Teaching Preservice K-8 Teachers Geometry and Reasoning Through Purposeful Play

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MA127 Geometrical Explorations:

- A general education course in the math department.
- Encouraged for Education majors; open to others.
- Required for the Math Specialization.
- Diverse mathematical background and comfort level.

Goals:

- Engage students in active learning, problem solving, and mathematical reasoning.
- Develop students' ability to discuss mathematical orally and in writing.
- Encourage prospective teachers and others to see mathematics as a creative endeavor.
- Create a deeper understanding and appreciation of geometry in students.

Guiding philosophy:

- All students can think mathematically.
- Simply stated problems can enable all students to enter yet still lead to deep discovery and understanding.
- Doing mathematics is the best way to learn and remember what was learned.
- Making mistakes in not only okay, but an important part of the learning process.
- Risk-taking, perseverance, reasoning, problem solving, communication are needed by all.

Level of mathematics:

- Geometrical concepts not advanced
  - Most topics appear (or could appear) in elementary or middles school mathematics classes
  - Explore connections and the why
- Challenge comes in the reasoning: making connections between ideas, applying what they've learned to new situations and solving non-routine problems.
- The biggest struggle for many students is communicating mathematics.

In the classroom:

- Little or no lecture
- Materials: pentominos, tangrams, geoboards, pattern blocks, tagboard Pythagorean puzzle pieces, tagboard regular polygon stencils, Polydron, colored pencils, scissors, rulers, graph paper, Miras, protractors, compasses, geogebra, plastic spheres, rubber bands, etc. (Many of these can be made with tagboard)
- Mix of open-ended exploration and guided worksheets
- Students apply what they know, dust off cobwebs, and teach others
- Short introductions and follow-up discussions

Selected Topics:

- Pentominos as an introduction to area model of multiplication and conservation of area.

- Geoboard activity-area of triangles, open-ended exploration, multiple solutions
- Writing for clarity
- Area formulas by dissection, duplication, and/or algebra--activity and paper
- Pythagorean theorem by dissection and algebra—activity and paper
- Regular polygons and tiling—sum of angles in a convex n-gon, 360 degrees surround each vertex
- Three-dimensional Solids
    - Regular and semiregular polyhedral
    - Prisms and cylinders, pyramids and cones
    - Volume and surface area, estimation and precision, growth factors
- Isometries (actions, properties, compositions, finding the isometry, connections to coordinate geometry). Paper and geogebra activities.
- Dilations and similarity (Centers and scale factors, compositions with isometries, similar triangles, proportionality proof of Pythagorean theorem.
- Star polygons, connecting fractions with geometry and art
- Golden triangles, golden rectangles, classical constructions
- Spherical and hyperbolic geometry-triangles and parallels

Assessment:

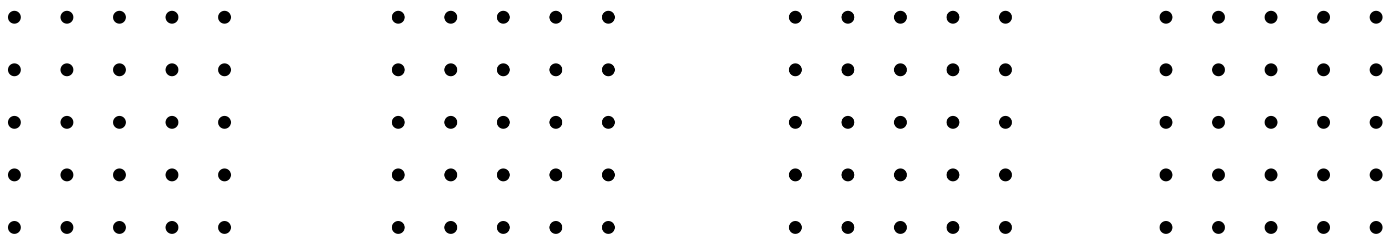
- Observation of in-class work (mix of open-ended exploration and guided worksheets)
- Graded homework assignments
- Journal essays
    - Mathematical autobiography
    - Polya's problem-solving strategies
    - A Mathematician's Lament by Paul Lockhart https://www.maa.org/external_archive/devlin/LockhartsLament.pdf
    - Growth Mindset, Taking Risks, and Perseverance in Mathematics
- Informal proofs (letters to a friend)
    - Deriving the area formula of a trapezoid
    - The Pythagorean theorem
- Two tests:
    - Routine problems on geometric facts
    - Questions related to class explorations
    - One reasoning problem on each test (options to choose from)
- Final portfolio
    - The Nature of Mathematics/Geometry (discovery, conjecture, proof, creative)
        - Proof of trapezoid area or Pythagorean theorem
        - Representative activity or homework
        - Reflection on how these artifacts demonstrate the nature of mathematics
    - Growth as a mathematician/geometer
        - Mathematical autobiography written on day one
        - Representative activity or homework
        - Essay about the artifact and your growth

Squares and Triangles on a Geoboard

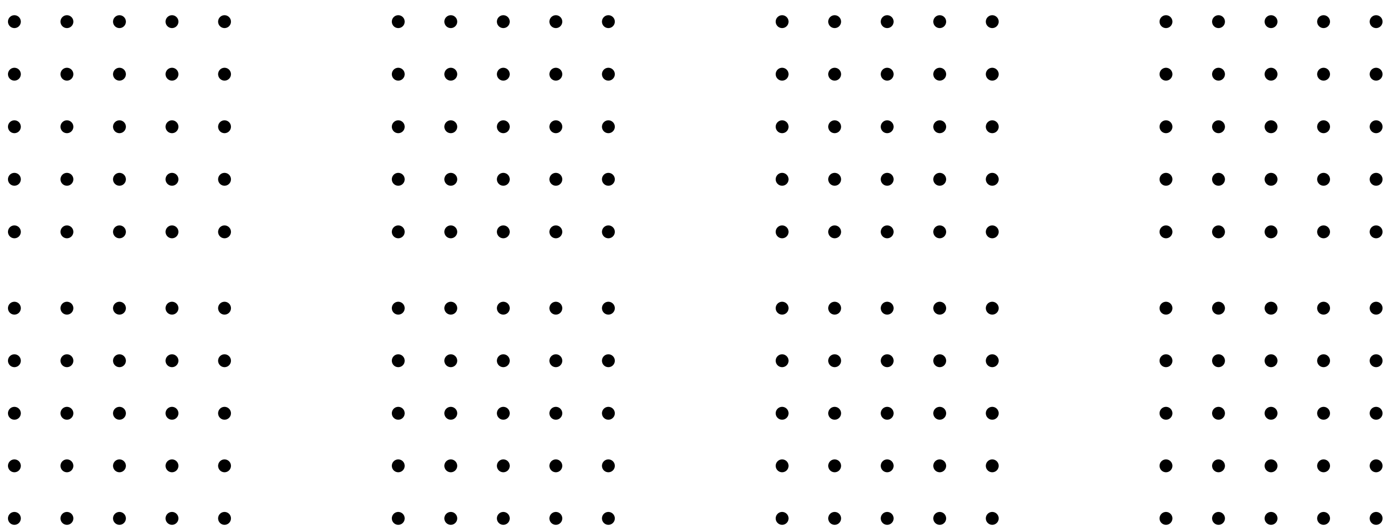
(Note: https://www.mathlearningcenter.org/web-apps/geoboard/ has virtual geoboards)

On a 5x5 geoboard (or dot paper), let the horizontal (and vertical) distance between neighboring points be 1 unit.

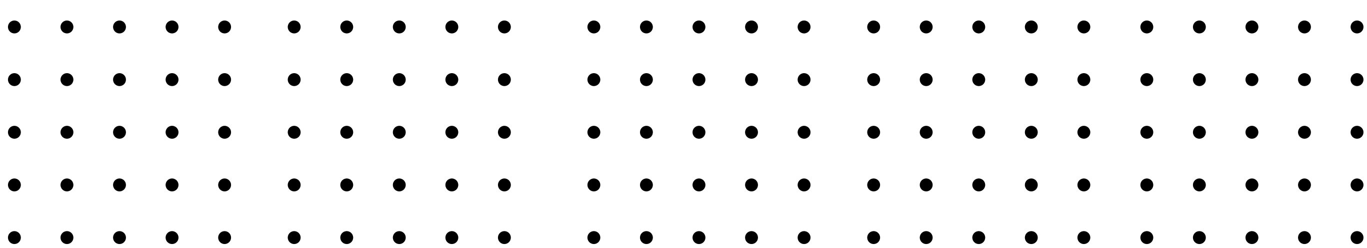
1. Eight squares, each with different whole number areas, can be shown on a 5x5 geoboard or a 5x5 dot paper grid. Find them! (Note that the vertices of the squares must be points on the grid.)



2. Also on a 5x5 geoboard (or dot paper), find triangles with areas ½, 1, 1 ½, 2, 2 ½, 3, 3 ½, 4, 4 ½, 5, 5 ½, 6, 6 ½, 7, 7 ½, and 8.



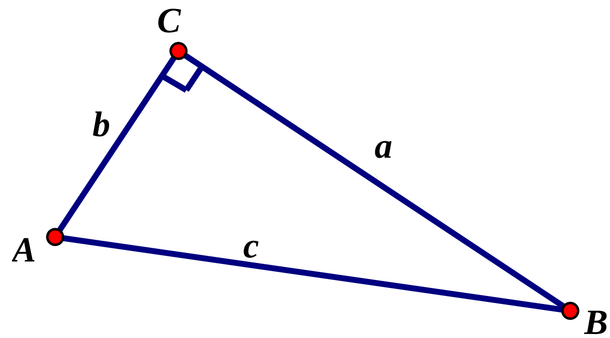
3. How many squares of different areas do you expect can be created on a 6 x 6 geoboard? Support your answer.



4. What other investigations might follow from this worksheet? If time permits, explore one!

The Pythagorean Theorem

Known prior to 500 B.C., the Pythagorean Theorem is a theorem that allows us to compute lengths associated with right triangles. The longest side, called the hypotenuse, of a right triangle is always opposite the right angle. The other two sides of the right triangle are called the legs.



1. Identify which lower case letter represents the hypotenuse in the triangle above.

2. Which angle appears to be the smallest in the triangle?

3. Which side appears to be the shortest in the triangle?

4. How are the answers to #2 and #3 related? Does this relationship also hold for the largest side and largest angle?

The Pythagorean Theorem states that if a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse then $a^2 + b^2 = c^2$.

5. If a triangle has legs of length 5 and 7, determine the length of the hypotenuse. You may leave the square root in your answer.

6. A square has a diagonal of length $\sqrt{80}$ inches. Determine the lengths of the sides.

There are over 350 different proofs of the Pythagorean Theorem. We will investigate a few of them.

7. On the other handout, you will find four congruent right triangles and four squares of varying sizes. Label the hypotenuse of the right triangle with the letter c and the legs with the letters a and b, respectively. Then determine which squares have area a^2, b^2, and c^2, respectively. Cut out the four triangles and the four squares; we will save the unlabeled square for later.
8. Take the four triangles and the square labeled c^2 and arrange them to form a single larger square to the right.
    a. Trace the shapes so that you will be able to refer to this sketch later.
    b. What is the length of a side of this large outer square in terms of lengths a, b, and/or c?

9. Now take the four triangles and the two squares labeled a^2 and b^2 and arrange them to form a single larger square below.
    a. Trace the shapes so that you will be able to refer to this sketch later.
    b. Is this large outer square congruent to the square to the right? How do you know?

10. a. Write the sum of the areas of the pieces used to fill the square above.

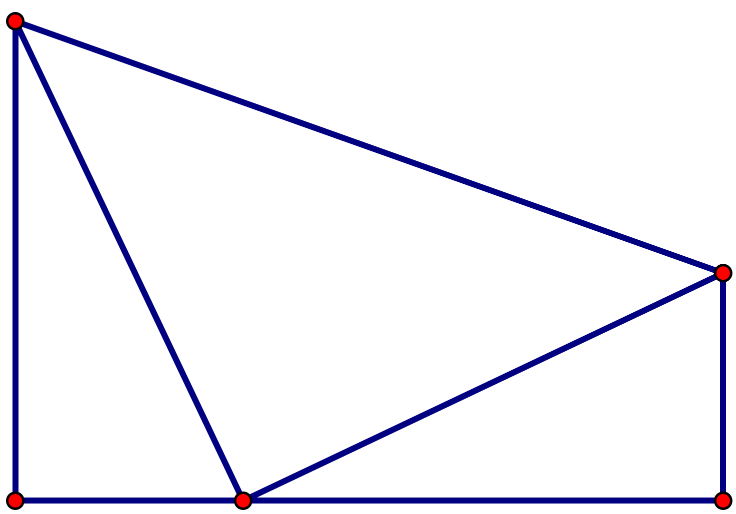
b. Write the sum of the areas of the pieces used to fill the square to the left.

c. Set the expressions in parts a and b equal to each other. How does this equation and the two puzzles prove the Pythagorean Theorem?

11. The next proof of the Pythagorean Theorem is attributed to James A. Garfield, the 20th President of the United States.
   a. Label the sides of the right triangles in the diagram below with the letters a, b, and c, as appropriate.
   b. Compute the area of the trapezoid. Be careful in selecting the parallel bases and height!

   c. Compute the areas of each of the three pieces.

   d. How does this prove the Pythagorean Theorem?



12. Now take the unmarked square together with the four triangles:

a. Arrange them so that they cover c^2.

b. Place a^2 and b^2 next to each other. Cover this design with the unmarked square and the four triangles.

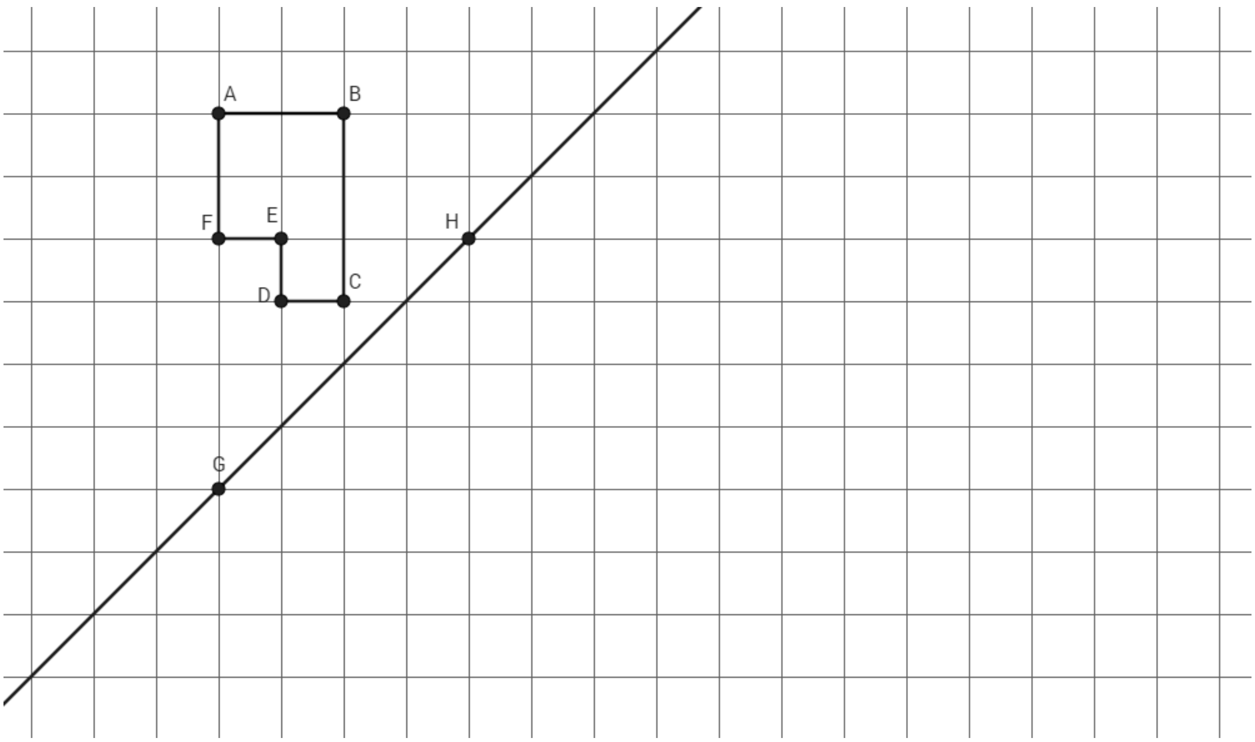
c. How does this prove the Pythagorean Theorem?

Reflecting Polygons and Polyominos

Activity-based lesson:

Recommended materials: colored pencils; Optional: miras, rulers, protractors.

1. Reflect pentamino ABCDEF across line $\overleftrightarrow{GH}$ by thinking of the line as either a mirror or a fold-and-trace line. In what ways are the reflected image and the original figure the same? In what ways are they different?



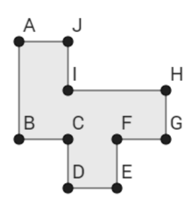
2. On your sketch ABOVE, label the vertices $A', B', C', D', E',$ and F', corresponding to the original pentominos vertices $A, B, C, D, E,$ and F.

a. Where does $\overline{BB'}$ intersect $\overleftrightarrow{GH}$?

b. What type of angle do these two lines create?

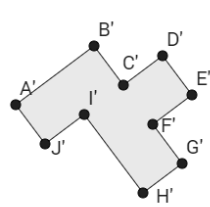
c. Using the 1 cm graph paper as a guide, what is the shortest distance from point B to line $\overleftrightarrow{GH}$? What is the distance from B to B'?

d. What happens to the other points in the reflection? Suppose P is any point in the plane and let P' be its image created by reflecting it across any line in the plane. What do you expect to be true about the relationship between the segment $\overline{PP'}$ and the reflecting line?



e. What happens if you reflect point G across line $\overleftrightarrow{GH}$? Is that true for any point on $\overleftrightarrow{GH}$?

3. To the right, you can see pentamino ABCDEFGHIJ and its reflection A'B'C'D'E'F'G'H'I'J'. Use what you discovered in part 2 to determine the reflecting line.



4. Recall that a point on the Cartesian plane can be identified by its coordinates (x, y) where x and y represent the number of units to the right and above the origin, respectively.
On the sketch below, do the following:

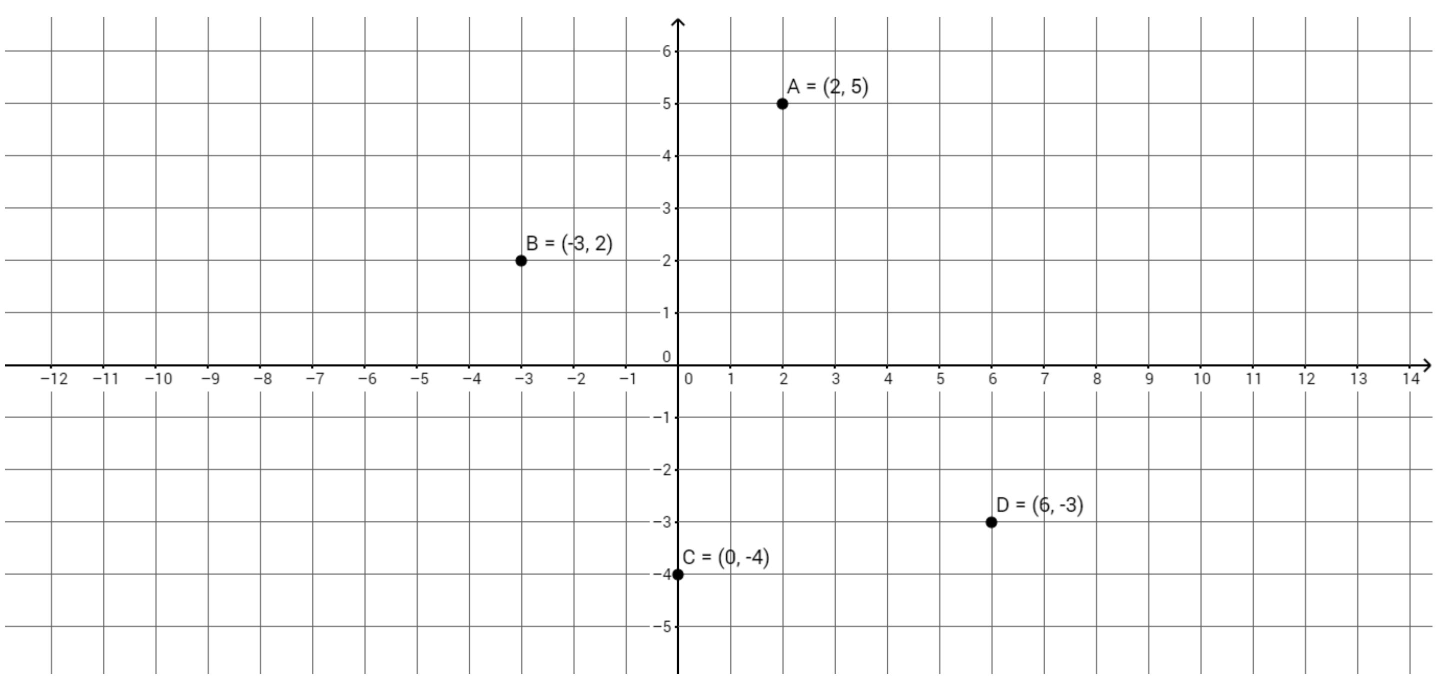
a. Use a red pencil to mark the reflected images A', B', C', and D' of the points A, B, C, and D across the horizontal (x-) axis. Determine their coordinates. What effect does reflecting across the x-axis have of the coordinates?

b. Use a green pencil to mark the reflections of the original points A, B, C, and D across the vertical (y-) axis. Determine the coordinates of their images A'', B'', C'', and D''. What effect does reflecting across the x-axis have of the coordinates?

c. Reflect the red points A', B', C', and D' that you sketched in part a across the vertical (y) axis, using a blue pencil to mark them. Determine the coordinates of the newest points A^*, B^*, C^*, and D^*. What effect does reflecting first across the x-axis and then across the y-axis have of the coordinates (from A to A^* for example)?

d. Focus on the original points A, B, C, and D and the final points A^*, B^*, C^*, and D^*. Is there a single reflecting line that gives this mapping? What type of transformation describes this mapping?

e. Suppose that you were to reflect the green points from part b across the horizontal (x) axis. What do you notice about their images?



Mastering Key Ideas: Complete the following statements:

1. The reflecting line is the ____ of the segment connecting a point and its reflected image.
2. If the names of the points of the image, read in the clockwise direction, are in alphabetical order, then the points in its reflected image____
____

3. Reflecting a point across the x-axis, changes the __________ of the _______-coordinate(s).
4. Reflecting a point across the y-axis, changes the __________ of the _______-coordinate(s).
5. Reflecting a point first across the horizontal axis and then across the vertical axis, changes the __________ of the original _______-coordinate(s). The composition of these two reflections is ___________________________________
____.
6. The following ideas, strategies, and techniques were helpful or insightful to me as I reflected points on a coordinate grid:

7. If I wanted to draw the reflection across a line but I only had a ruler (possibly a compass), but no mirror, Mira, grid, or foldable paper, I could ...