Transformational Approach to Congruence Proofs in Geometry – Session 1157

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Introduction

The Common Core introduces geometric transformations in 8th grade. I will assume you are familiar with those basics and focus this presentation on proof from a transformational point of view. A key innovation of the CCSSM in geometry is the redefinition of congruence on a transformational basis (and a parallel change for similarity).
From the CCSSM: "In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other."

Minimal interpretation of the CCSSM

1. Use transformations to justify SSS, SAS, ASA
2. Continue along traditional path

Ambitious interpretation (ours)

- New definitions and assumptions
- New approaches to proof

(I was inspired to try this approach by reading *Geometry, A Transformational Approach,* by Oxford and Usiskin, ©1975. More information about texts is available on a document you can get from me by following the instructions at the end of this handout.)

One hour is nowhere near enough time for you to learn this approach. The best I can do is give you a brief introduction and let you try a few proofs on your own. Henri and I are available for physical or virtual workshops on this subject — our contact information is above.

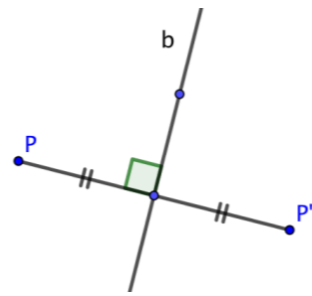
In order to save time, precise definitions and our assumptions follow on the next page.

Precise Definitions of Reflection, Rotation, and Translation

These are the three most basic rigid transformations. They can all be done with patty paper and other physical and software tools, including GeoGebra. A glide reflection is an important fourth type, because any rigid transformation can be done with just one of these four. (The proof is hard. It is included on Henri's website.)

Reflection

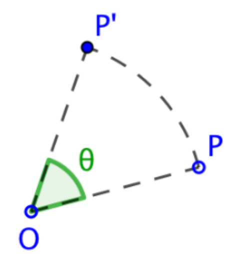
A reflection in a line b maps any point on b to itself and any other point P to a point $\boldsymbol{P'}$ so that b is the perpendicular bisector of $\boldsymbol{PP'}$.



A consequence of the definition of reflection important in proofs:
If $\boldsymbol{A'} = B$ then $\boldsymbol{B'} = A$.

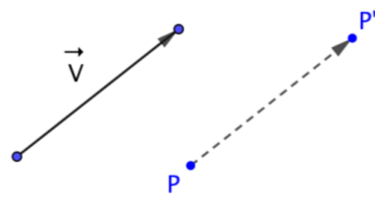
Rotation

Given a point O and a directed angle θ (positive for counterclockwise, negative for clockwise), the image of a point $P \neq O$ under a rotation with center O and angle θ is a point P' on the circle centered at O with radius OP, such that $\angle POP' = \theta$. The image of O is O.



Translation

Given a vector $\vec{V}$, the image of a point P under a translation by $\vec{V}$ is a point P' such that the vector $\overrightarrow{PP'} = \vec{V}$.
Traditionally, vectors are not discussed in geometry. This is a good opportunity to change that. The informal concept is not too difficult for students: A directed line segment whose position doesn't matter. Using vectors is easier than defining translation in any other way. Defining a vector formally is harder and unnecessary.



Assumptions

1. The parallel postulate: Through a point outside a given line, one and only one line can be drawn parallel to the given line.
2. **Reflection preserves distance and angle measure**. (Students have no trouble accepting this assumption because reflection can be done with patty paper.)

And the *construction postulates:*

3. Two distinct lines meet in at most one point.
4. A circle and a line meet in at most two points.
5. Two distinct circles meet in at most two points.

This list is not “Hilbert complete,” but it doesn’t need to be.

The presentation slides for this session are in a folder titled NCTM 2018 in my Dropbox. The session handouts are also in this folder as .docx files, so you can modify them for your classes as you wish. Please be sure to acknowledge Henri Picciotto and Lew Douglas if you use our materials.

Here's the link to the folder:

Henri and I have been working on mapping out a logical sequence for a transformational approach to high school geometry. We put together a large document summarizing our work so far. We hope it will be useful to teachers and curriculum developers. If you use this work, please acknowledge accordingly.

On Henri's Geometric Transformations page, there is more information, all of which is available under the same conditions. Look down the page. Here are the sections:

- Transformational Proof in High School Geometry – this is the comprehensive document.

The rigorous proofs in the document are based on our assumptions and definitions. We avoided the use of congruent triangles, even though that use would be legitimate since we have proved the triangle congruence criteria. The idea was to add a different style of proof to our repertoire.

Ideas for some of the proofs came from the online course *Geometry Transformed!* that was given online from Nov. 7, 2017 through Dec. 5, 2017. The course was sponsored by the Park City Math Institute. You can find their academic year outreach programs here: http://projects.ias.edu/pcmi/outreach/ .

These interesting topics on Henri's Geometric Transformations page were not discussed in our session:

- Similar Graphs
- The Glide Reflection
- Only Four Kinds of Isometries

Available Geometry Texts that use a Transformational Approach (there may be others)

1. UCSMP: University of Chicago School Mathematics Project - Geometry http://ucsmp.uchicago.edu/secondary/curriculum/geometry/
2. Pearson enVision Geometry https://www.pearsonrealize.com/index.html#/

Personally, I like the UCSMP text better, but both are too traditional to be ideal.
Illustrative Mathematics (https://www.illustrativemathematics.org) has already completed a curriculum for grades 6-8 that is available free of charge. They are working on curriculum for grades 9-11 that will incorporate a transformational approach. At the top of their home page is a link to subscribe to their newsletter, which includes contact methods. The newsletter will update people when the HS curriculum is ready.