Strategies and Tasks to Build Procedural Fluency from Conceptual Understanding

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Please take a handout as you enter

2018 NCTM Annual Conference and Exhibition
Washington, D.C.
April 26, 2018

FYI

Electronic copies of slides are available
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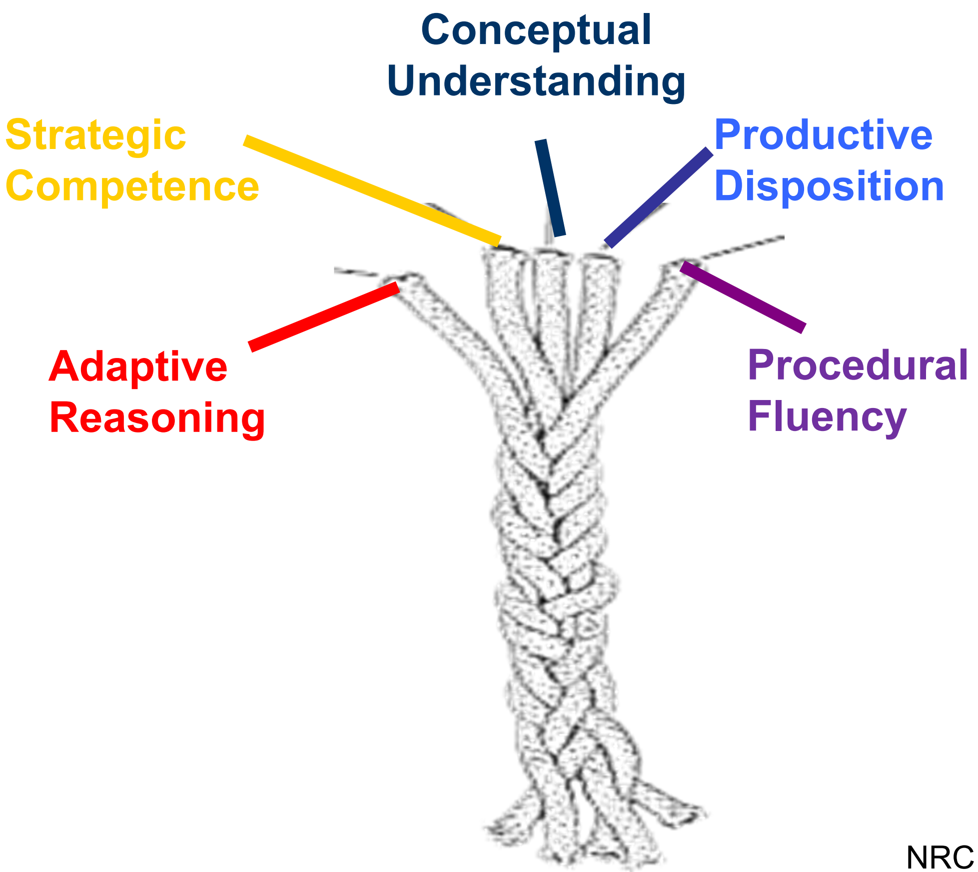
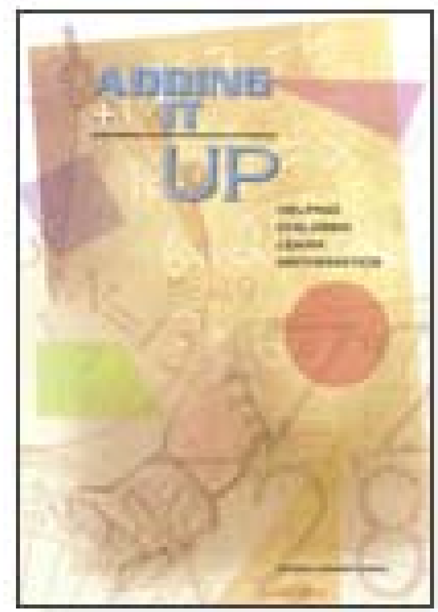
Key Questions

- What is procedural fluency?
- What tasks and strategies help students build fluency?
- What common pitfalls should I avoid?

Strands of Mathematical Proficiency





NRC (2001). *Adding It Up*. Washington, D.C.: National Academies Press.

Key Features of CCSS-M

- **Focus:** Focus strongly where the standards focus.
- **Coherence**: Think across grades, and link to major topics
- **Rigor:** In major topics, pursue conceptual understanding, procedural skill and fluency, and application
- **Standards for Mathematical Practice**

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- **Standards for Mathematical Practice**

Discuss

What does it mean to be fluent with procedures?

What is Procedural Fluency?

Which students demonstrate procedural fluency? Evidence for your answer?

- Alan
- Ana
- Marissa

Alan

1000 - 98

100 - 18

2650_MS_OS_Alan_clip_01_v01-WMV_640x360.wmv

Alan_clip3_100_minus_18.wmv

Source: The Marilyn Burns Math Reasoning Inventory

Ana

1000 - 98

[Ana_Clip5_1000_minus_998.wmv]

99 + 17

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Source: The Marilyn Burns Math Reasoning Inventory

What is Procedural Fluency?

Which students demonstrate procedural fluency? Evidence for your answer?

- Alan
- Ana

Procedural Fluency

- **Efficiency**—can carry out easily, keep track of subproblems, and make use of intermediate results to solve the problem.
- **Accuracy**—reliably produces the correct answer.
- **Flexibility**—knows more than one approach, chooses appropriate strategy, and can use one method to solve and another method to double-check.

NCTM, 2014; Russell, 2000

Marissa

295 students, 25 on each bus

Marissa_clip4_bus.wmv

Source: The Marilyn Burns Math Reasoning Inventory

Procedural Fluency

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Adapted from NCTM, 2014; Russell, 2000

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? $2 + 3 = ?$	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? $2 + ? = 5$	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? $? + 3 = 5$
Take from	Five apples were on the table. I ate two apples. How many apples are on the table now? $5 - 2 = ?$	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? $5 - ? = 3$	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? $? - 2 = 3$
	Total Unknown	**Addend Unknown**	**Both Addends Unknown[1]**
Put Together/ Take Apart[2]	Three red apples and two green apples are on the table. How many apples are on the table? $3 + 2 = ?$	Five apples are on the table. Three are red and the rest are green. How many apples are green? $3 + ? = 5,\ 5 - 3 = ?$	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? $5 = 0 + 5, 5 = 5 + 0$ $5 = 1 + 4, 5 = 4 + 1$ $5 = 2 + 3, 5 = 3 + 2$
	Difference Unknown	**Bigger Unknown**	**Smaller Unknown**
Compare[3]	("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? $2 + ? = 5,\ 5 - 2 = ?$	(Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? $2 + 3 = ?,\ 3 + 2 = ?$	(Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? $5 - 3 = ?,\ ? + 3 = 5$

Procedural Fluency Beyond Whole Number Computation

Order each set of fractions from least to greatest.

What strategies did you use?

1. $\frac{1}{8} \quad \frac{1}{4} \quad \frac{1}{5}$

2. $\frac{3}{5} \quad \frac{3}{4} \quad \frac{3}{8}$

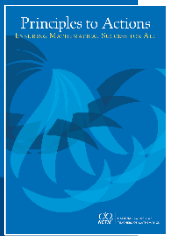
3. $\frac{3}{12} \quad \frac{1}{12} \quad \frac{8}{12}$

4. $\frac{7}{8} \quad \frac{3}{4} \quad \frac{4}{5}$

5. $\frac{17}{25} \quad \frac{3}{10} \quad \frac{4}{8}$

Comparing & Ordering Fractions

- Common denominators
- Common numerators
- Benchmark fractions, e.g. 1, ½
- Other equivalent representations, e.g., decimals, percents



Solving Proportions

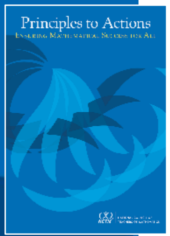
<u>Find the Missing Value</u>

Find the value of the unknown in each of the proportions shown below. What method did you use?

$$\frac{5}{2} = \frac{y}{10} \qquad \frac{a}{24} = \frac{7}{8}$$

$$\frac{n}{8} = \frac{3}{12} \qquad \frac{30}{6} = \frac{b}{7}$$

$$\frac{5}{20} = \frac{3}{d} \qquad \frac{3}{x} = \frac{4}{28}$$



Solving Proportions

- Scale Factor?
- Unit Rate?

$$\frac{5}{2} = \frac{y}{10} \qquad \frac{a}{24} = \frac{7}{8}$$

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Common Core State Standards 7.RP

2. Recognize and represent proportional relationships between quantities.

a. **Decide whether two quantities are in a proportional relationship**, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

b. **Identify the constant of proportionality (unit rate) in tables**, graphs, equations, diagrams, and verbal descriptions of proportional relationships.

c. **Represent proportional relationships by equations**. For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as t = pn.

d. Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and (1, r) where r is the unit rate.

Procedural Fluency Beyond Whole Number Computation

Solve for *x*.

$$3(x + 5) = 45$$

$$3x + 15 = 45$$
$$3x = 30$$
$$x = 10$$

$$x + 5 = 15$$
$$x = 10$$

TV Sales-Part A
(PARCC Grade 7)

A store is advertising a sale with 10% off all items in the store. Sales tax is 5%.

A 32-inch television is regularly priced at $295.00. What is the total price of the television, including sales tax, if it was purchased on sale? Fill in the blank to complete the sentence. Round your answer to the nearest cent.

The total cost of the television is $______.

Submit Answer

TV Sales-Part B
(PARCC Grade 7)

Write your answers to the following problem in your answer booklet.

A store is advertising a sale with 10% off all items in the store. Sales tax is 5%.

Adam and Brandi are customers discussing how the discount and tax will be calculated.

Here is Adam's process for finding the total cost for any item in the store.

- Take 10% off the original price.
- Then, add the sales tax to the discounted price.

Adam represents his process as:

$$T = \underbrace{0.9p}_{\text{sale price}} + \underbrace{0.05(0.9p)}_{\text{sales tax}}$$

Here is Brandi's process for finding the total cost for any item in the store.

- Determine the original price of the item, including sales tax.
- Then, take 10% off.

Brandi represents her process as:

$$T = \underbrace{1.05p}_{\text{T.V. price plus tax}} - \underbrace{0.10(1.05p)}_{\text{10\% off discount}}$$

In both equations, *T* represents the total cost of the television and *p* represents the regular price.

Are they both correct? Use the properties of operations to justify your answer.

TV Sales-Part B
(PARCC Grade 7)

Adam’s Process

$$T = \underbrace{0.9p}_{\text{sale price}} + \underbrace{0.05(0.9p)}_{\text{sales tax}}$$

$$\begin{aligned} T &= (0.9p)(1.05) \\ &= (0.9)(1.05)p \\ &= 0.945p \end{aligned}$$

Brandi’s Process

$$T = \underbrace{1.05p}_{\text{T.V. price plus tax}} - \underbrace{0.10(1.05p)}_{\text{10\% off discount}}$$

$$\begin{aligned} T &= (1.05p)(0.9) \\ &= (1.05)(0.9)p \\ &= 0.945p \end{aligned}$$

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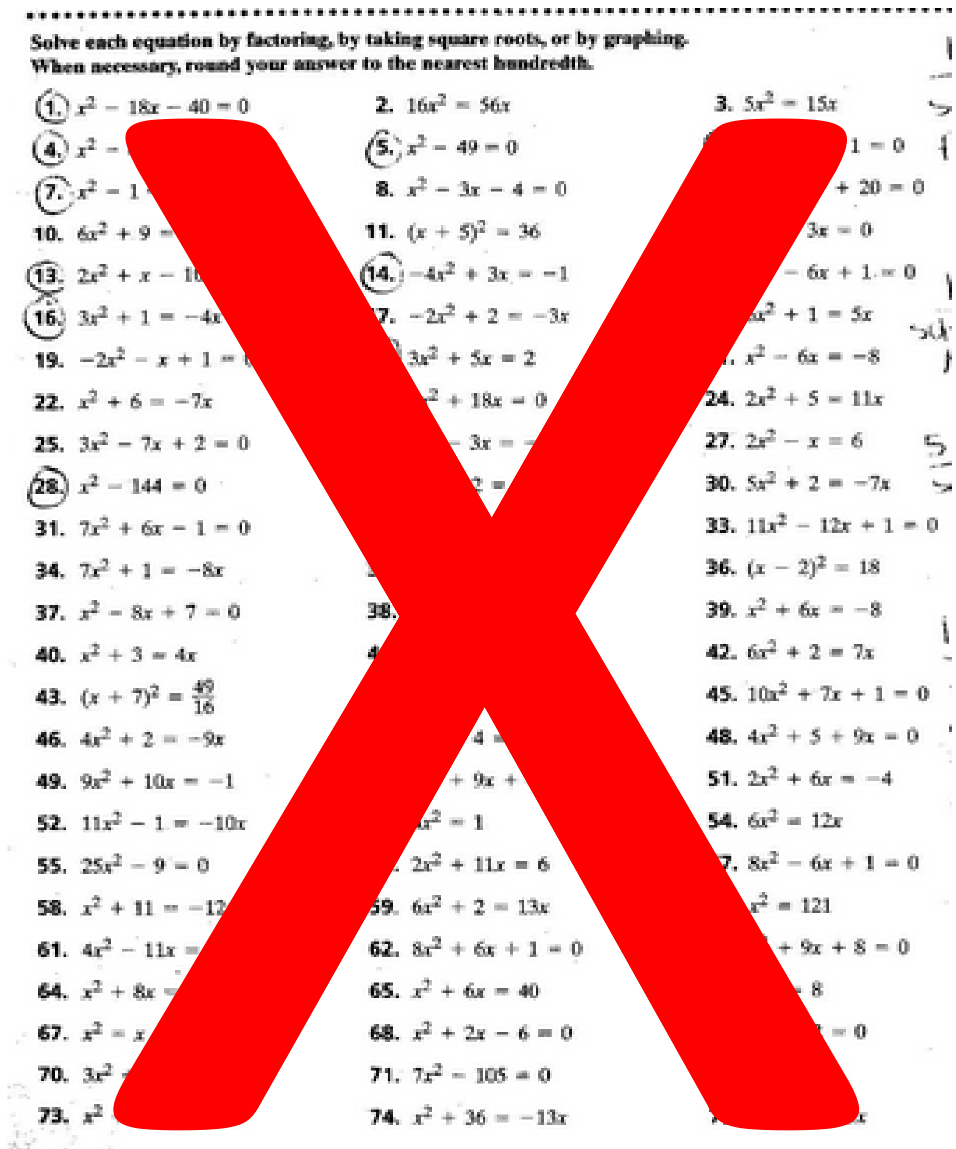
Adapted from NCTM, 2014; Russell, 2000

How Can We Develop Students' Proficiency?



Solve each equation by factoring, by taking square roots, or by graphing. When necessary, round your answer to the nearest hundredth.

1. $x^2 - 18x - 40 = 0$
2. $16x^2 = 56x$
3. $5x^2 = 15x$
4. $x^2 -$
5. $x^2 - 49 = 0$
8. $x^2 - 3x - 4 = 0$
10. $6x^2 + 9$
11. $(x + 5)^2 = 36$
13. $2x^2 + x - 1$
14. $-4x^2 + 3x = -1$
16. $3x^2 + 1 = -4x$
19. $-2x^2 - x + 1 =$
22. $x^2 + 6 = -7x$
24. $2x^2 + 5 = 11x$
25. $3x^2 - 7x + 2 = 0$
27. $2x^2 - x = 6$
28. $x^2 - 144 = 0$
30. $5x^2 + 2 = -7x$
31. $7x^2 + 6x - 1 = 0$
33. $11x^2 - 12x + 1 = 0$
34. $7x^2 + 1 = -8x$
36. $(x - 2)^2 = 18$
37. $x^2 - 8x + 7 = 0$
38.
39. $x^2 + 6x = -8$
40. $x^2 + 3 = 4x$
42. $6x^2 + 2 = 7x$
43. $(x + 7)^2 = \frac{49}{16}$
45. $10x^2 + 7x + 1 = 0$
46. $4x^2 + 2 = -9x$
48. $4x^2 + 5 + 9x = 0$
49. $9x^2 + 10x = -1$
51. $2x^2 + 6x = -4$
52. $11x^2 - 1 = -10x$
54. $6x^2 = 12x$
55. $25x^2 - 9 = 0$
58. $x^2 + 11 = -12$
59. $6x^2 + 2 = 13x$
61. $4x^2 - 11x =$
62. $8x^2 + 6x + 1 = 0$
64. $x^2 + 8x =$
65. $x^2 + 6x = 40$
67. $x^2 = x$
68. $x^2 + 2x - 6 = 0$
70. $3x^2$
71. $7x^2 - 105 = 0$
73. x^2
74. $x^2 + 36 = -13x$

Principles to Actions: Ensuring Mathematical Success for All

- Describes the **supportive conditions, structures, and policies** required to give each and every student the power of mathematics
- Focuses on **teaching and learning**
- Emphasizes engaging students in **mathematical thinking**
- Describes how to ensure that mathematics achievement is maximized **for every student**
- Is not specific to any standards; **it's universal**

Principles to Actions: Ensuring Mathematics Success for All

Guiding Principles for School Mathematics

1. **Teaching and Learning**
2. **Access and Equity**
3. **Curriculum**
4. **Tools and Technology**
5. **Assessment**
6. **Professionalism**

→ Essential Elements of Effective Math Programs

Teaching and Learning Principle

An excellent mathematics program requires effective teaching that engages students in meaningful learning through individual and collaborative experiences that promote their ability to make sense of mathematical ideas and reason mathematically.

Effective
Mathematics Teaching Practices

1. Establish mathematics **goals** to focus learning.
2. Implement **tasks** that promote reasoning and problem solving.
3. Use and connect mathematical **representations.**
4. Facilitate meaningful mathematical **discourse**.
5. Pose purposeful **questions**.
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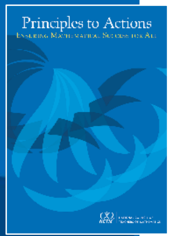
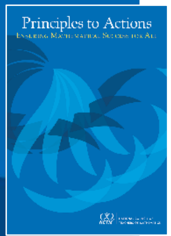


Build Procedural Fluency from Conceptual Understanding

Procedural Fluency should:

- Build on a foundation of conceptual understanding;
- Result in generalized methods for solving problems; and
- Enable students to flexibly choose among methods to solve contextual and mathematical problems.



Build Procedural Fluency from Conceptual Understanding

Students must be able to do much more than carry out mathematical procedures. They must know which procedure is appropriate and most productive in a given situation, what a procedure accomplishes, and what kind of results to expect. ***Mechanical execution of procedures without understanding their mathematical basis often leads to bizarre results.***

Martin, 2009, p. 165

What Research Tells Us

- When procedures are connected with the underlying concepts, students have better retention of the procedures and are more able to apply them in new situations
- Informal methods → general methods → formal algorithms is more effective than rote instruction.
- Engaging students in solving challenging problems is essential to build conceptual understanding.

Looking into the Classroom

Select a problem:

Band Concert Problem (Grade 3)

The third-grade class is responsible for setting up the chairs for their spring band concert. In preparation, they need to determine the total number of chairs that will be needed and ask the school's engineer to retrieve that many chairs from the central storage area. The class needs to set up 7 rows of chairs with 20 chairs in each row, leaving space for a center aisle. How many chairs does the school's engineer need to retrieve from the central storage area?

Candy Jar Problem (Grade 7)

A candy jar contains 5 Jolly Ranchers (JRs) and 13 Jawbreakers (JBs). Suppose you had a new candy jar with the same ratio of Jolly Ranchers to Jawbreakers, but it contained 100 Jolly Ranchers. How many Jawbreakers would you have? Explain how you know.

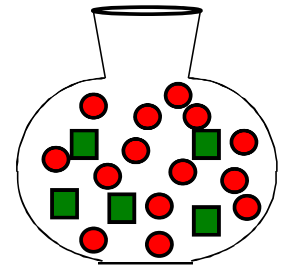
Band Concert Task

Common Core State Standards:

Standard 3.OA.3. Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

Standard 3.NBT. 3. Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9 x 80, 5 x 60) using strategies based on place value and properties of operations.

Candy Jar Task



Common Core State Standards 7.RP:

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a. **Decide whether two quantities are in a proportional relationship**, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

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c. **Represent proportional relationships by equations**. For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as $t = pn$.

d. Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and (1, r) where r is the unit rate.

Looking into the Classroom

Select a problem:

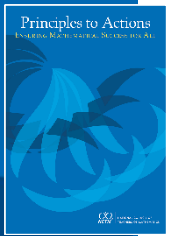
- Band Concert—Grade 3
- Candy Jar—Grade 7

Please work the problem as if you were a student.

When done, share your work with people at your table.

Discuss:

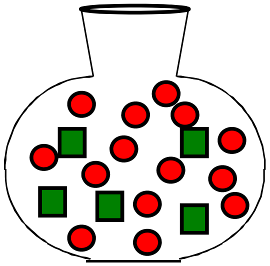
- How might this task support students' development of procedural fluency?



Looking into the Classroom

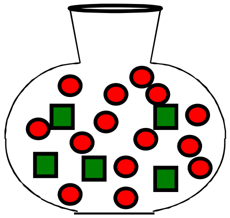
- Read the case for your problem:
  - Mr. Harris
  - Mr. Donnelly
- Make note of what the teacher did before or during instruction to support his/her students' learning and understanding, paying special attention to actions that promote procedural fluency.
- Talk with a neighbor about the actions and interactions that you identified as supporting procedural fluency.

Candy Jar Problem



A candy jar contains 5 Jolly Ranchers (squares) and 13 Jawbreakers (circles). Suppose you had a new candy jar with the same ratio of Jolly Ranchers to Jawbreakers, but it contained 100 Jolly Ranchers. How many Jawbreakers would you have? Explain how you know.

Candy Jar Problem



Group 1 (incorrect, additive)

100 JRs is 95 more than the 5 we started with. So we will need 95 more JBs than the 13 we started with.

5 JRs + 95 JRs = 100 JRs
13 JBs + 95 JBs = 108 JBs

Groups 3 and 5 (scale factor)

You had to multiply the five JRs by 20 to get 100, so you'd also have to multiply the 13 JBs by 20 to get 260.

$$\text{5 JRs} \xrightarrow{(\times 20)} \text{100 JRs}$$
$$\text{13 JBs} \xrightarrow[(\times 20)]{} \text{260 JBs}$$

Groups 4 and 7 (scaling up)

JR	JB	JR	JB
5	13	55	143
10	26	60	156
15	39	65	169
20	52	70	182
25	65	75	195
30	78	80	208
35	91	85	221
40	104	90	234
45	117	95	247
50	130	100	260

Group 2 (unit rate)

Since the ratio is 5 JRs for 13 JBs, we divided 13 by 5 and got 2.6. So that would mean that for every 1 JR there are 2.6 JBs. So then you just multiply 2.6 by 100.

$$\text{1 JR} \xrightarrow{(\times 100)} \text{100 JRs}$$
$$\text{2.6 JBs} \xrightarrow[(\times 100)]{} \text{260 JBs}$$

Group 6 (scaling up)

JRs	5	10	20	40	80	100
JBs	13	26	52	104	208	260

We started by doubling both the number of JRs and JBs. But then when we got to 80 JRs, we didn't want to double it anymore because we wanted to end up at 100 JRs, and doubling 80 would give us too many. So we noticed that if we added 20 JRs:52 JBs and 80 JRs:208 JBs, we would get 100 JRs:260 JBs.

Group 8 (scaling up)

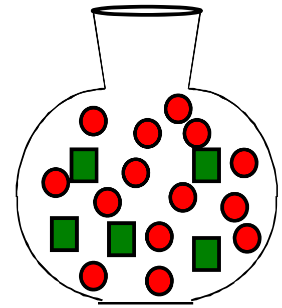
We drew 100 JRs in groups of 5. Then we put 13 JBs with each group of 5 JRs. We then counted the number of JBs and found we had used 260 of them.

Procedural Fluency

Extending learning from the original Candy Jar Task:

$$\frac{5}{13} = \frac{127}{x}$$

Procedural Fluency



$$\frac{5}{13} = \frac{127}{x}$$

$13 \div 5 = 2.6.$

$2.6 \cdot 127 = 330.2$

$\frac{5}{13} = \frac{127}{330.2}$

because

$127 \div = 25.4$

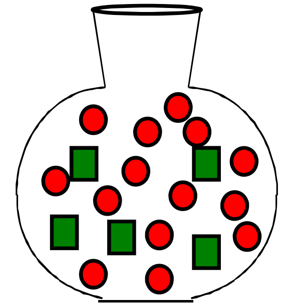
$330.2 \div 13 = 25.4$

$\times 25.4$

JR	5	10	15	20	127
~~S~~B	13	26	39	52	330.2

$\times 25.4$

Procedural Fluency



$$\frac{5}{13} = \frac{127}{x}$$

Unit Rate

$13 \div 5 = 2.6.$

$2.6 \cdot 127 = 330.2$

Scale Factor

$\frac{5}{13} = \frac{127}{330.2}$

because

$127 \div 5 = 25.4$

$330.2 \div 13 = 25.4$

Scaling Up

×25.4

JR	5	10	15	20	127
JB	13	26	39	52	330.2

×25.4

Building Procedural Fluency

Finding the Missing Value

Find the value of the unknown in each of the proportions shown below.

$$\frac{5}{2}=\frac{y}{10} \qquad \frac{a}{24}=\frac{7}{8}$$

$$\frac{n}{8}=\frac{3}{12} \qquad \frac{30}{6}=\frac{b}{7}$$

$$\frac{5}{20}=\frac{3}{d} \qquad \frac{3}{x}=\frac{4}{28}$$

What might we expect students to be able to do when presented with a missing value problem, after they have had the opportunity to develop a set of strategies through solving a variety of contextual problems like the Candy Jar Task?

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Developing Procedural Fluency

1. Develop conceptual understanding building on students' informal knowledge
2. Develop informal strategies to solve problems
3. Refine informal strategies to develop fluency with standard methods and procedures (algorithms)

Effective Mathematics Teaching Practices

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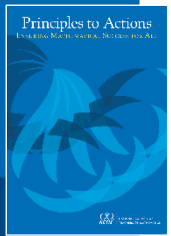
Implement Tasks that Promote Reasoning and Problem Solving

Mathematical tasks should:

- Provide opportunities for students to engage in exploration or encourage students to use procedures in ways that are connected to concepts and understanding;
- Build on students’ current understanding; and
- Have multiple entry points.



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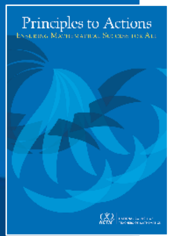
NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS



Evaluate and Compare Methods

<u>Finding the Missing Value</u>

Find the value of the unknown in each of the proportions shown below using both unit rates and scale factors. Which strategy do you prefer for each one? Why?

$$\frac{5}{2} = \frac{y}{10} \qquad \frac{a}{24} = \frac{7}{8}$$

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IES Algebra Practice Guide, 2015

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Evaluate and Compare Methods

	Unit Rate	Scale Factor	Which do you prefer? Why?
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IES Algebra Practice Guide, 2015

Evaluate and Compare Methods

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IES Algebra Practice Guide, 2015

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Build Procedural Fluency from Conceptual Understanding

What are teachers doing?

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Striving to use procedures appropriately and efficiently.

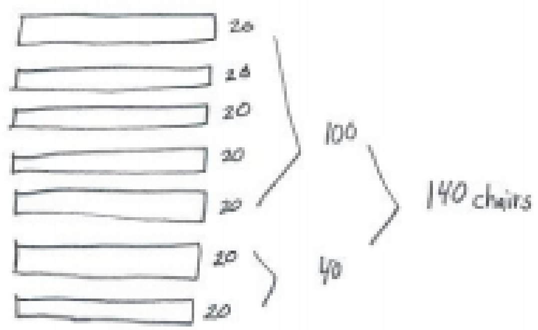
The Band Concert

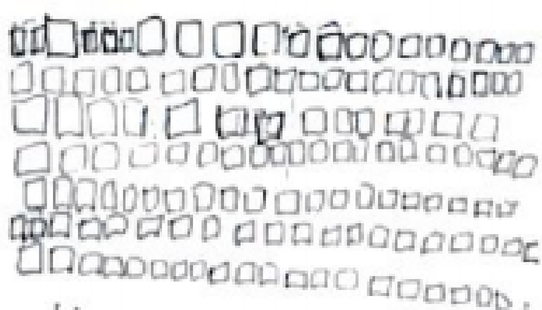
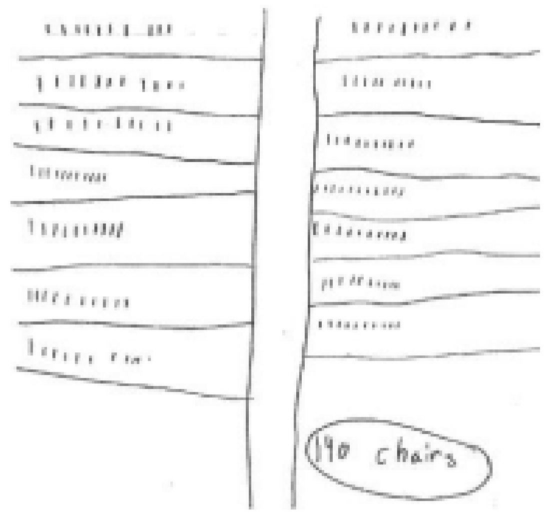
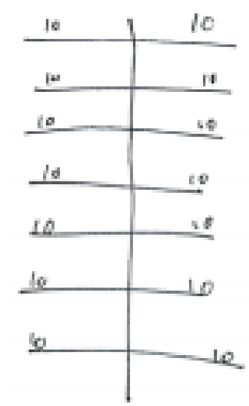
The third-grade class is responsible for setting up the chairs for their spring band concert. In preparation, they need to determine the total number of chairs that will be needed and ask the school's engineer to retrieve that many chairs from the central storage area.

The class needs to set up 7 rows of chairs with 20 chairs in each row, leaving space for a center aisle.

How many chairs does the school's engineer need to retrieve from the central storage area?

The Band Concert

Jasmine	Kenneth	Teresa
	20+20+20+20+20+20+20 40+40=80 80+20=100 100+20=120 120+20=140 140 chairs	1 2 3 4 5 6 7 20, 40, 60, 80, 100, 120, 140

Molly	Tyrell	Ananda
 160		 70 + 70 = 140 chairs

Starting Point to Build Procedural Fluency

The Band Concert

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Find the product

5 x 20 =

6 x 80 =

4 x 70 =

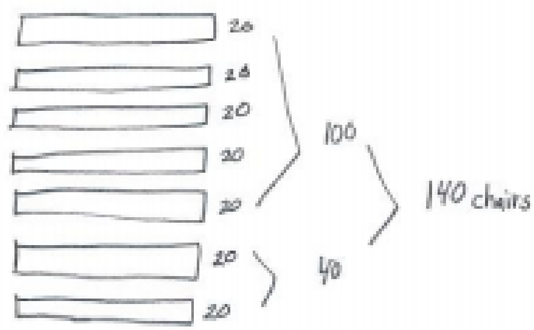
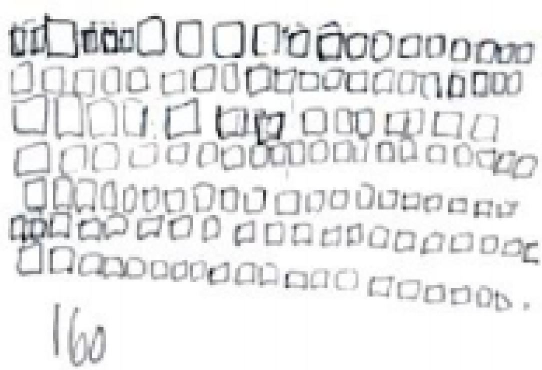
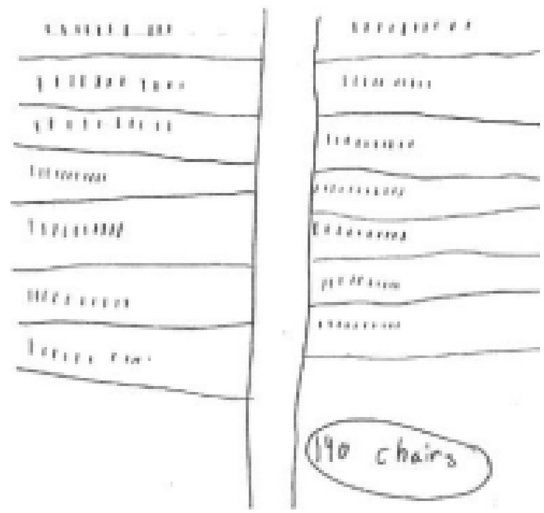
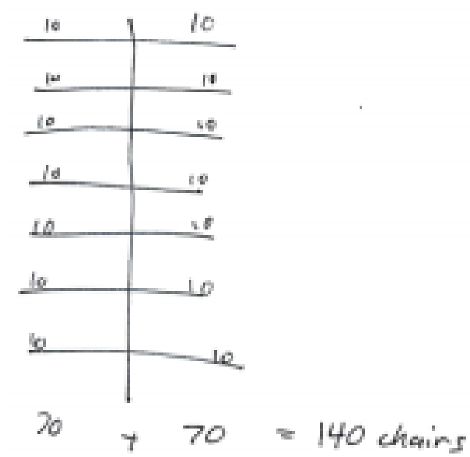
3 x 50 =

9 x 20 =

2 x 60 =

8 x 30 =

The Band Concert

Jasmine	Kenneth	Teresa
 	20+20+20+20+20+20+20 40+40=80 80+20=100 100+20=120 120+20=140 140 chairs	1 2 3 4 5 6 7 20, 40, 60, 80, 100, 120, 140
Molly	**Tyrell**	**Ananda**
 	 	

Developing Procedural Fluency
Whole Number Multiplication

1. Develop conceptual understanding building on students’ informal knowledge
2. Develop informal strategies to solve problems
3. Refine informal strategies to develop fluency with standard methods and procedures (algorithms)

	Numbers and Operations in Base Ten
1	Use place value understanding and properties of operations to add and subtract.
2	Use place value understanding and properties of operations to add and subtract.
3	Use place value understanding and properties of operations to perform multi-digit arithmetic. *A range of algorithms may be used.*
4	Use place value understanding and properties of operations to perform multi-digit arithmetic. *Fluently add and subtract multi-digit whole numbers using the standard algorithm.*
5	Perform operations with multi-digit whole numbers and with decimals to hundredths. *Fluently multiply multi-digit whole numbers using the standard algorithm.*
6	Compute fluently with multi-digit numbers and find common factors and multiples. *Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.*

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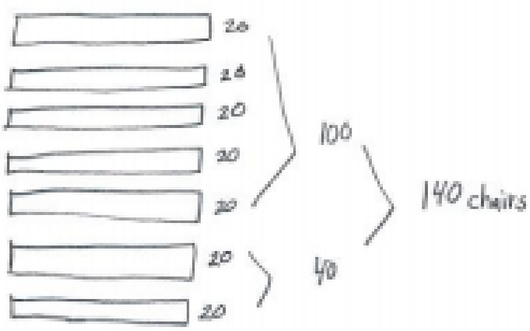
The Band Concert

<table>
<tr><th>Jasmine</th><th>Kenneth</th><th>Teresa</th></tr>
<tr>
<td>
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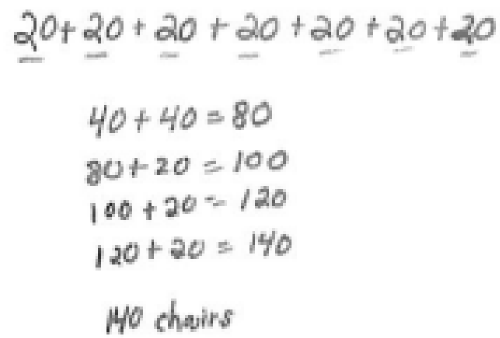
<td>
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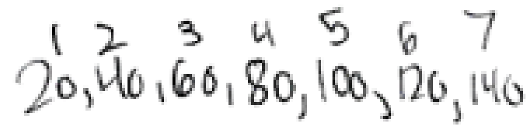
<td>
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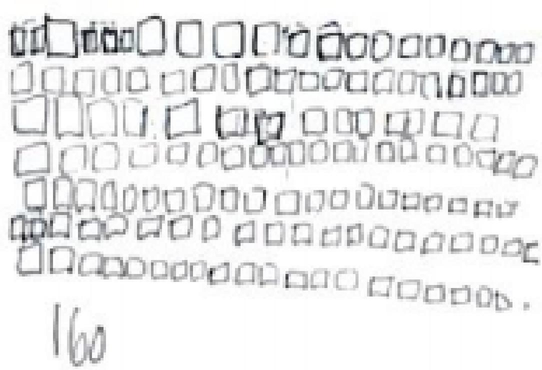
<td>
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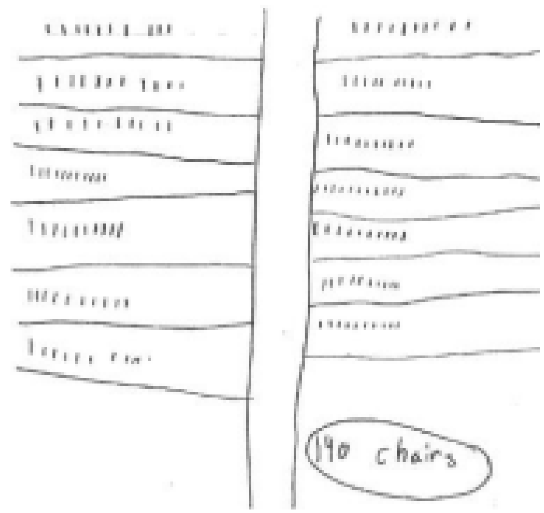
<td>
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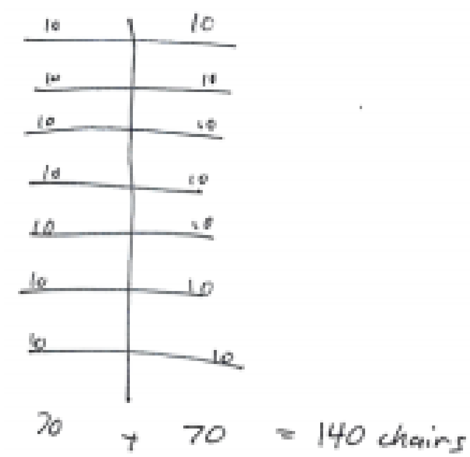
<td>
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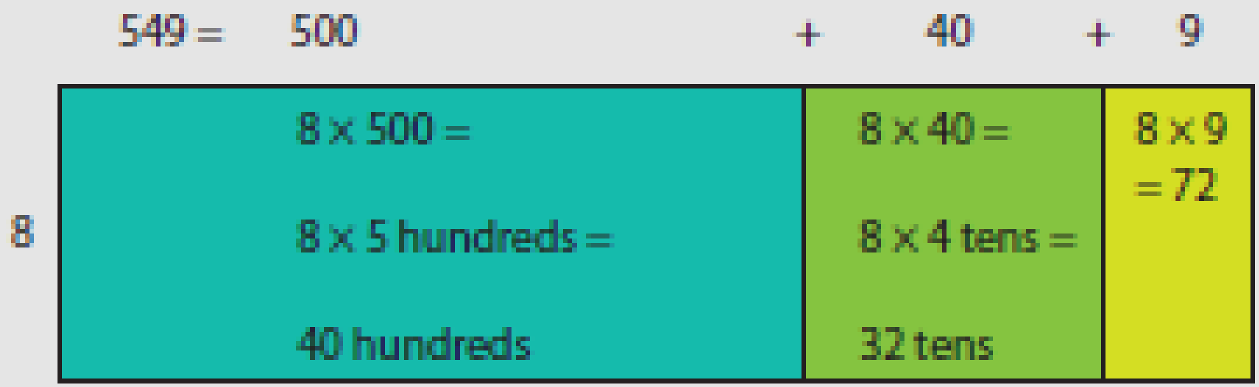
Multiplication Algorithms

Computation of 8×549 connected with an area model

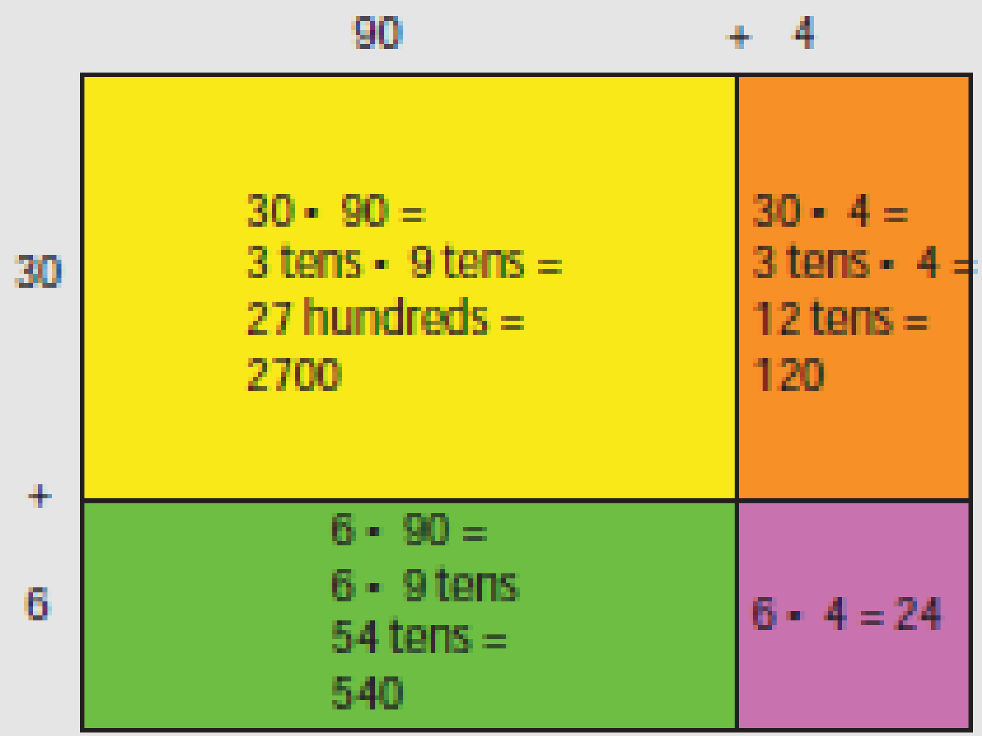


Each part of the region above corresponds to one of the terms in the computation below.

$$\begin{aligned} 8 \times 549 &= 8 \times (500 + 40 + 9) \\ &= 8 \times 500 + 8 \times 40 + 8 \times 9. \end{aligned}$$

Multiplication Algorithms

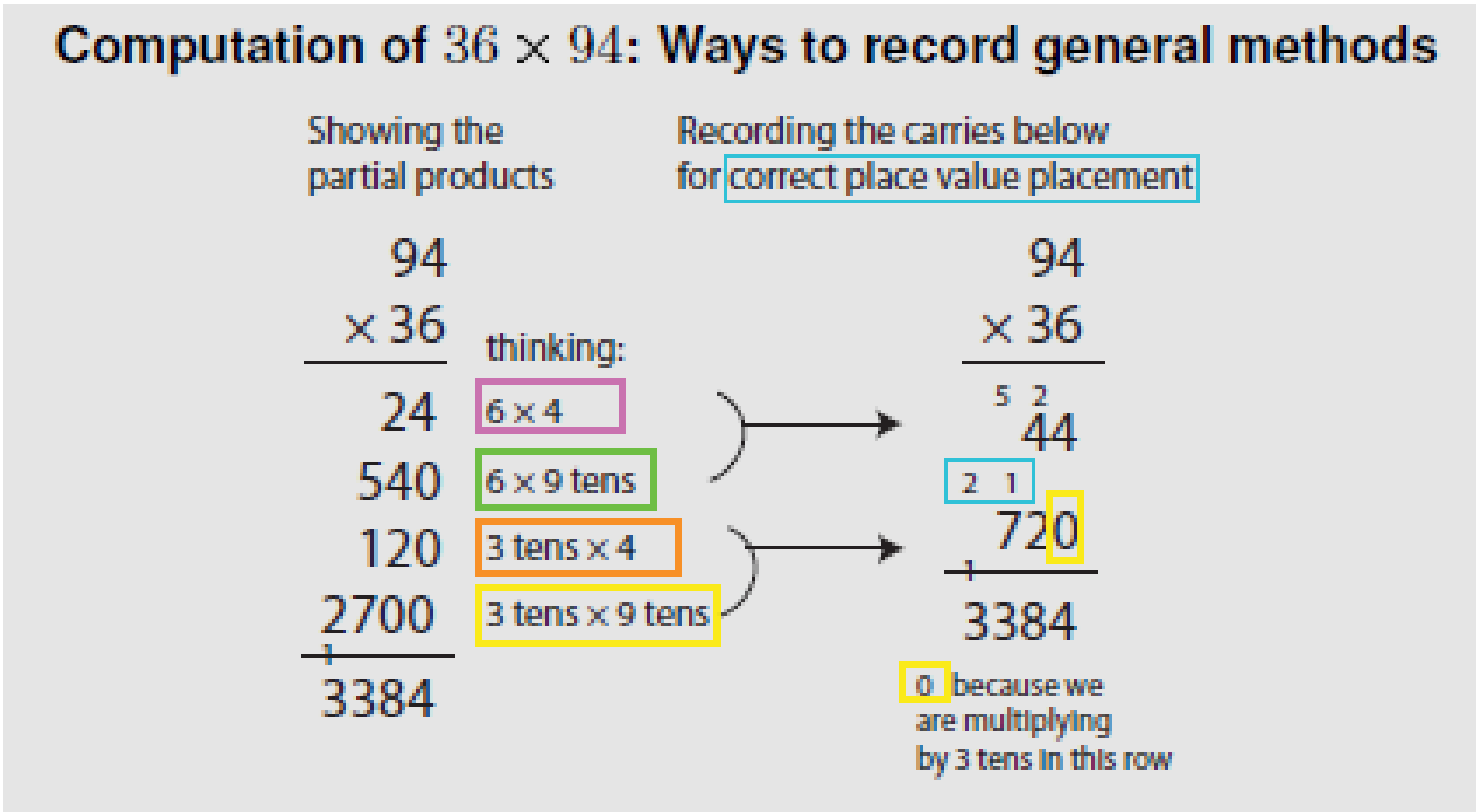
Computation of 36 × 94 connected with an area model



The products of like base-ten units are shown as parts of a rectangular region.

CCSS Numbers and Operations in Base-Ten Progression, April 2011

Multiplication Algorithms



CCSS Numbers and Operations in Base-Ten Progression, April 2012

What is Meant by “Standard Algorithm?”

“In mathematics, an algorithm is defined by its steps and not by the way those steps are recorded in writing. With this in mind, minor variations in methods of recording standard algorithms are acceptable.”

Fuson & Beckmann, 2013; NCSM Journal, p. 14

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Multiplication Algorithms

$$\begin{array}{r@{}l} 1 & \\ 2 & \\ 94 & \\ \times 36 & \\ \hline 564 & \\ {}^{1}282 & \\ \hline 3384 & \end{array}$$

From $30 \times 4 = 120$.
The 1 is 1 hundred,
not 1 ten.

$$\begin{array}{rcrcl} 94 & = & 90 & + & 4 \\ \times 36 & = & 30 & + & 6 \\ \hline 6 \times 4 & = & 24 & & \\ 6 \times 90 & = & 540 & & \\ 30 \times 4 & = & 120 & & \\ 30 \times 90 & = & 2700 & & \\ \hline & & {}^{1} & & \\ & & 3384 & & \end{array}$$

Fuson & Beckmann, 2013, p. 25

Multiplication Algorithms

$$
\begin{array}{r}
\scriptstyle 1 \\
\scriptstyle 2 \\
94 \\
\times\ 36 \\
\hline
564 \\
{}^{1}282 \\
\hline
3384
\end{array}
\qquad\qquad
\begin{array}{r}
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\hline
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120 \\
2700 \\
\hline
{}^{1} \\
3384
\end{array}
$$

Fuson & Beckmann, 2013, p. 25

Providing a Basis for Future Learning

36×94

	90	+ 4	
30	2700	120	→ 2820
+ 6	540	24	→ 564
			3384

$(x + 6)(x + 4)$

	x	$+ 4$	
x	x^2	$4x$	→ x^2+4x
$+ 6$	$6x$	24	→ $6x+24$
			$\mathbf{x^2+10x+24}$

A Functions Approach to Equation Solving

$y = 12x + 10$

- Solve for y when x = 3, 10, 100.
- Solve $70 = 12x + 10$

U.S. Shirts charges $12 per shirt plus $10 set-up charge for custom printing.

- What is the total cost of an order for 3 shirts?
- What is the total cost of an order for 10 shirts?
- What is the total cost of an order for 100 shirts?
- A customer spends $70 on T-shirts. How many shirts did the customer buy?

Analyze Worked Examples

Eliza solved this problem **correctly**. Here is her work:

$6 - k = -3$

$6 - k = -3$
$-6 \qquad\quad -6$
$-k = -9$
$\div -1 \quad \div -1$
$k = 9$

Why did Eliza subtract 6 FROM BOTH SIDES of the equation?

Why did Eliza divide by -1?

Your Turn:

$-6 - k = 3$

McGinn, Lange & Booth, MTMS, 2015

Analyze Worked Examples

Helaina tried to simplify this expression, but she **didn't** do it correctly. Here is her first step:

$5 - 4x + 2$

$5 - 4x + 2$

$4x - 5 + 2$

What did Helaina do wrong in her first step?

Would it have been okay to write 5+2-4x? Explain why or why not.

Your Turn:

$12x + 4 - 5x$

McGinn, Lange & Booth, MTMS, 2015

Effective
Mathematics Teaching Practices

1. Establish mathematics **goals** to focus learning.
2. Implement **tasks** that promote reasoning and problem solving.
3. Use and connect mathematical **representations.**
4. Facilitate meaningful mathematical **discourse**.
5. Pose purposeful **questions**.
6. Build **procedural fluency** from conceptual understanding.
7. Support **productive struggle** in learning mathematics.
8. **Elicit and use evidence** of student thinking.

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http://www.nctm.org/PtA/

Classroom Resources | Publications | Standards & Positions | Research & Advocacy | Conferences & Professiona

Principles to Actions

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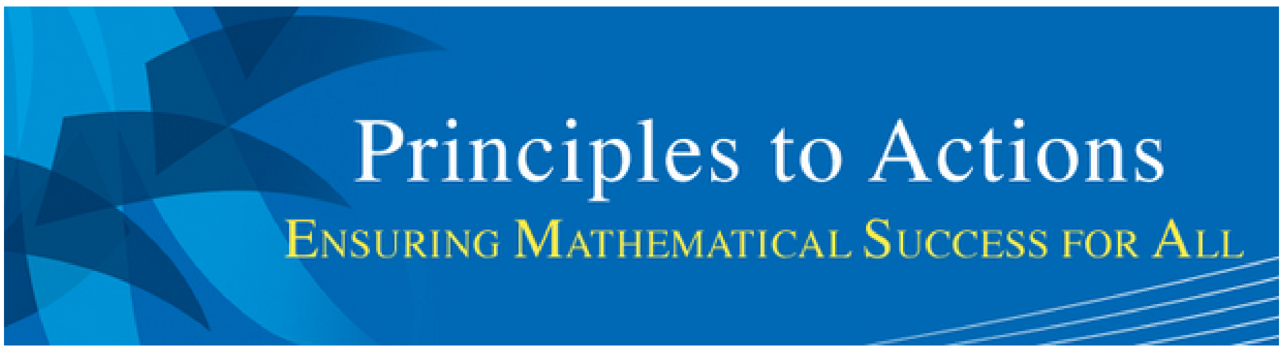


Specific, research-based teaching practices that are essential for a high-quality mathematics education for all students are combined with core principles to build a successful mathematics program at all levels.

Principles to Actions offers guidance to teachers, mathematics coaches, administrators, parents, and policymakers.

Principles to Actions Resources

- *Principles to Actions* Executive Summary (in English and Spanish)
- *Principles to Actions* overview presentation
- *Principles to Actions* professional development guide (Reflection Guide)
- Mathematics Teaching Practices presentations
  - Elementary case, multiplication (Mr. Harris)
  - Middle school case, proportional reasoning (Mr. Donnelly) (in English and Spanish)
  - High school case, exponential functions (Ms. Culver)
- *Principles to Actions* Spanish translation

http://www.nctm.org/PtAToolkit/

Principles to Actions Professional Learning Toolkit

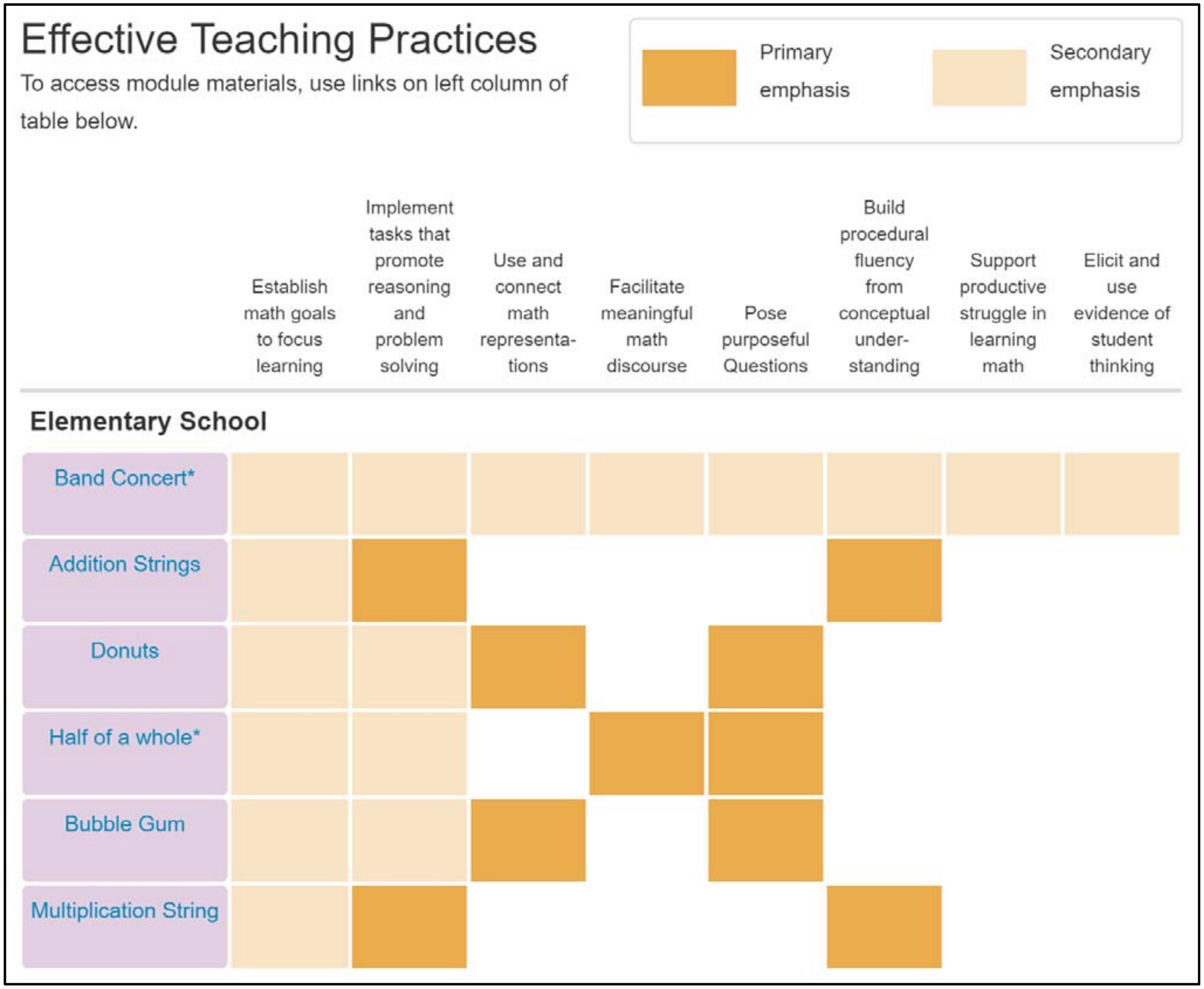
These grade-band specific professional learning modules are focused on the Effective Teaching Practices and Guiding Principles from *Principles to Actions: Ensuring Mathematical Success for All*.

Presentation, presenter notes, and required materials are provided in each module to support professional learning with teachers through analyzing artifacts of teaching (e.g., mathematical tasks, narrative and video cases, student work samples, vignettes) and abstracting from the specific examples general ideas about how to effectively support student learning.

The Teaching and Learning Modules were developed in collaboration with the Institute for Learning (IFL) at the University of Pittsburgh.

Learning modules are available exclusively to NCTM members. Limited open examples are provided for each grade level (denoted with *).

http://www.nctm.org/PtAToolkit/



Effective Teaching Practices

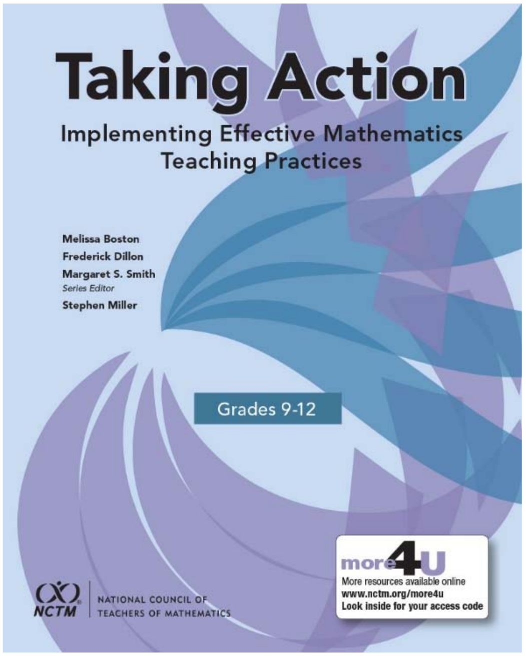
To access module materials, use links on left column of table below.

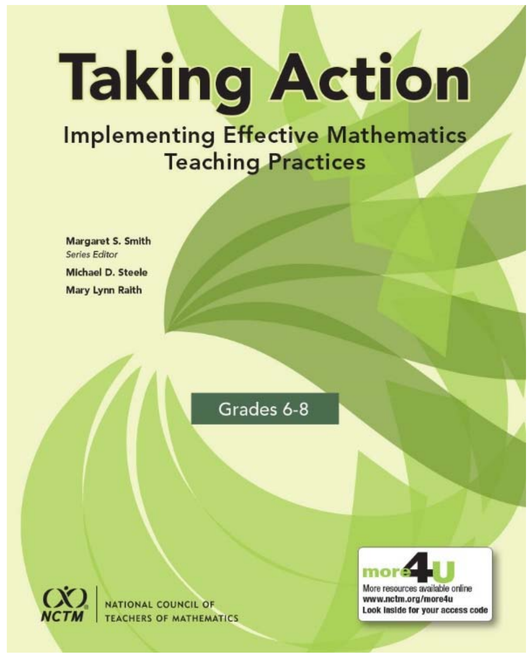
Primary emphasis

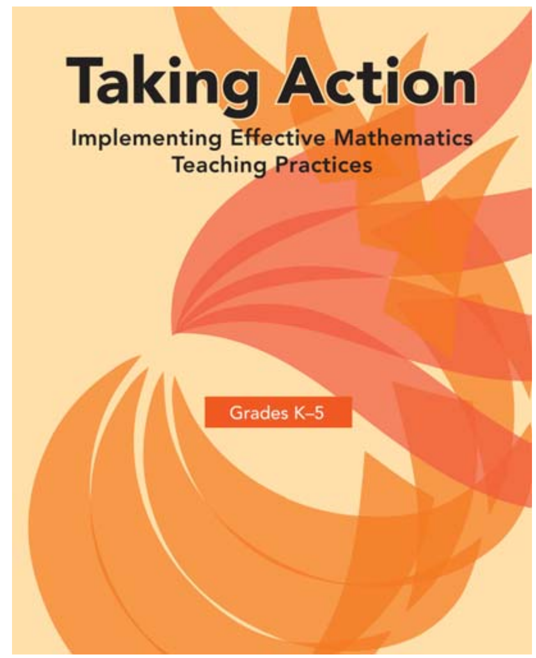
Secondary emphasis

	Establish math goals to focus learning	Implement tasks that promote reasoning and problem solving	Use and connect math representa-tions	Facilitate meaningful math discourse	Pose purposeful Questions	Build procedural fluency from conceptual under-standing	Support productive struggle in learning math	Elicit and use evidence of student thinking
Elementary School								
Band Concert*								
Addition Strings								
Donuts								
Half of a whole*								
Bubble Gum								
Multiplication String								

Taking Action with *Principles to Actions*







The Title Is Principles to *Actions*

Your Actions?

Thank You!

Diane Briars
djbmath@comcast.net