Nurturing Inquiry in Your Calculus Class:

three sets of three questions, with implications for Taylor Series

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Finish this statement:

- I believe our primary concern in teaching students is…

"…**to empower** them to lead meaningful adult lives."

Empowered learners: what do they look like?

Motivated, Resourceful, Self-Regulated Learners are…

- self-appraising (who am I?)
- self-directed (who do I want to become?)
- self-assessing (how am I doing?)
- self-advocating (what is my voice?)

The Students are Watching…

- Do we want students to ask questions?
- Do we want them to grapple with difficulty?
- Do we want them to have some templates/scripts with which they can approach difficult tasks?

…then we need to provide these models in our classrooms.

Let us then begin with this end in mind.

Mathematics…

- $\mu\alpha\theta\eta\tau\eta\varsigma$ – means 'one who is training, learning a discipline,' (a *disciple*, actually!)

- What patterns do I see in the world?

observation

- What's happening at a deeper level?

conjecture

- Can I be sure of this, in general?

proof

- What good is it, anyway?

application

Purposes of today's talk

- To make the case that our students' ***deep cognitive engagement*** is at the heart of learning.
- To explore some ways in which we can ***stimulate <u>student</u> inquiry*** by modeling good questions in class.
- To provide specific moves that assist our work, specifically with ***the generative, creative work*** of proofs and problem-solving.

Which proofs? In our Calculus classes…

…we prove most results along the way (as we need them) including:

$\lim_{\theta\to 0}\frac{sin(\theta)}{\theta}=1$ $\lim_{\theta\to\infty}\frac{sin(\theta)}{\theta}=0$ $\lim_{x\to\infty}\frac{a_0\cdot x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+k}{b_0\cdot x^n+b_1x^{n-1}+b_2x^{n-2}+\cdots+k}=\frac{a_0}{b_0}$

$\frac{d}{dx}[x^n]=n\cdot x^{n-1}$ $\frac{d}{dx}[sin(x)]=cos(x)$, and five other trig functions' derivatives

Derivatives of other function models: $\frac{d}{dx}[e^x]$, $\frac{d}{dx}[a^x]$, $\frac{d}{dx}[ln(x)]$, $\frac{d}{dx}[log_a(x)]$

…and those of various function operations: $\frac{d}{dx}[k\cdot f(x)]$, $\frac{d}{dx}[f(x)+g(x)]$, $\frac{d}{dx}[f(x)\cdot g(x)]$

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$, $\frac{d}{dx}[f(g(x))]$, $\frac{d}{dx}[f^{-1}(x)]$, $\frac{d}{dx}\int_a^x f(t)dt=f(x)$, $\int_a^b f'(t)dt=f(b)-f(a)$

…endeavoring at all times to use the language of function operations (composites, products, etc.) so as to enhance students' capacity to connect this work to previous learning.

Punchline unveiled:

The "three sets of three questions"

- During proofs and problem-solving

(1) How is that legal? (2) How is that helpful? (3) How is that intuitive?

- When generating/initiating creative action

(A) Where am I? (B) Where do I want to be? (C) What tools do I have to get there?

- When motivating Taylor series

(i) Can we model this? (ii) For what values is the model useful? (iii) How good is the model?

One Observation, First

What do our students typically ask when learning?

- The "how to tutor a friend" myth…

(the wrong three questions to ask)

(1) Did you get problem #17?

(2) Can I see your solution?

(3) Oh, I get how this one works…!

Do we want students to be able to read math? Write it? Both?
They need our help to debunk myths about learning and studying.

The first set of three:

replacing their questions with better ones
during proofs and problem-solving

(1) How is that legal?

(what's the mathematics behind that action?)

(2) How is that helpful?

(did that get us closer to our solution/goal?)

(3) How is that intuitive?

(how would I have [in a million years] thought to do that…?)

Let's prove…(on paper)

$$\frac{d}{dx}[k \cdot f(x)] = \ \dots ?$$

$$\frac{d}{dx}[f(x) \cdot g(x)] = \ \dots ?$$

$$\frac{d}{dx}[sin(x)] = \ \dots ?$$

We'll need to refer to earlier results for this last one…namely:

$sin(A+B) = sin(A)cos(B) + cos(A)sin(B)$ and $\lim_{\theta \to 0} \frac{sin(\theta)}{\theta} = 1$

The second set:
"How would I have thought of that?"

(A) What do you see? (where am I?)

(B) What do you want? (where do I want to be?)

(C) What tool(s) do you have to get there?

Let's prove…(on paper)

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The third set:
Why even study Taylor series, anyway?

(i) Can we model this function with an easier one?

We wish to have a function whose properties are identical to our generating function at $x = a$ and similar to it near there. Let's examine $y = e^x$.

(ii) For what values is the model 'good?'

If there are boundaries on the usefulness of this model, we need to know those. Nice motivator: $y = \frac{1}{1-x}$

(iii) How good is the model?

Once we've built this function, we need to know, at least roughly, how close we are to the actual values. Let's generate Taylor's Theorem and Lagrange Error bound

Maclaurin series for $y = e^x$

The idea is that our lovely, algebraic polynomial will MODEL the more complicated transcendental function. This allows us to use simple addition to compute values, which (for modern computing in particular) is awesome.

So, we wish to let $T(0) = f(0)$, making $T(x) \approx f(x)$ for x near to 0

We then make sure that $T'(0) = f'(0)$, making $T'(x) \approx f'(x)$ for x near to 0

Next we force $T''(0) = f''(0)$, making $T''(x) \approx f''(x)$ for x near to 0

…and so on…

Framing the question this way helps our students see why we even wish to create these models in the first place!

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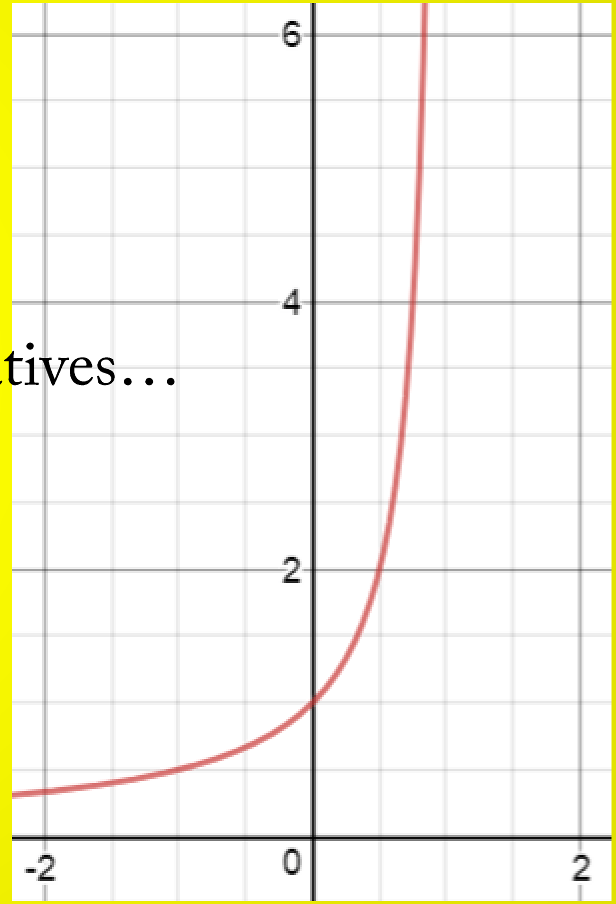
Maclaurin series for $y = \frac{1}{1-x}$

- First, look at the generating function itself
- Find the Maclaurin series
  - polynomial long division
  - Factoring
  - Maclaurin-style, writing the polynomial to specifications, determined by the derivatives…
- Conceptualize 'convergence'
- Look at the successive partial sums



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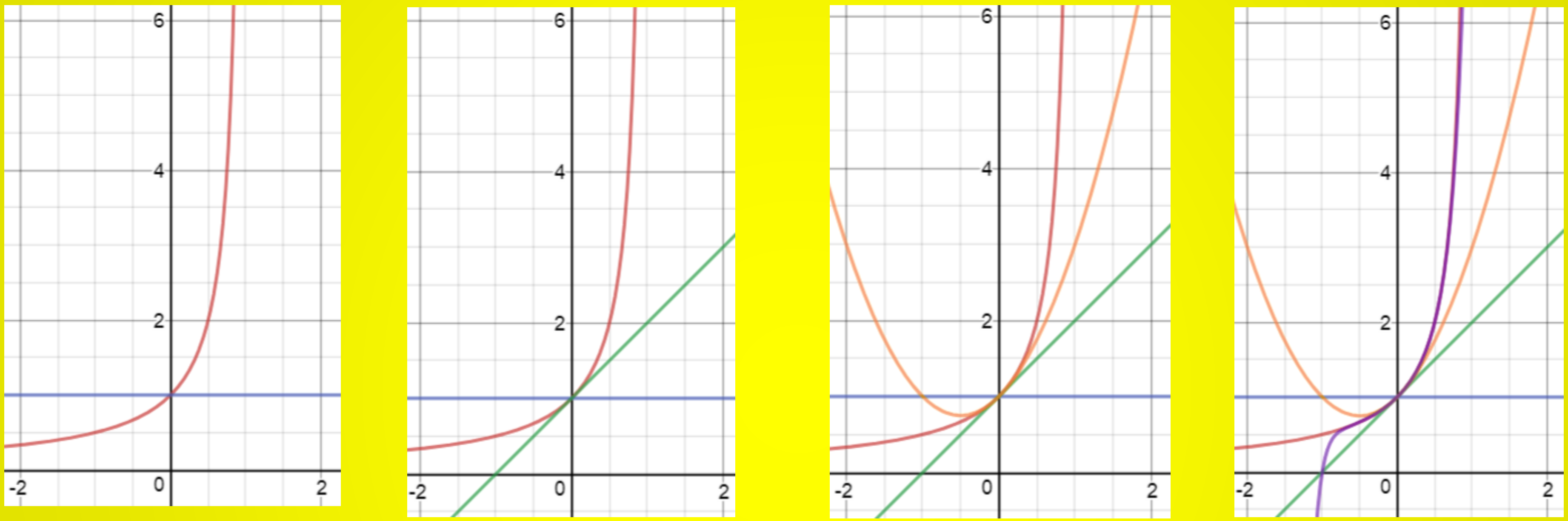

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Taylor's Theorem: a proof with a synthesis of FTC, product rule, & substitution

Begin with: $f(x) = f(a) + \int_a^x f'(t)dt$

(NOTE: This means that $f'(x) = f'(a) + \int_a^x f''(t)dt$;

further, $x \cdot f'(x) = x \cdot f'(a) + x \cdot \int_a^x f''(t)dt$; we'll use this below)

$$f(x) = f(a) + \int_a^x f'(t)dt$$

now use the integration product rule: u=f'(t) and dv=dt

$$= f(a) + t \cdot f'(t)\Big|_{t=a}^{t=x} - \int_a^x t \cdot f''(t)dt$$

$$= f(a) + x \cdot f'(x) - a \cdot f'(a) - \int_a^x t \cdot f''(t)dt$$

$$= f(a) + x \cdot f'(a) + \boldsymbol{x \cdot \int_a^x f''(t)dt} - \boldsymbol{a \cdot f'(a)} - \int_a^x t \cdot f''(t)dt$$

$$= f(a) + x \cdot f'(a) - \boldsymbol{a \cdot f'(a)} + \int_a^x \boldsymbol{x \cdot f''(t)dt} - \int_a^x t \cdot f''(t)dt$$

$$= f(a) + (x - a) \cdot f'(a) + \int_a^x (x - t) \cdot f''(t)dt$$

Taylor's Theorem: an 'integrative' proof

Recall that starting from: $f(x) = f(a) + \int_a^x f'(t)dt$, we generated , from the previous page:

$$f(x) = f(a) + (x-a)\cdot f'(a) + \int_a^x (x-t)\cdot f''(t)dt$$

$$= f(a) + f'(a)\cdot(x-a) + \int_a^x f''(t)\cdot(x-t)dt$$ **use product rule again!**

$$= f(a) + f'(a)\cdot(x-a) + \left[-f''(t)\cdot\frac{1}{2}(x-t)^2\right]\Big|_{t=a}^{t=x} + \int_a^x \frac{1}{2}(x-t)^2\cdot f'''(t)dt$$

$$= f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2 + \int_a^x f'''(t)\cdot\frac{1}{2}(x-t)^2 dt$$ **product rule**

$$= f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2 + \left[-f'''(t)\cdot\frac{1}{2}\frac{1}{3}(x-t)^3\right]\Big|_{t=a}^{t=x} + \int_a^x \frac{1}{3!}(x-t)^3\cdot f^{(4)}(t)dt$$

$$= f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdot\frac{f'''(a)}{3!}(x-a)^3 + \int_a^x f^{(4)}(t)\cdot\frac{1}{3!}(x-t)^3 dt$$

=…and so on: $f(x) = \sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i + \int_a^x f^{(n+1)}(t)\cdot\frac{1}{n!}(x-t)^n dt$

…further notes on Taylor's remainder term:

NOTE: with respect to a positive-valued function *f*,

$$\int_a^x f^{(n+1)}(t) \cdot \frac{1}{n!}(x-t)^n dt \le \int_a^x \left|f^{(n+1)}(t)\right|_{max} \cdot \frac{1}{n!}(x-t)^n dt$$

This remainder integral now has a constant factor, $\left|f^{(n+1)}(t)\right|_{max\ on\ [a,x]}$, which we can distribute out, making the remaining integrand very easy to integrate! Moreover, this gives us the supposedly-tricky Lagrange form of the upper-bound for the error.

$$\left|\int_a^x f^{(n+1)}(t) \cdot \frac{1}{n!}(x-t)^n dt\right| \le \int_a^x \left|f^{(n+1)}(t)\right|_{max\ on\ [a,x]} \cdot \frac{1}{n!}(x-t)^n dt$$

$$= \left|f^{(n+1)}(t)\right|_{max} \cdot \int_a^x \frac{1}{n!}(x-t)^n dt$$

$$= \left|f^{(n+1)}(t)\right|_{max} \cdot \frac{1}{n!} \frac{(x-a)^{n+1}}{n+1}$$

$$= \left|f^{(n+1)}(t)\right|_{max} \cdot \frac{(x-a)^{n+1}}{(n+1)!}$$ …as needed!

Other practices for success in Calculus

- Math histories: students write 2-4 pages about their relationship with math over the years…a trove of information!
- **Formative feedback, decoupled in time from delivery of grades. Required revisitation of all test problems**, and modeling/practice in class: homework revision.
- Keen attention in the first weeks to the form of students' written work.
- Strong exhortation to use study groups and TALK
- Music, videos, and bad jokes distributed liberally.
- Use of tau, the appropriate circle constant. It helps. Read Michael Hartl's TAU MANIFESTO, please.

Resources/authors for further study and reflection:

- Brown, Roedinger, and McDaniel, *Make It Stick: The Science of Successful Learning*
- Daniel Willingham, *Why Don't Students Like School? A Cognitive Scientist Answers Questions About How the Mind Works and What It Means*
- Carol Dweck, *Mindset: The New Psychology of Success*
  - http://www.youtube.com/watch?v=ICILzbB1Obg
- Malcolm Gladwell, *Outliers: The Story of Success*
- Dan Pink, *Drive: The Surprising Truth About What Motivates Us*
- Claude Steele, *Whistling Vivaldi: How Stereotypes Affect Us, and What We Can Do*
- Angela Duckworth's extensive work on grit development

A Message on 'Fit'

"Good teachers join self and subject and students in the fabric of life."

–Parker Palmer, p.11, *The Courage to Teach*

Thank you for your interest and participation in this NCTM 2018 session.

Please influence the Ferg'(t) function; send feedback to:

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Appendix note: a few other juicy results that implement our inquiry strategies

- Maclaurin series for sin(x) and Vieta's formulas provide us with a solution to the 1644 'Basel Problem'

$(1 + ax)(1 + bx)(1 + cx) \cdot \cdots (1 + kx) = ???$ in standard form

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = ???$$