Algebraic Procedures in Need of a Conceptual Makeover

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$$\square^2$$

Identify a real number that when placed into the blank will produce the smallest possible value.

What is the smallest possible value?

Thumbs up when you have a number.

$$\square^2 + 3$$

Identify a real number that when placed into the blank will produce the smallest possible value.

What is the smallest possible value?

Thumbs up when you have a number.

$$\square^2 - 5$$

Identify a real number that when placed into the blank will produce the smallest possible value.

What is the smallest possible value?

Thumbs up when you have a number.

$$(\square + 7)^2$$

Identify a real number that when placed into the blank will produce the smallest possible value.

What is the smallest possible value?

👍 Thumbs up when you have a number.

$$(\square + 6)^2 + 1$$

Identify a real number that when placed into the blank will produce the smallest possible value.

What is the smallest possible value?

Thumbs up when you have a number.

$$2(\square - 3)^2 + 1$$

Identify a real number that when placed into the blank will produce the smallest possible value.

What is the smallest possible value?

Thumbs up when you have a number.

$$5(\square - h)^2 - 2$$

Describe the value that when placed into the blank will produce the smallest possible value.

What is the smallest possible value?

Thumbs up when you have a description.

$$5(\square - h)^2 + k$$

Describe the value that when placed into the blank will produce the smallest possible value.

What is the smallest possible value?

Thumbs up when you have a description.

$$a(x-h)^2+k$$

Describe the value of x that will produce the smallest possible value.

What is the smallest possible value?

Thumbs up when you have a description.

Vertex Form of a Quadratic

Identify the vertex of the quadratic given the quadratic expression

$$a(x - h)^2 + k$$

Describe the conditions for when the vertex is a minimum and when it is a maximum.

Instructional Shifts in Mathematics

Focus strongly where the standards focus

Coherence think across the grades and link to major topics within the grade

Rigor in major topics pursue with equal intensity:

- conceptual understanding,
- procedural skill and fluency, and
- application

Mathematics Teaching Practices

- Establish mathematics goals to focus learning.
- Implement tasks that promote reasoning and problem solving.
- Use and connect mathematical representations.
- Facilitate meaningful mathematical discourse.
- Pose purposeful questions.
- Build procedural fluency from conceptual understanding.
- Support productive struggle in learning mathematics.
- Elicit and use evidence of student thinking.

So I began to wonder…

- How do you pursue the three aspects of rigor with equal intensity?
- What are the concepts that need to be understood?
- What are the procedures that build from those understandings?
- What is does it mean to be procedurally fluent?
- How do you build procedural fluency from conceptual understanding?

Procedural Fluency Defined

Accurate

correct solutions

Efficient

selecting and implementing an appropriate strategy quickly

Flexible

adapting strategies and applying to different problems / context

Conceptual Understanding → **Procedural Fluency**

WARNING:
a rush to fluency neglects students' development of reasoning strategies & undermines students' confidence and interest in math

Writing the Equation of a Line

Traditional Procedure

Write the equation of the line that passes through the points (-1,2) and (7, 6).

$$m = \frac{6-2}{7-(-1)} = \frac{4}{8} = 0.5$$

- Use the slope formula to find m.

$$y = mx + b$$
$$6 = 0.5\,(7) + b$$

- Use the formula y=mx+b and substitute slope for m and one of the points for (x, y).

$$6 = 3.5 + b$$
$$6 - 3.5 = b$$
$$2.5 = b$$

- Solve for b.

$$y = 0.5x + 2.5$$

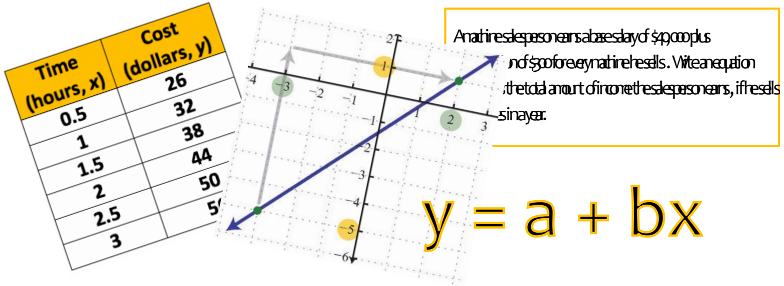
- Substitute m and b back into the equation.

Writing the Equation of a Line

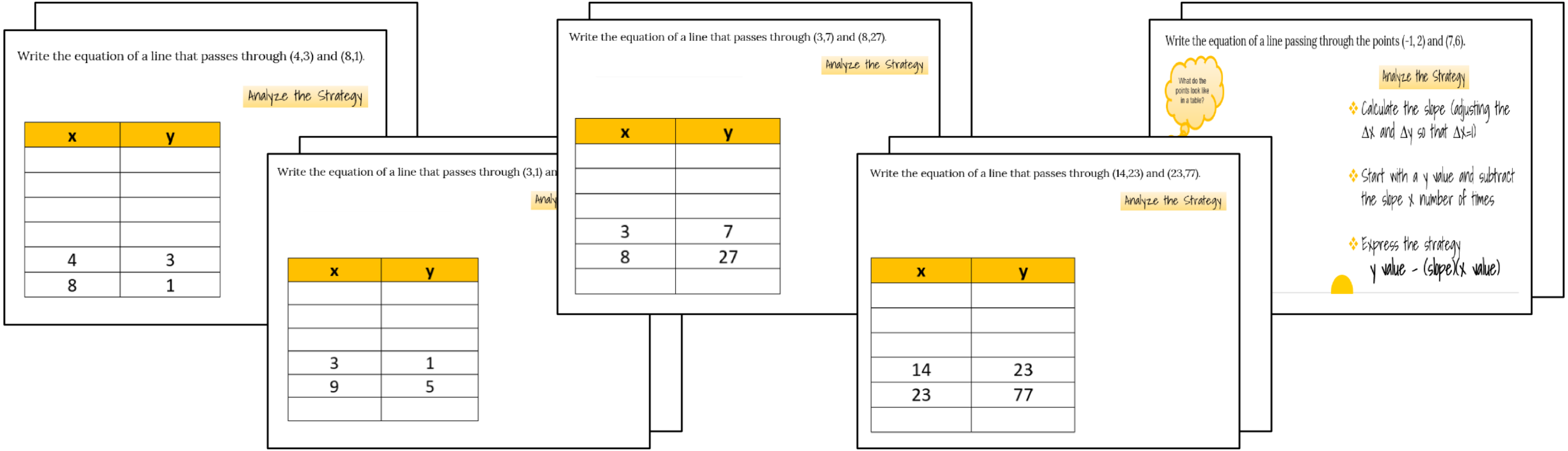
Prior Understandings

- Defining characteristics of a linear relationship are the constant rate of change (slope) and the initial value (y-intercept)

- Connections between how the slope and y-intercept are represented among mathematical representations (graph, table, description and symbols)



Intentionally building informal reasoning strategies



Series of problems where students are given time to reason and develop strategies. Each problem is followed up with discussion of strategy and additional practice before moving on.

Write the equation of a line that passes through (4,3) and (8,1).

x	y
4	3
8	1

Analyze the Strategy

- understanding the y intercept is the y value when x=0
- using the Δx and Δy separately to find the y-intercept

Write the equation of a line that passes through (3,1) and (9,5).

x	y
3	1
9	5

Analyze the Strategy

- Adjusting the Δx and Δy by taking half of each
- Using the Δx and Δy separately to find the y-intercept

Write the equation of a line that passes through (3,7) and (8,27).

x	y
3	7
8	27

Analyze the Strategy

- Adjusting the Δx and Δy so that $\Delta x=1$ and relate to slope
- Using the Δx and Δy separately to find the y-intercept
- Connect to using the slope three times and relate to the x value

Write the equation of a line that passes through (14,23) and (23,77).

Analyze the Strategy

x	y
14	23
23	77

- Calculate the slope
- Start with a y value, subtract the slope fourteen times and relate to the x value

Write the equation of a line passing through the points (-1, 2) and (7,6).

What do the points look like in a table?

What is the slope of the line?

What calculations produce the y value for when x = 0?

Analyze the Strategy

- Calculate the slope (adjusting the Δx and Δy so that $\Delta x = 1$)
- Start with a y value and subtract the slope x number of times
- Express the strategy

  y value - (slope)(x value)

Writing the Equation of a Line

How do the two procedures compare?

Traditional Procedure

- Use the slope formula to find m.
- Use the formula y=mx+b and substitute slope for m and one of the points for (x, y).
- Solve for b.
- Substitute m and b back into the equation.

Procedure with a Conceptual Makeover

- Calculate the slope (adjusting the Δx and Δy so that $\Delta x=1$).
- Start with a y value and subtract the slope x number of times
- Write the equation

Solving Exponential Equations Using Logarithms

Traditional Procedure

Solve $5(2^x) = 240$

$2^x = 48$	Isolate the exponential part of the expression
$\log_2 48 = x$	Convert the equation to logarithmic form
$x = \dfrac{\log 48}{\log 2}$	Use change of base rule since 48 is not a rational power of 2
$x \approx 5.585$	Evaluate the expression using calculator

Solving Exponential Equations Using Logarithms

Prior Understandings

- Writing an expression into equivalent forms reveals properties of the quantity it represents.
- Power to a Power property for exponents $(b^m)^n = b^{mn}$

Express each of the numbers as a *Power of* 10. You can find exact values for some of the required exponents. Others might require some calculator exploration.

10^x

Power of 10

a. 100

10^2

b. 1,000

10^3

c. 10,000

10^4

Multiples of 10

d. 2

$10^{.3}$

e. 500

$10^{2.7}$

f. 4,782

$10^{3.68}$

Non-integer exponents

g. 0.01

10^{-2}

h. -0.001

not possible

i. 0.0023

$10^{-2.64}$

Negative exponents

j. 3.45

$10^{.54}$

k. 34.5

$10^{1.54}$

l. 345

$10^{2.54}$

Extension

Use your results from rewriting numbers as a *Power of* 10 to solve each of the following equations.

a. $10^x = 100$

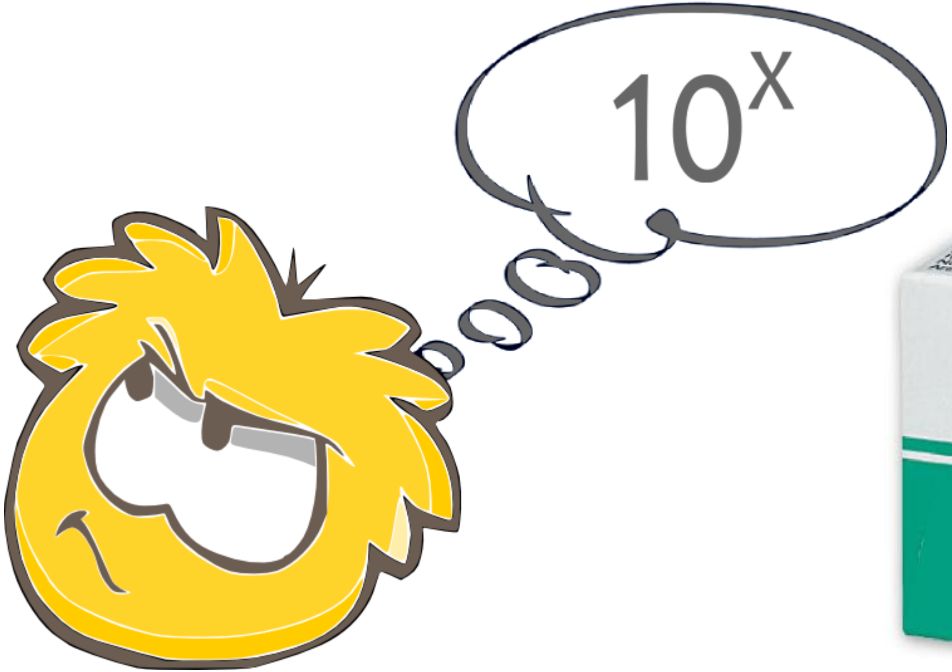
b. $10^{x+2} = 1{,}000$

c. $10^{3x+1} = 10{,}000$

d. $3(10)^{x+3} = 300$

e. $10^{2x} = 500$

f. $2(10)^x = 6.90$


10^x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Logarithms

$$\log_{10} a = b \text{ if and only if } 10^b = a$$

Without the use of a calculator, find each of the following logarithms.

a. $\log 10^{2.5}$

b. $\log 100$

c. $\log 10{,}000$

d. $\log 3.45$

e. $\log 500$

f. $\log -0.001$

Use what you know about logarithms to help find the exact solution to these equations.

a. $10^x = 15$

b. $5(10^{2x}) = 60$

c. $3(10)^{x+4} = 21$

d. $3^x = 729$

e. $5(2^x) = 240$

c. $3(5^x + 12) = 60$

Solving Exponential Equations Using Logarithms

Procedure with a Conceptual Makeover

Solve $5(2^x) = 240$

$$2^x = 48$$

$$10^{x \cdot \log 2} = 10^{\log 48}$$

$$x \log 2 = \log 48$$

$$x = \frac{\log 48}{\log 2}$$

$$x \approx 5.585$$

- Isolate the exponential part of the expression
- Rewrite each expression into a common base
- Same structure and base so exponents are equivalent
- Divide both sides by log 2
- Evaluate the expression using calculator

Solving Exponential Equations Using Logarithms

How do the two procedures compare?

Traditional Procedure

Solve $5(2^x) = 48$

$$2^x = 48$$
$$\log_2 48 = x$$
$$x = \frac{\log 48}{\log 2}$$
$$x \approx 5.585$$

Procedure with a Conceptual Makeover

Solve $5(2^x) = 48$

$$2^x = 48$$
$$10^{x \cdot \log 2} = 10^{\log 48}$$
$$x \log 2 = \log 48$$
$$x = \frac{\log 48}{\log 2}$$
$$x \approx 5.585$$

Conceptual Understanding → **Procedural Fluency**

WARNING:

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Here's the Challenge

When you are teaching a procedure

- approach it through the perspective of a student
- focus on why and when
- make connections explicit
- compare procedures and problems
- reach out to colleagues & do this work together

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