Strategies and Tasks to Build Procedural Fluency from Conceptual Understanding

Diane J. Briars

NCTM Past President

2018 NCTM Annual Meeting and Exposition

Washington, D.C.

April 26, 2018

The Problems

The Band Concert Problem

The third-grade class is responsible for setting up the chairs for their spring band concert. In preparation, they need to determine the total number of chairs that will be needed and ask the school's engineer to retrieve that many chairs from the central storage area. The class needs to set up 7 rows of chairs with 20 chairs in each row, leaving space for a center aisle. How many chairs does the school's engineer need to retrieve from the central storage area?

How might third grade students approach this problem?

Candy Jar Problem

A candy jar contains 5 Jolly Ranchers (JRs) and 13 Jawbreakers (JBs). Suppose you had a new candy jar with the same ratio of Jolly Ranchers to Jawbreakers, but it contained 100 Jolly Ranchers. How many Jawbreakers would you have? Explain how you know.

How might seventh grade students who had worked with ratios, but not proportions, approach this problem?

Exploring Representations for Multiplication
The Case of Mr. Harris and the Band Concert Task

Mr. Harris wanted his third-grade students to understand the structure of multiplication and decided to develop a task that would allow students to explore multiplication as equal groups through a familiar context—the upcoming spring band concert. He thought that the Band Concert Task (shown below) would prompt students to make or draw arrays and provide an opportunity to build conceptual understanding toward fluency in multiplying one-digit whole numbers by multiples of 10 using strategies based on place value and properties of operations—all key aspects of the standards for third grade students. He felt that the task aligned well with his math goals for the lesson and supported progress along math learning progressions, had multiple entry points, would provide opportunities for mathematical discourse, and it would challenge his students. As students worked on the task he would be looking for evidence that his students could identify the number of equal groups and the size of each group within visual or physical representations, such as collections or arrays, and connect these representations to multiplication equations.

> The third-grade class is responsible for setting up the chairs for the spring band concert. In preparation, the class needs to determine the total number of chairs that will be needed and ask the school's engineer to retrieve that many chairs from the central storage area. The class needs to set up 7 rows of chairs with 20 chairs in each row, leaving space for a center aisle. How many chairs does the school's engineer need to retrieve from the central storage area?

Mr. Harris began the lesson by asking students to consider how they might represent the problem. "Before you begin working on the task, think about a representation you might want to use and why, and then turn and share your ideas with a partner." The class held a short conversation sharing their suggestions, such as using cubes or drawing a picture. Then the students began working individually on the task.

As Mr. Harris made his way around the classroom, he noticed many students drawing pictures. Some students struggled to organize the information, particularly those who tried to represent each individual chair. He prompted these students to pause and review their work by asking, "So, tell me about your picture. How does it show the setup of the chairs for the band concert?" Other students used symbolic approaches, such as repeated addition or partial products, and a few students chose to use cubes or grid paper. He made note of the various approaches so he could decide which students he wanted to present their work, and in which order, later during the whole class discussion.

In planning for the lesson, Mr. Harris prepared key questions that he could use to press students to consider critical features of their representations related to the structure of multiplication. As the students worked, he often asked: "How does your drawing show the seven rows?" "How does your drawing show that there are 20 chairs in each row?" "Why are you adding all those twenties?" "How many twenties are you adding and why?"

He also noticed a few students changed representations as they worked. Dominic started to draw tally marks, but switched to using a table. When Mr. Harris asked her why, she explained she got tired of making all those marks. Similarly, Jamal started to build an array with cubes, but then switched to drawing an array. Their initial attempts were valuable, if not essential, in helping each of these students make sense of the situation.

Before holding a whole class discussion, Mr. Harris asked the students to find a classmate who had used a different representation and directed them to take turns explaining and comparing their work, as well as their solutions. He encouraged them to also consider how their representations were similar and different. For example, Jasmine who had drawn a diagram compared her work with Kenneth who had used equations (see reverse for copies of their work). Jasmine noted that they had gotten the same answer and Kenneth said they both had the number 20 written down seven times. Molly, in particular, was a student who benefited from this sharing process because she was able to acknowledge how confused she had gotten in drawing all those squares (see reverse side) and had lost track of her counting. Her partner helped her mark off the chairs in each row in groups of ten and recount them. The teacher repeated this process once more as students found another classmate and held another sharing and comparing session.

Written by DeAnn Huinker (University of Wisconsin-Milwaukee), drawing on experiences with teachers and students in the Milwaukee area. This case is intended to support the Guiding Principle for Teaching and Learning in *Principles to Actions: Ensuring Mathematical Success for All* (Reston, Va.: National Council of Teachers of Mathematics 2014).

During the whole class discussion, Mr. Harris asked the presenting students to explain what they had done and why and to answer questions posed by their peers. He asked Jasmine to present first since her diagram accurately modeled the situation and it would likely be accessible to all students. Kenneth went next as his approach was similar to Jasmine's but without the diagram. Both clearly showed the number 20 written seven times. Then Teresa presented. Her approach allowed the class to discuss how skip counting by twenties was related to the task and to multiplication, a connection not apparent for many students. Below is an excerpt from this discussion.

Mr. H: So, Teresa skip counted by twenties. How does this relate to the Band Concert situation?

Connor: She counted seven times like she wrote on her paper.

Mr. H: I'm not sure I understand, can someone add on to what Connor was saying?

Grace: Well each time she counted it was like adding 20 more chairs, just like what Kenneth did.

Mr. H: Do others agree with what Grace is saying? Can someone explain it in their own words?

Mason: Yeah, the numbers on top are like the 7 rows and the numbers on the bottom are the total number of chairs for that many rows.

Mr. H: This is interesting. So what does the number 100 mean under the 5?

Mason: It means that altogether five rows have 100 total chairs, since there are 20 chairs in each row.

Mr. H: Then what does the 140 mean?

Mason: It means that seven rows would have a total of 140 chairs.

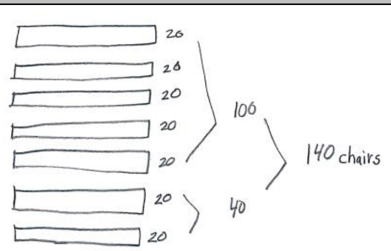
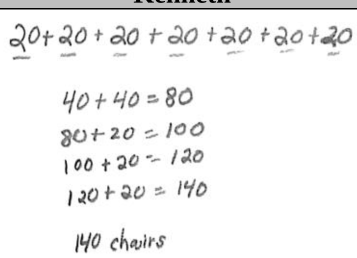
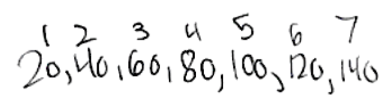
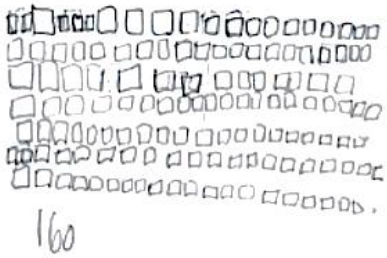
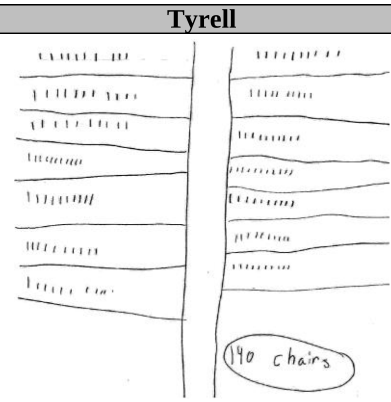
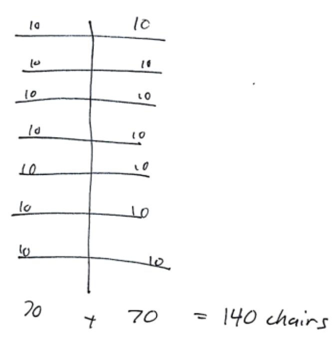
[Mr. Harris paused to write this equation on the board: 7 x 20 = 140.]

Mr. H: Some of you wrote this equation on your papers. How does this equation relate to each of the strategies that we have discussed so far? Turn and talk to a partner about this equation.

[After a few minutes, the whole class discussion continued and Grace shared what she talked about with her partner.]

Grace: Well, we talked about how the 7 means seven rows like Jasmine showed in her picture and how Teresa showed. And the 20 is the number of chairs that go in each row like Jasmine showed, and like how Kenneth wrote down. Teresa didn't write down all those twenties but we know she counted by twenty.

Toward the end of the lesson, Mr. Harris had Tyrell and Ananda present their representations because they considered the aisle and worked with tens rather than with twenties. After giving the students a chance to turn and talk with a partner, he asked them to respond in writing whether it was okay to represent and solve the task using either of these approaches and to justify their answers. He knew this informal experience with the distributive property would be important in subsequent lessons and the student writing would provide him with some insight into whether or not his students understood that quantities could be decomposed as a strategy in solving multiplication problems.

Jasmine	**Kenneth**	**Teresa**
		
Molly	**Tyrell**	**Ananda**
		

Exploring Proportional Relationships: *The Case of Mr. Donnelly*

Mr. Donnelly wanted his students to understand that quantities that are in a proportional (multiplicative) relationship grow at a constant rate and that there were three key strategies that could be used to solve problems of this type – scaling up, scale factor, and unit rate. He selected the Candy Jar task for the lesson since it was aligned with his goals, was cognitively challenging, and had multiple entry points.

> A candy jar contains 5 Jolly Ranchers (JRs) and 13 Jawbreakers (JBs). Suppose you had a new candy jar with the same ratio of Jolly Ranchers to Jawbreakers, but it contained 100 Jolly Ranchers. How many Jawbreakers would you have? Explain how you know.

As students began working with their partners on the task, Mr. Donnelly walked around the room stopping at different groups to listen in on their conversations and to ask questions as needed (e.g., How did you get that? How do you know that the new ratio is equivalent to the initial ratio?). When students struggled to figure out what to do he encouraged them to look at the work they had done the previous day that included producing a table of ratios equivalent to 5 JRs: 13 JBs and a unit rate of 1 JR to 2.6 JBs. He also encouraged students to consider how much bigger the new candy jar must be when compared to the original jar.

As he made his way around the room Mr. Donnelly also made note of the strategies students were using (see reverse side) so he could decide which groups he wanted to have present their work. After visiting each group, he decided that he would ask Groups 4, 5, and 2 to share their approaches (in this order) since each of these groups used one of the strategies he was targeting and the sequencing reflected the sophistication and frequency of strategies.

During the discussion he asked the presenters (one student from each of the targeted groups) to explain what their group did and why and he invited other students to consider whether the approach made sense and to ask questions. He made a point of labeling each of the three strategies, asking students which strategy was most efficient in solving this particular task, and asking students questions that helped them make connections between the different strategies and to the key ideas he was targeting. Specifically he wanted students to see that that the scale factor identified by Group 5 was the same as the number of entries in the table created by Group 4 (or the number of small candy jars that it would take to make the new candy jar) and that the unit rate identified by Group 2 was the factor that connected the JRs and JBs in each row of the table.

Below is an excerpt from the discussion that took place around the unit rate solution that was presented by Jerry from Group 2.

Jerry: We figured that there was 1 JR for 2.6 JBs so that a jar with 100 JRs would have 260 JBs. So we got the same thing as the other groups.
Mr. D.: Can you tell us how you figured out that there was 1 JR for 2.6 JBs?
Jerry: We divided 13 by 5.
Mr. D.: Does anyone have any questions for Jerry? (pause) Danielle?
Danielle: How did you know to do 13 ÷ 5?
Jerry: See we wanted to find out the number of JBs for 1 JR. So if we wanted JRs to be 1, we needed to divide it by 5. So now we needed to do the same thing to the JBs.
Danielle: So how did you then get 260 JBs?
Jerry: Well once we had 1 JR to 2.6 JBs it was easy to see that we needed to multiply each type of candy by 100 so we could get 100 JRs.

Written by Margaret Smith (University of Pittsburgh), drawing on two sources: *Principles to Actions: Ensuring Mathematical Success for All* (Reston, Va.: National Council of Teachers of Mathematics, 2014) and *Improving Instruction in Rational Numbers and Proportionality: Using Cases to Transform Mathematics Teaching and Learning,* vol. 1, by Margaret S. Smith, Edward A. Silver, and Mary Kay Stein (New York: Teachers College Press, 2005). This activity is intended to support the Teaching and Learning Guiding Principle in *Principles to Actions: Ensuring Mathematical Success for All* (NCTM 2014).

Mr. D.: So Jerry's group multiplied by 100 but Danielle and her group (Group 5) multiplied by 20. Can they both be right? Amanda?

Amanda: Yes. Jerry's group multiplied 1 and 2.6 by 100 and Danielle and her group multiplied 5 and 13 by 20. Jerry's group multiplied by a number 5 times bigger than Danielle's group because their ratio was 1/5 the size of the ratio Danielle's group used. So it is the same thing.

Mr. D.: Do others agree with what Danielle is saying? *(Students are nodding their heads and giving Danielle a thumbs up.)* So what is important here is that both groups kept the ratio constant by multiplying both the JRs and JBs by the same amount. We call what Jerry and his group found the **unit rate**. A unit rate describes how many units of one quantity (in this case JBs) correspond to one unit of another quantity (in this case JRs). *(Mr. Donnelly writes this definition on the board.)*

Mr. D.: I am wondering if we can relate the unit rate to the table that Group 4 shared. Take 2 minutes and talk to your partner about this. (*2 minutes pass*)

Mr. D.: Kamiko and Jerilyn (from Group 4), can you tell us what you were talking about?

Kamiko: We noticed that if we looked at any row in our table that the number of JBs in the row was always 2.6 times the number of JRs in the same row.

Mike: Yeah we saw that too. So it looks like any number of JRs times 2.6 will give you the number of JBs.

Mr. D.: So what if we were looking for the number of JBs in a jar that had 1000 JRs?

Mike: Well the new jar would be 1000 times bigger so you multiply by 1000.

Mr. D.: So take 2 minutes and see if you and your partner can write a rule that we could use to find the number of JBs in a candy jar no matter how many JRs are in it.

(After 2 minutes the discussion continues.)

Towards the end of the lesson Mr. Donnelly placed the solution produced by Group 1 on the document camera and asked students to decide whether or not this was a viable approach to solving the task and to justify their answer. He told them they would have five minutes to write a response that he would collect as they exited the room. He thought that this would give him some insight as to whether or not individual students were coming to understand that ratios needed to grow at constant rate that was multiplicative not additive.

<table>
<tr><th>Group 1 (1st solution)
(incorrect additive)</th><th>Groups 3 and 5
(scale factor)</th><th>Groups 1 (2nd solution), 4 and 7
(scaling up)</th></tr>
<tr><td>100 JRs is 95 more than the 5 we started with. So we will need 95 more JBs than the 13 I started with.

5 JRs + 95 JRs = 100 JRs
13 JBs + 95 JBs = 108 JBs</td><td>You had to multiply the five JRs by 20 to get 100, so you'd also have to multiply the 13 JBs by 20 to get 260.

(x20)
5 JRs → 100 JRs
13 JBs → 260 JBs
(x20)</td><td><table><tr><th>JR</th><th>JB</th><th>JR</th><th>JB</th></tr><tr><td>5</td><td>13</td><td>55</td><td>143</td></tr><tr><td>10</td><td>26</td><td>60</td><td>156</td></tr><tr><td>15</td><td>39</td><td>65</td><td>169</td></tr><tr><td>20</td><td>52</td><td>70</td><td>182</td></tr><tr><td>25</td><td>65</td><td>75</td><td>195</td></tr><tr><td>30</td><td>78</td><td>80</td><td>208</td></tr><tr><td>35</td><td>91</td><td>85</td><td>221</td></tr><tr><td>40</td><td>104</td><td>90</td><td>234</td></tr><tr><td>45</td><td>117</td><td>95</td><td>247</td></tr><tr><td>50</td><td>130</td><td>100</td><td>260</td></tr></table></td></tr>
<tr><th>Group 2 (unit rate)</th><th colspan="2">Group 6 (scaling up)</th></tr>
<tr><td>Since the ratio is 5 JRs for 13 JBs, we divided 13 by 5 and got 2.6. So that would mean that for every 1 JR there are 2.6 JBs. So then you just multiply 2.6 by 100.

(x100)
1 JR → 100 JRs
2.6 JBs → 260 JBs
(x100)</td><td colspan="2"><table><tr><td>JRs</td><td>5</td><td>10</td><td>20</td><td>40</td><td>80</td><td>100</td></tr><tr><td>JBs</td><td>13</td><td>26</td><td>52</td><td>104</td><td>208</td><td>260</td></tr></table>
We started by doubling both the number of JRs and JBs. But then when we got to 80 JRs we didn't want to double it anymore because we wanted to end up at 100 JRs and doubling 80 would give me too many. So we noticed that if we added 20 JRs: 52 JBs and 80 JRs: 208 JBs we would get 100 JRs:260 JBs.</td></tr>
<tr><th colspan="3">Group 8 (scaling up)</th></tr>
<tr><td colspan="3">We drew 100 JRs in groups of 5. Then we put 13 JBs with each group of 5 JRs. We then counted the number of JBs and found we had used 260 of them.</td></tr>
</table>

Effective Teaching Practice:
Build Procedural Fluency from Conceptual Understanding.

Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.

What are ***teachers*** doing?	What are ***students*** doing?
Providing students with opportunities to use their own reasoning strategies and methods for solving problems. Asking students to discuss and explain why the procedures that they are using work to solve particular problems. Connecting student-generated strategies and methods to more efficient procedures as appropriate. Using visual models to support students' understanding of general methods. Providing students with opportunities for distributed practice of procedures.	Making sure that they understand and can explain the mathematical basis for the procedures that they are using. Demonstrating flexible use of strategies and methods while reflecting on which procedures seem to work best for specific types of problems. Determining whether specific approaches generalize to a broad class of problems. Striving to use procedures appropriately and efficiently.

From *Principles to Actions: Ensuring Mathematical Success for All*, NCTM, 2014, pp 47-48.