help his students to understand the benefits of using a strategy.
- Model It
  - Show his students how to perform the steps in a strategy.
  - Demonstrate to his students the reasons the steps in a strategy are necessary.
- Memorize It
  - Learn both the steps of the strategy and what action is performed during each step.
- Support It
  - Use strategy charts and graphic organizers.
- Establish Independent Practice
  - Incorporate activities in his lesson plans to allow his students to maintain and generalize their new strategy skills in various settings and across several tasks.

STAR Learning Tactic

- **S**earch the word problem.
  1. Read the problem carefully.
  2. Ask, “What do I know, and what do I need to find?”
  3. Write down the facts.
- **T**ranslate the words into an equation in picture form.
  1. Choose a variable to solve for.
  2. Identify the operations necessary (look for cue words).
  3. If possible, represent the problem with manipulatives.
  4. Draw a picture of the equation, including know facts and operations.
- **A**nswer the problem.
  1. Perform the necessary operations, solving for the unknown.
- **R**eview the solution.
  1. Reread the problem.
  2. Ask, “Does the answer make sense? Why or why not?”
  3. Check the answer.

What Works Clearinghouse (WWC) Best Evidence Encyclopedias (BEE).

- Improving Mathematical Problem Solving Recommendations
  - Prepare problems and use them in whole-class instruction—MINIMAL EVIDENCE
  - Assist students in monitoring and reflecting on the problem-solving process—**STRONG EVIDENCE**
  - Teach students how to use visual representations—**STRONG EVIDENCE**
  - Expose students to multiple problem-solving strategies— **MODERATE EVIDENCE**
  - Help students recognize and articulate mathematical concepts and notation—**MODERATE EVIDENCE**
  - http://ies.ed.gov/ncee/wwc/PracticeGuide.aspx?sid=16

Monitoring and Reflecting

- Assist students in monitoring and reflecting on the problem-solving process.
  - Provide students with a list of prompts to help them monitor and reflect during the problem-solving process.
  - Model how to monitor and reflect on the problem-solving process.
  - Use student thinking about a problem to develop students' ability to monitor and reflect.

List of Prompts for Problem Solving

Sample Question List	Sample Task List

Sample Question List

- What is the story in this problem about?
- What is the problem asking?
- What do I know about the problem so far ? What information is given to me? How can this help me?
- Which information in the problem is relevant?
- In what way is this problem similar to problems I have previously solved?
- What are the various ways I might approach the problem?
- Is my approach working? If I am stuck, is there another way I can think about solving this problem?
- Does the solution make sense? How can I verify the solution?
- Why did these steps work or not work?
- What would I do differently next time?

Sample Task List

- Identify the givens and goals of the problem.
- Identify the problem type.
- Recall similar problems to help solve the current problem.
- Use a visual to represent and solve the problem.
- Solve the problem.
- Check the solution.

Model Problem Solving Process

Problem

Last year was unusually dry in Colorado. Denver usually gets 60 inches of snow per year. Vail, which is up in the mountains, usually gets 350 inches of snow. Both places had 10 inches of snow less than the year before. Kara and Ramon live in Colorado and heard the weather report. Kara thinks the decline for Denver and Vail is the same. Ramon thinks that when you compare the two cities, the decline is different. Explain how both people are correct.

Model Problem Solving Process

Solution

TEACHER: First, I ask myself, **"What is this story about, and what do I need to find out?"** I see that the problem has given me the usual amount of snowfall and the change in snowfall for each place, and that it talks about a decline in both cities. I know what *decline* means: "a change that makes something less." Now I wonder how the decline in snowfall for Denver and Vail can be the same for Kara and different for Ramon. I know that a decline of 10 inches in both cities is the same, so I guess that's what makes Kara correct. How is Ramon thinking about the problem?

I ask myself, **"Have I ever seen a problem like this before?"** As I think back to the assignments we had last week, I remember seeing a problem that asked us to calculate the discount on a $20 item that was on sale for $15. I remember we had to determine the percent change. This could be a similar kind of problem. This might be the way Ramon is thinking about the problem.

Before I go on, I ask myself, **"What steps should I take to solve this problem?"** It looks like I need to divide the change amount by the original amount to find the percent change in snowfall for both Denver and Vail.

Denver: $10 \div 60 = 0.166$ or 16.67% or 17% when we round it to the nearest whole number

Vail: $10 \div 350 = 0.029$ or 2.9% or 3% when we round it to the nearest whole number

So the percent decrease in snow for Denver was much greater (17%) than for Vail (3%). Now I see what Ramon is saying! It's different because the percent decrease for Vail is much smaller than it is for Denver.

Finally, I ask myself, **"Does this answer make sense when I reread the problem?"** Kara's answer makes sense because both cities did have a decline of 10 inches of snow. Ramon is also right because the percent decrease for Vail is much smaller than it is for Denver. Now, both of their answers make sense to me.

Use Student Thinking to Help Monitor and Reflect

Problem

Find a set of five different numbers whose average is 15.

Solution

TEACHER: Jennie, what did you try?

STUDENT: I'm guessing and checking. I tried 6, 12, 16, 20, 25 and they didn't work. The average is like 17.8 or something decimal like that.

TEACHER: That's pretty close to 15, though. Why'd you try those numbers?

STUDENT: What do you mean?

TEACHER: I mean, where was the target, 15, in your planning? It seems like it was in your thinking somewhere. If I were choosing five numbers, I might go with 16, 17, 20, 25, 28.

STUDENT: But they wouldn't work—you can tell right away.

TEACHER: How?

STUDENT: Because they are all bigger than 15.

TEACHER: So?

STUDENT: Well, then the average is going to be bigger than 15.

TEACHER: Okay. That's what I meant when I asked "Where was 15 in your planning?" You knew they couldn't all be bigger than 15. Or they couldn't all be smaller either?

STUDENT: Right.

TEACHER: Okay, so keep the target, 15, in your planning. How do you think five numbers whose average is 15 relate to the number 15?

STUDENT: Well, some have to be bigger and some smaller. I guess that is why I tried the five numbers I did.

TEACHER: That's what I guess, too. So, the next step is to think about how much bigger some have to be, and how much smaller the others have to be. Okay?

STUDENT: Yeah.

TEACHER: So, use that thinking to come up with five numbers that work.

Visual Representations

- Teach students how to use visual representations.
  - Select visual representations that are appropriate for students and the problems they are solving.
  - Use think-alouds and discussions to teach students how to represent problems visually.
  - Show students how to convert the visually represented information into mathematical notation.

Visual Representations for Problem Solving

Problem

Cheese costs $2.39 per pound. Find the cost of 0.75 pounds of cheese.[69]

Sample table

	Cost of Cheese		Pounds of Cheese
(a)	2.39		1
	?		0.75
(b)	2.39	← 2.39 ←	1
	x	← 2.39 ←	0.75

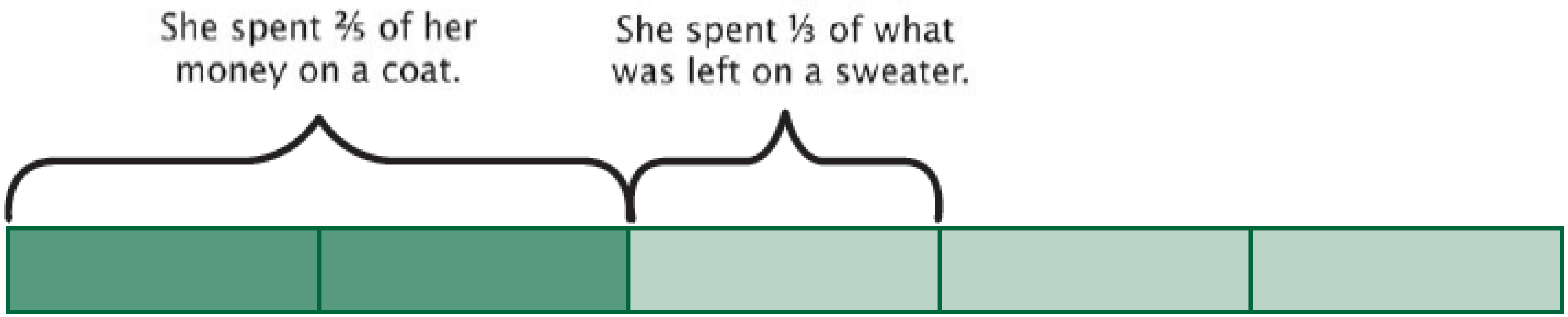
This table depicts the relationship between the weight of cheese and its cost. Every pound of cheese will cost $2.39, and this relationship can be used to determine the cost of 0.75 pounds of cheese by using the rule "times 2.39," which can be stated in an equation as $x = 0.75 \times 2.39$.

Visual Representations for Problem Solving

Problem

Eva spent ⅖ of the money she had on a coat and then spent ⅓ of what was left on a sweater. She had $150 remaining. How much did she start with?

Sample strip diagram



This strip diagram depicts the money Eva spent on a coat and a sweater. It shows how the amount of money she originally had is divided into 5 equal parts and that 2 of the 5 parts are unspent. The problem states that the unspent amount equals $150. Several strategies can then be employed to make use of this information in an equation, such as $\frac{2}{5} \times x = 150$, to determine the original amount.

Visual Representations for Problem Solving

Problem

John recently participated in a 5-mile run. He usually runs 2 miles in 30 minutes. Because of an ankle injury, John had to take a 5-minute break after every mile. At each break he drank 4 ounces of water. How much time did it take him to complete the 5-mile run?

Sample schematic diagram

This schematic diagram depicts the amount of time John needed to run 5 miles when each mile took him 15 minutes to run and he took a 5-minute break after every mile. The total time (x) it took him to complete the run is equal to the total number of minutes in this diagram, or $x = (5 \times 15) + (4 \times 5)$.

Use Think-Alouds

Problem

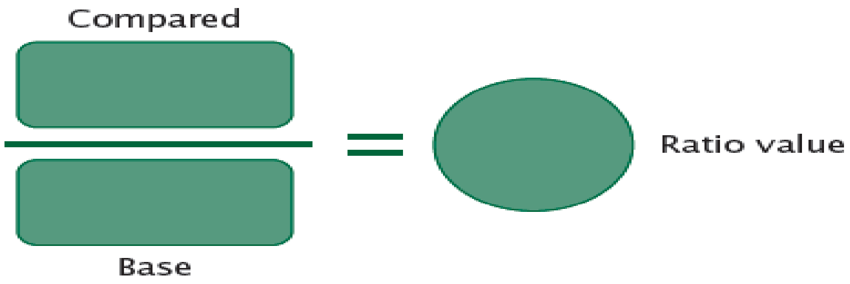
Monica and Bianca went to a flower shop to buy some roses. Bianca bought a bouquet with 5 pink roses. Monica bought a bouquet with two dozen roses, some red and some yellow. She has 3 red roses in her bouquet for every 5 yellow roses. How many red roses are in Monica's bouquet?

Solution

TEACHER: I know this is a ratio problem because two quantities are being compared: the number of red roses and the number of yellow roses. I also know the ratio of the two quantities. There are 3 red roses for every 5 yellow roses. This tells me I can find more of each kind of rose by multiplying.

I reread the problem and determine that I need to solve the question posed in the last sentence: "How many red roses are in Monica's bouquet?" Because the question is about Monica, perhaps I don't need the information about Bianca. The third sentence says there are two dozen red and yellow roses. I know that makes 24 red and yellow roses, but I still don't know how many red roses there are. I know there are 3 red roses for every 5 yellow roses. I think I need to figure out how many red roses there are in the 24 red and yellow roses.

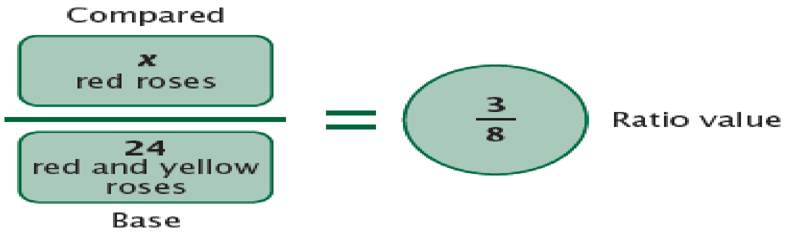
Let me reread the problem… That's correct. I need to find out how many red roses there are in the bouquet of 24 red and yellow roses. The next part of the problem talks about the ratio of red roses to red and yellow roses. I can draw a diagram that helps me understand the problem. I've done this before with ratio problems. These kinds of diagrams show the relationship between the two quantities in the ratio.



Use Think-Alouds

TEACHER: I write the quantities and units from the problem and an *x* for what must be solved in the diagram. First, I am going to write the ratio of red roses to yellow roses here in the circle. This is a part-to-whole comparison—but how can I find the whole in the part-to-whole ratio when we only know the part-to-part ratio (the number of red roses to the number of yellow roses)?

I have to figure out what the ratio is of red roses to red and yellow roses when the problem only tells about the ratio of red roses to yellow roses, which is 3:5. So if there are 3 red roses for every 5 yellow roses, then the total number of units for red and yellow roses is 8. For every 3 units of red roses, there are 8 units of red and yellow roses, which gives me the ratio 3:8. I will write that in the diagram as the ratio value of red roses to red and yellow roses. There are two dozen red and yellow roses, and that equals 24 red and yellow roses, which is the base quantity. I need to find out how many red roses (*x*) there are in 24 red and yellow roses.



I can now translate the information in this diagram to an equation like this:

$$\frac{x \text{ red roses}}{24 \text{ red-and-yellow roses}} = \frac{3}{8}$$

Then, I need to solve for *x*.

$$\frac{x}{24} = \frac{3}{8}$$

$$24\left(\frac{x}{24}\right) = 24\left(\frac{3}{8}\right)$$

$$x = \frac{72}{8}$$

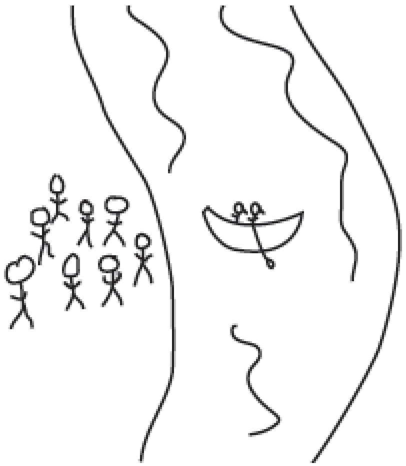
$$x = 9$$

Convert Visuals to Mathematical Notation

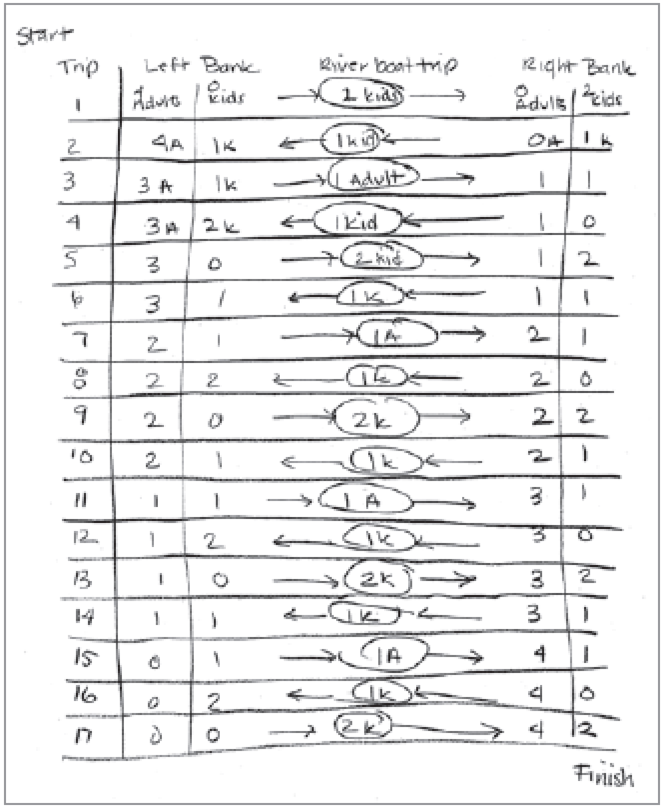
Problem

There are 4 adults and 2 children who need to cross the river. A small boat is available that can hold either 1 adult or 1 or 2 small children. Everyone can row the boat. How many one-way trips does it take for all of them to cross the river?

Using a representation without relevant details and with irrelevant details to represent the problem:



Using a schematic diagram with relevant details and without irrelevant details to represent the problem:



Start

Trip	Left Bank Adults	Left Bank Kids	River boat trip	Right Bank Adults	Right Bank Kids
1	4	0	→ 2 kids →	0	2
2	4A	1K	← 1 kid ←	0A	1K
3	3A	1K	→ 1 Adult →	1	1
4	3A	2K	← 1 Kid ←	1	0
5	3	0	→ 2 kid →	1	2
6	3	1	← 1K ←	1	1
7	2	1	→ 1A →	2	1
8	2	2	← 1K ←	2	0
9	2	0	→ 2K →	2	2
10	2	1	← 1K ←	2	1
11	1	1	→ 1A →	3	1
12	1	2	← 1K ←	3	0
13	1	0	→ 2K →	3	2
14	1	1	← 1K ←	3	1
15	0	1	→ 1A →	4	1
16	0	2	← 1K ←	4	0
17	0	0	→ 2K →	4	2

Finish

Multiple Problem Solving Strategies

- Expose students to multiple problem-solving strategies.
  - Provide instruction in multiple strategies.
  - Provide opportunities for students to compare multiple strategies in worked examples.
  - Ask students to generate and share multiple strategies for solving a problem.

Multiple Strategies

Problem

Ramona's furniture store has a choice of 3-legged stools and 4-legged stools. There are five more 3-legged stools than 4-legged stools. When you count the legs of the stools, there are exactly 29 legs. How many 3-legged and 4-legged stools are there in the store?

Solution 1: Guess and check

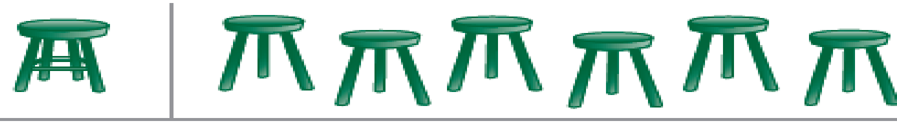
4 × 4 legs = 16 legs	9 × 3 legs = 27 legs	total = 43 legs
3 × 4 legs = 12 legs	8 × 3 legs = 24 legs	total = 36 legs
2 × 4 legs = 8 legs	7 × 3 legs = 21 legs	total = 29 legs

TEACHER: This works; the total equals 29, and with two 4-legged stools and seven 3-legged stools, there are five more 3-legged stools than 4-legged stools.

Multiple Strategies

Solution 2

TEACHER: Let's see if we can solve this problem logically. The problem says that there are five more 3-legged stools than 4-legged stools. It also says that there are 29 legs altogether. If there are five more 3-legged stools, there has to be at least one 4-legged stool in the first place. Let's see what that looks like.

stools		
total legs	$4 \times 1 = 4$	$+$ $3 \times 6 = 18$
	$4 + 18 = 22$	

TEACHER: We can add a stool to each group, and there will still be a difference of five stools.

stools		
total legs	$4 \times 2 = 8$	$+$ $3 \times 7 = 21$
	$8 + 21 = 29$	

TEACHER: I think this works. We have a total of 29 legs, and there are still five more 3-legged stools than 4-legged stools. We solved this by thinking about it logically. We knew there was at least one 4-legged stool, and there were six 3-legged stools. Then we added to both sides so we always had a difference of five stools.

Compare Multiple Strategies

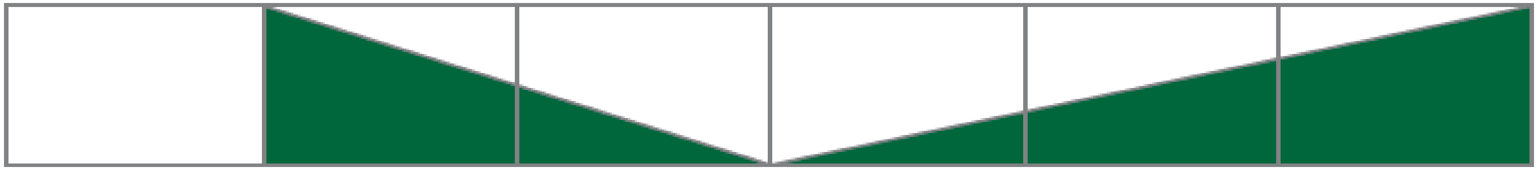
Mandy's solution		Erica's solution	
$5(y + 1) = 3(y + 1) + 8$		$5(y + 1) = 3(y + 1) + 8$	
$5y + 5 = 3y + 3 + 8$	Distribute	$2(y + 1) = 8$	Subtract on both
$5y + 5 = 3y + 11$	Combine	$y + 1 = 4$	Divide on both
$2y + 5 = 11$	Subtract on both	$y = 3$	Subtract on both
$2y = 6$	Subtract on both		
$y = 3$	Divide on both		

TEACHER: Mandy and Erica solved the problem differently, but they got the same answer. Why? Would you choose to use Mandy's way or Erica's way? Why?

Generate and Share Multiple Strategies

Problem[116]

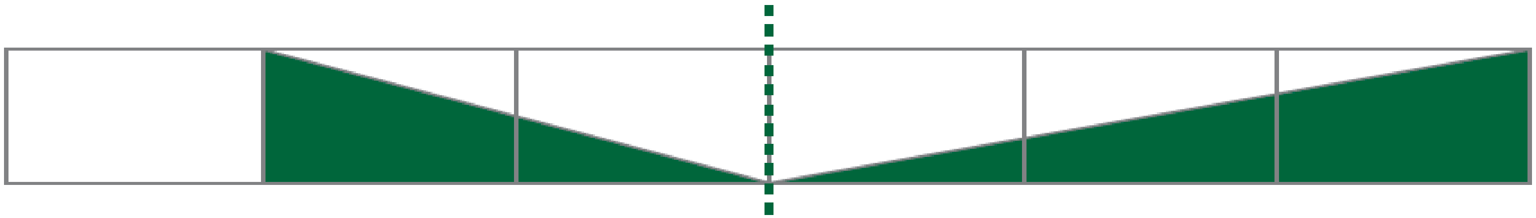
What fraction of the whole rectangle is green?



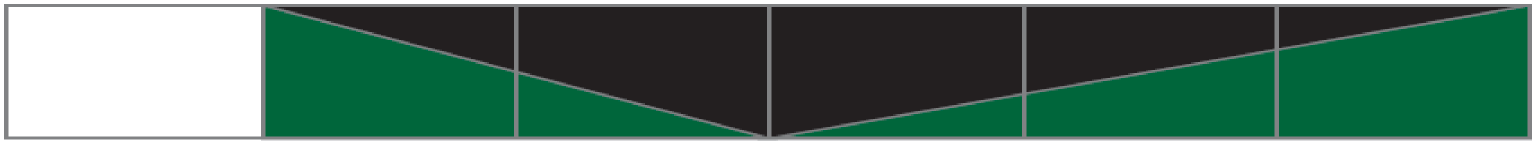
Solution strategies

STUDENT 1: If I think of it as what's to the left of the middle plus what's to the right of the middle, then I see that on the left, the green part is ⅓ of the area; so that is ⅓ of ½ of the entire rectangle. On the right, the green part is ½ of the area; so it is ½ of ½ of the entire rectangle. This information tells me that the green part is

$(\frac{1}{3} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{2}) = \frac{1}{6} + \frac{1}{4} = \frac{2}{12} + \frac{3}{12} = \frac{5}{12}$ of the entire rectangle.



STUDENT 2: I see that the original green part and the part I've colored black have the same area. So the original green part is ½ of the parts black and green, or ½ of ⅚ of the entire rectangle. This tells me that the green and black part is $\frac{1}{2} \times \frac{5}{6} = \frac{5}{12}$ of the entire rectangle.



Articulate Mathematical Concepts

- Help students recognize and articulate mathematical concepts and notation.
  - Describe relevant mathematical concepts and notation, and relate them to the problem-solving activity.
  - Ask students to explain each step used to solve a problem in a worked example.
  - Help students make sense of algebraic notation.

Describe Relevant Math Concepts

Problem

Is the sum of two consecutive numbers always odd?

Solution

STUDENT: Yes.

TEACHER: How do you know?

STUDENT: Well, suppose you take a number, like 5. The next number is 6.

For 5, I can write five lines, like this:

| | | | |

For 6, I can write five lines and one more line next to it, like this:

| | | | | |

Then, I can count all of them, and I get 11 lines.

See? It's an odd number.

TEACHER: When you say, "It's an odd number," you mean the sum of the two consecutive numbers is odd. So, can you do that with any whole number, like n? What would the next number be?

STUDENT: It would be $n + 1$.

TEACHER: So, can you line them up like you did for 5 and 6?

STUDENT: You mean, like this?

$$\begin{array}{l} n \\ \underline{n + 1} \end{array}$$

TEACHER: Right. So, what does that tell you about the sum of n and $n + 1$?

STUDENT: It's 2 n's and 1, so it's odd.

TEACHER: Very good. The sum, which is $n + n + 1 = 2n + 1$, is always going to be odd.

Explain Each Step

Problem

Are $\frac{2}{3}$ and $\frac{8}{12}$ equivalent fractions? Why or why not?

A correct description, but still not a complete explanation

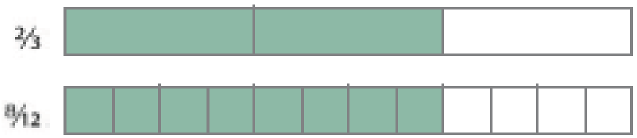
STUDENT: Whatever we multiply the top of $\frac{2}{3}$ by, we must also multiply the bottom by.

This rule is correct, but it doesn't explain why we get an equivalent fraction this way.

A mathematically valid explanation

STUDENT: You can get an equivalent fraction by multiplying the numerator and denominator of $\frac{2}{3}$ by the same number. If we multiply the numerator and denominator by 4, we get $\frac{8}{12}$.

If I divide each of the third pieces in the first fraction strip into 4 equal parts, then that makes 4 times as many parts that are shaded and 4 times as many parts in all. The 2 shaded parts become $2 \times 4 = 8$ smaller parts and the 3 total parts become $3 \times 4 = 12$ total smaller parts. So the shaded amount is $\frac{2}{3}$ of the strip, but it is also $\frac{8}{12}$ of the strip.



This explanation is correct, complete, and logical.

Making Sense of Algebra Notation

Problem

A plumbing company charges $42 per hour, plus $35 for the service call.

Solution

TEACHER: How much would you pay for a 3-hour service call?

STUDENT: $\$42 \times 3 + \$35 = \$161$ for a 3-hour service call.

TEACHER: What will the bill be for 4.5 hours?

STUDENT: $\$42 \times 4.5 + \$35 = \$224$ for 4.5 hours.

TEACHER: Now, I'd like you to assign a variable for the number of hours the company works and write an expression for the number of dollars required.

STUDENT: I'll choose h to represent the number of hours the company works.

$42h + 35 = \$$ required

TEACHER: What is the algebraic equation for the number of hours worked if the bill comes out to $140?

STUDENT: $42h + 35 = 140$

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