Algebraic Procedures in Need of a Conceptual Makeover
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Instructional Shifts in Mathematics
- Focus strongly where the standards focus
- Coherence think across the grades and link to major topics within the grade
- Rigor in major topics pursue with equal intensity:
  - conceptual understanding,
  - procedural skill and fluency,
  - application

Mathematics Teaching Practices
- Establish mathematics goals to focus learning.
- Implement tasks that promote reasoning and problem solving.
- Use and connect mathematical representations.
- Facilitate meaningful mathematical discourse.
- Pose purposeful questions.
- Build procedural fluency from conceptual understanding.
- Support productive struggle in learning mathematics.
- Elicit and use evidence of student thinking.

Procedural Fluency Defined

Conceptual Understanding
- Initial Exploration
- Discussion

Procedural Fluency

WARNING:
Procedure: Identifying Vertex of a Quadratic from Vertex Form

Traditional Procedure

**Vertex Form of a Quadratic Function**
\[ f(x) = a(x-h)^2 + k \]

- \( a \) indicates a reflection across the x-axis and/or a vertical stretch or compression.
- \( h \) indicates a horizontal translation.
- \( k \) indicates a vertical translation.

The vertex of the parabola is at \((h, k)\).

A Conceptual Makeover

*Identify the smallest quantity an expression can represent by substituting values for \( x \)*

**Prior Understandings and Skills**
- Understand expressions represent quantities
- Evaluate an expression by substituting values for the unknown variable

Identify a real number that when placed into the blank will produce the smallest possible value. Also, identify the smallest value of the expression.

\[
\begin{align*}
\boxed{ }^2 & \quad \boxed{ }^2 + 3 & \quad \boxed{ }^2 - 5 & \quad (\boxed{ } + 7)^2 \\
(\boxed{ } + 6)^2 + 1 & \quad 2(\boxed{ } - 3)^2 + 1 & \quad 5(\boxed{ } - h)^2 - 2 \\
5(\boxed{ } - h)^2 + k & \quad a(\boxed{ } - h)^2 + k
\end{align*}
\]

**Discussion**

What strategy did you use to identify the real number to substitute into the expression? Justify why the strategy works to provide the smallest possible value for the expression.
Identify a real number that when placed into the blank will produce the largest possible value. Also, identify the largest value of the expression.

The expression is:

\[-x^2 \quad -x^2 + 8 \quad -(x - 4)^2 \]

\[-(x + 3)^2 - 6 \quad -2(x - h)^2 + k\]

What strategy did you use to identify the real number to substitute into the expression? Justify why the strategy works to provide the largest possible value for the expression.

**Vertex Form of a Quadratic**

Identify the vertex of the quadratic given the quadratic expression

\[a(x - h)^2 + k\]

Describe the conditions for when the vertex is a minimum and when it is a maximum.

**Reasoning Strategies**

- Focus on the value for x for which the evaluation of \((x - h)\) is 0.
- Understand that when \(x - h\) is 0 then the evaluation of the entire expression is k.
- Recognize when the expression represents a minimum or a maximum quantity and consider how adding k to the expression \(a(x - h)^2\) affects the quantity for various values of x.
Procedure: Writing the equation of a line given two points.

Traditional Procedure

Write the equation of the line that passes through the points (-1,2) and (7, 6).

\[ m = \frac{6 - 2}{7 - (-1)} = \frac{4}{8} = 0.5 \]

❖ Use the slope formula to find m.

\[ y = mx + b \]
\[ 6 = 0.5 (7) + b \]
\[ 6 = 3.5 + b \]
\[ 6 - 3.5 = b \]
\[ 2.5 = b \]

❖ Use the formula \( y = mx + b \) and substitute slope for m and one of the points for \((x, y)\).

❖ Solve for b.

\[ y = 0.5x + 2.5 \]
❖ Substitute m and b back into the equation

A Conceptual Makeover: Use a Table to Build a Generalized Procedure

Prior Understandings and Skills

❖ Defining characteristics of a linear relationship are the constant rate of change (slope) and the initial value (y-intercept)
❖ Connections between how the slope and y-intercept are represented among mathematical representations (graph, table, description and symbols)

For each of the following tables, write the equation of a line that passes through the two given points.

\[
\begin{array}{c|c}
\text{x} & \text{y} \\
\hline
4 & 3 \\
8 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c}
\text{x} & \text{y} \\
\hline
3 & 1 \\
9 & 5 \\
\end{array}
\]
Algebraic Procedures in Need of a Conceptual Makeover

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<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>27</td>
</tr>
</tbody>
</table>

Analyze the Strategy

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>23</td>
</tr>
<tr>
<td>23</td>
<td>77</td>
</tr>
</tbody>
</table>

Analyze the Strategy

Generalized Procedure
Write the equation of a line passing through the points (-1, 2) and (7,6).

Reasoning Strategies
❖ Students transition from using $\Delta x$ and $\Delta y$ separately to the $\Delta y$ when $\Delta x=1$ and relate this to slope.
❖ The $y$ value changes as the $x$ value changes. The amount of change in the $y$ value from the $y$-intercept is $mx$ where $m$ is the slope
❖ The $y$-intercept is any given $y$ value minus the amount of change in the $y$ value.

What do the points look like in a table?
What is the slope of the line?
What calculations produce the $y$ value for when $x = 0$?
Procedure: Solving Exponential Equations Using Logarithms

Traditional Procedure
Solve $5(2^x) = 240$

$2^x = 48 \quad \checkmark$ Isolate the exponential part of the expression
$log_2 48 = x \quad \checkmark$ Convert the equation to logarithmic form
$x = \frac{\log 48}{\log 2} \quad \checkmark$ Use change of base rule since 48 is not a rational power of 2
$x \approx 5.585 \quad \checkmark$ Evaluate the expression using calculator

A Conceptual Makeover: Rewriting Quantities as a Power of 10

Prior Understandings and Skills
❖ Writing an expression into equivalent forms reveals properties of the quantity it represents.
❖ Power to a Power property for exponents; $(b^m)^n = b^{mn}$

Express each of the numbers as a *Power of 10*. You can find exact values for some of the required exponents. Others might require some calculator exploration.

<table>
<thead>
<tr>
<th>10^x</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. 100</td>
</tr>
<tr>
<td>d. 2</td>
</tr>
<tr>
<td>g. 0.01</td>
</tr>
<tr>
<td>j. 3.45</td>
</tr>
</tbody>
</table>

**Discussion**
Describe the numbers that can be written as a power of 10. What numbers cannot be written as a power of 10 and why?
Use your results from rewriting numbers as a *Power of 10* to solve each of the following equations.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. (10^x = 100)</td>
<td>(x = 2)</td>
</tr>
<tr>
<td>b. (10^{x+2} = 1,000)</td>
<td>(x = 1)</td>
</tr>
<tr>
<td>c. (10^{3x+1} = 10,000)</td>
<td>(x = 3)</td>
</tr>
<tr>
<td>d. (3(10)^{x+3} = 300)</td>
<td>(x = -1)</td>
</tr>
<tr>
<td>e. (10^{2x} = 500)</td>
<td>(x = 2.295)</td>
</tr>
<tr>
<td>f. (2(10)^x = 6.90)</td>
<td>(x = 0.8)</td>
</tr>
</tbody>
</table>

### Common Logarithms

Look for and make use of structure

\[ \log_{10} a = b \text{ if and only if } 10^b = a \]

Without the use of a calculator, find each of the following logarithms.

<table>
<thead>
<tr>
<th>Logarithm</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. (\log 10^{2.5})</td>
<td>(2.5)</td>
</tr>
<tr>
<td>b. (\log 100)</td>
<td>(2)</td>
</tr>
<tr>
<td>c. (\log 10,000)</td>
<td>(4)</td>
</tr>
<tr>
<td>d. (\log 3.45)</td>
<td>(0.538)</td>
</tr>
<tr>
<td>e. (\log 500)</td>
<td>(2.699)</td>
</tr>
<tr>
<td>f. (\log -0.001)</td>
<td>Not defined</td>
</tr>
</tbody>
</table>

Use what you know about logarithms to help find the exact solution for these equations. (Note: The exact solution will use “\(\log 45\)” rather than the decimal approximation of 1.6532)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. (10^x = 15)</td>
<td>(x = \log 15\approx 1.176)</td>
</tr>
<tr>
<td>b. (5(10^x) = 60)</td>
<td>(x = \log 60/5 \approx 1.447)</td>
</tr>
<tr>
<td>c. (3(10)^{x+4} = 21)</td>
<td>(x = \log 21/30 \approx -0.464)</td>
</tr>
<tr>
<td>d. (3^x = 729)</td>
<td>(x = \log 729/3 \approx 4)</td>
</tr>
<tr>
<td>e. (5(2^x) = 240)</td>
<td>(x = \log 240/5 \approx 4.322)</td>
</tr>
<tr>
<td>f. (3(5^x + 12) = 60)</td>
<td>(x = \log 60/3 - 12 \approx 0.670)</td>
</tr>
</tbody>
</table>

### Reasoning Strategies

- Rewriting values as a Power of 10.
- Reasoning that when there are equivalent expressions with like bases then the exponents must be equivalent.
Here’s the Challenge

When you are teaching a procedure
- approach it through the perspective of a student
- focus on why and when
- make connections explicit
- compare procedures and problems
- reach out to colleagues & do this work together

References


