

Building meaning into algebra equations using multiple representations and progressive formalization

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**INTERNATIONAL CONSORTIUM
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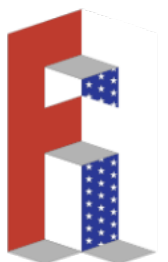
$$3x + 2 = 8$$

- How can we help students see structure and meaning in these symbols?
- How can we help students develop meaningful strategies to manipulate these symbols?

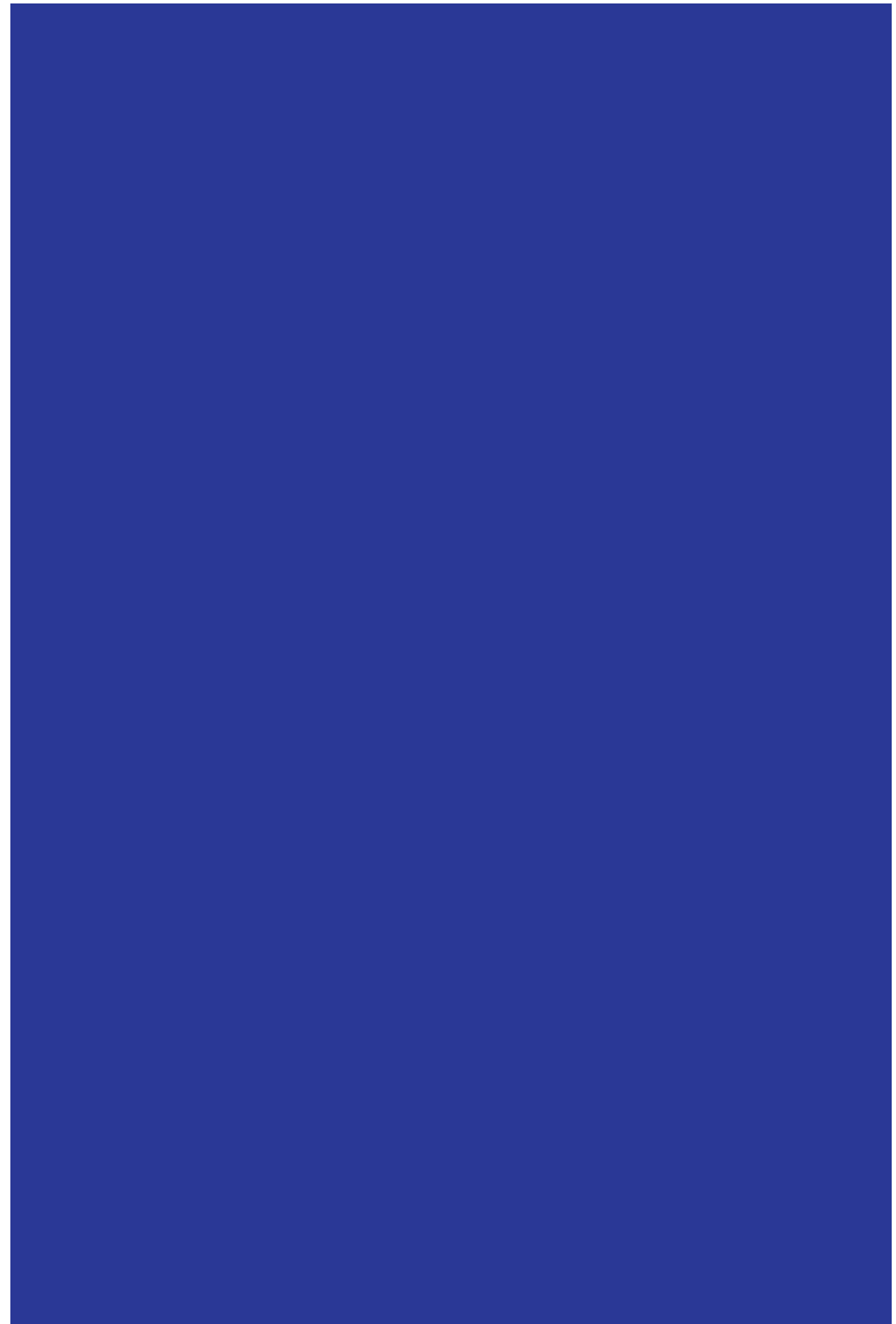


Session outline

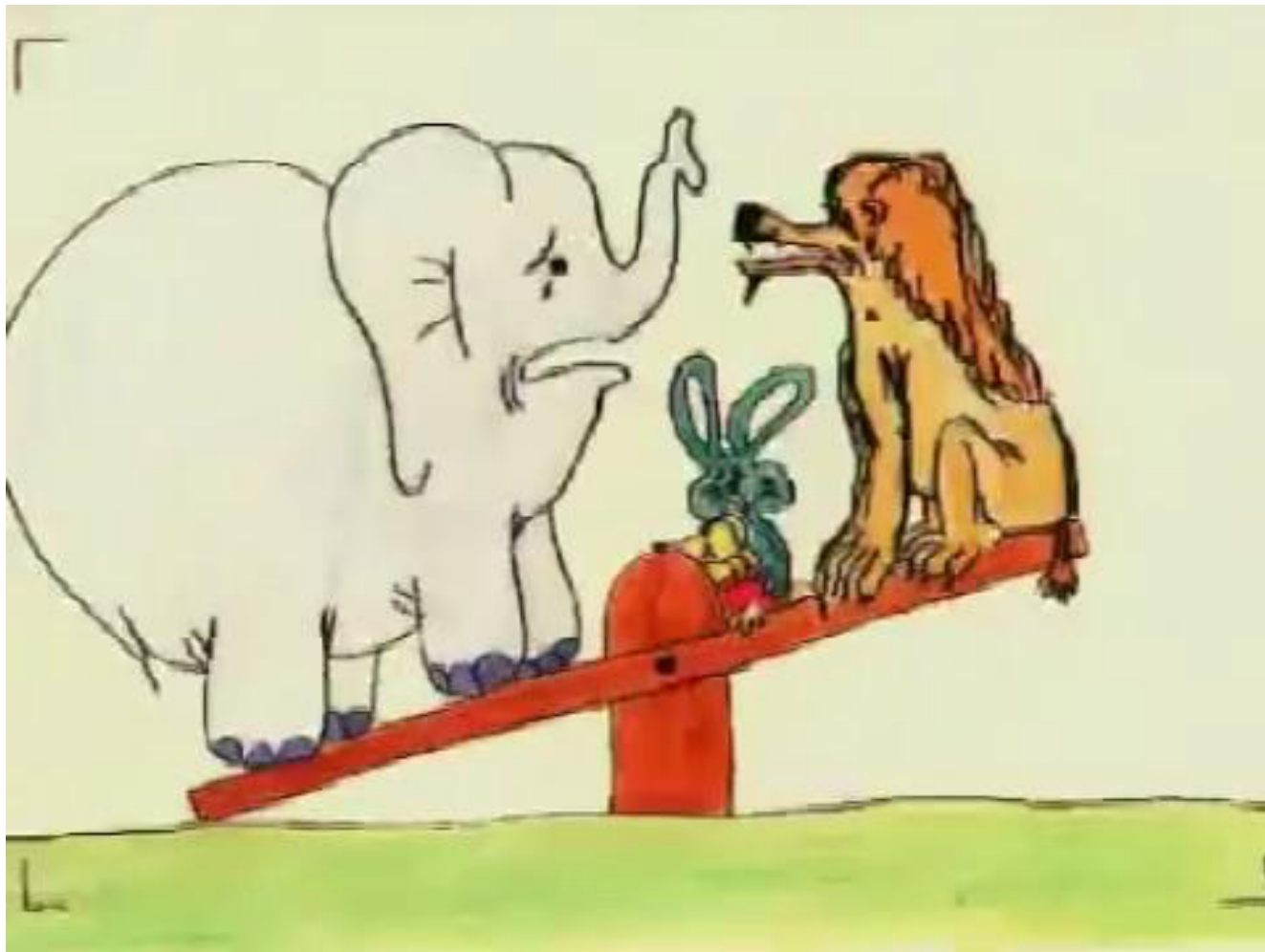
1. The balance sequence
2. Using models and progressive formalization to help students *learn* and *do* math meaningfully.
3. The arrow chain sequence
4. In the classroom



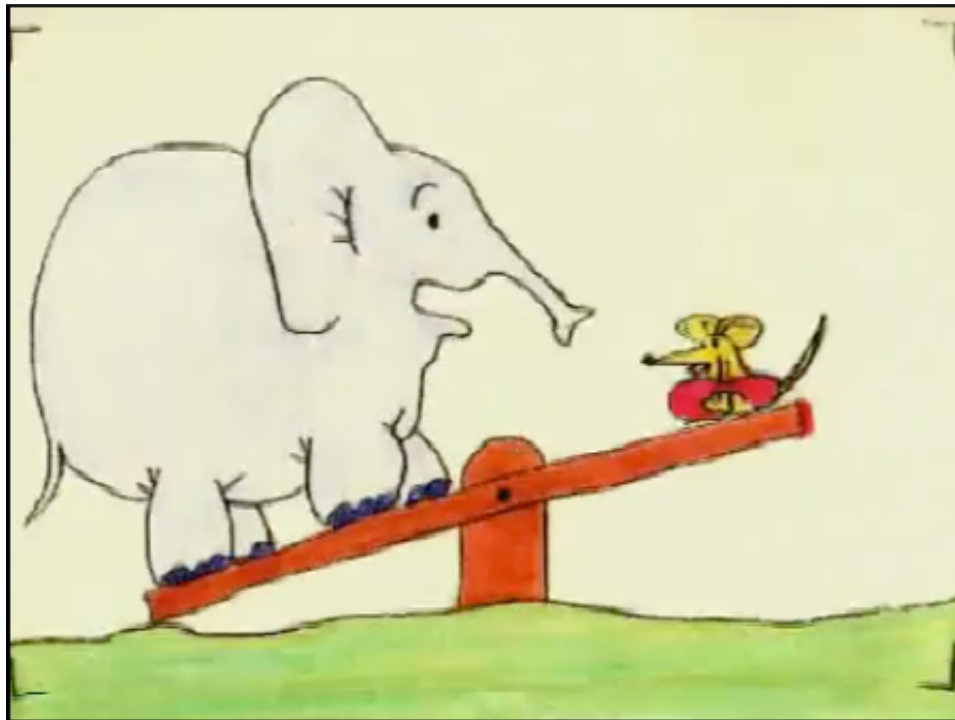
The balance sequence

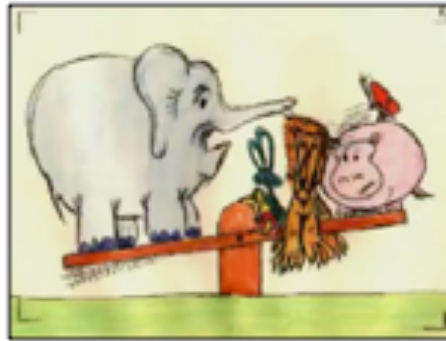
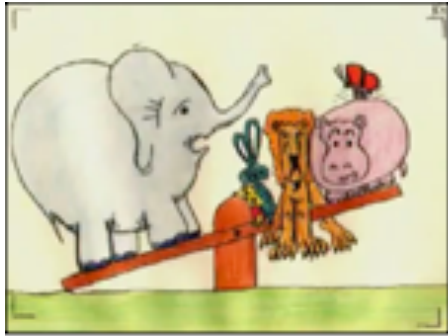


The see-saw



What can you say about the elephant and the mouse?





What happened when the butterfly landed? What does that tell you?



The balance model

What does the balance model do for students?

$$2x + 9 = 5x + 2$$

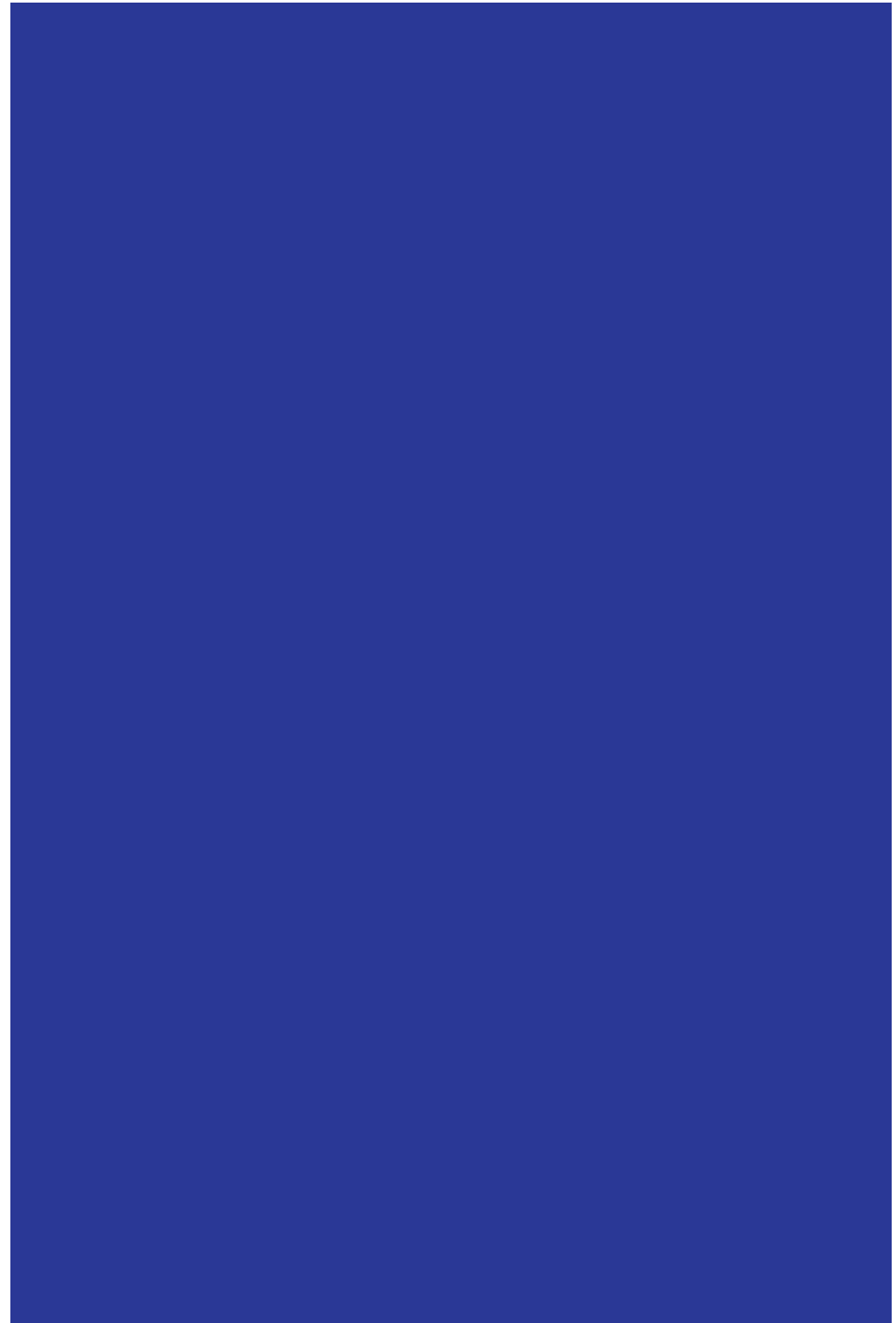
- What kinds of **structure** and **meaning** does the balance model help students see in these symbols?
- What kinds of **meaningful strategies** for symbol manipulation does it help students develop?

The balance model

$$2x + 9 = 5x + 2$$

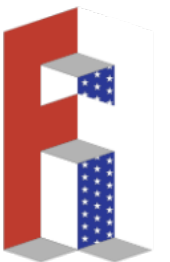
- **Structure and meaning?**
 - The equation describes a relationship between *objects*
 - The equals sign means “balanced”
- **Strategies for symbol manipulation**
 - Manipulate the objects
 - Do the same manipulation to both sides to maintain balance

Perspectives on Modeling

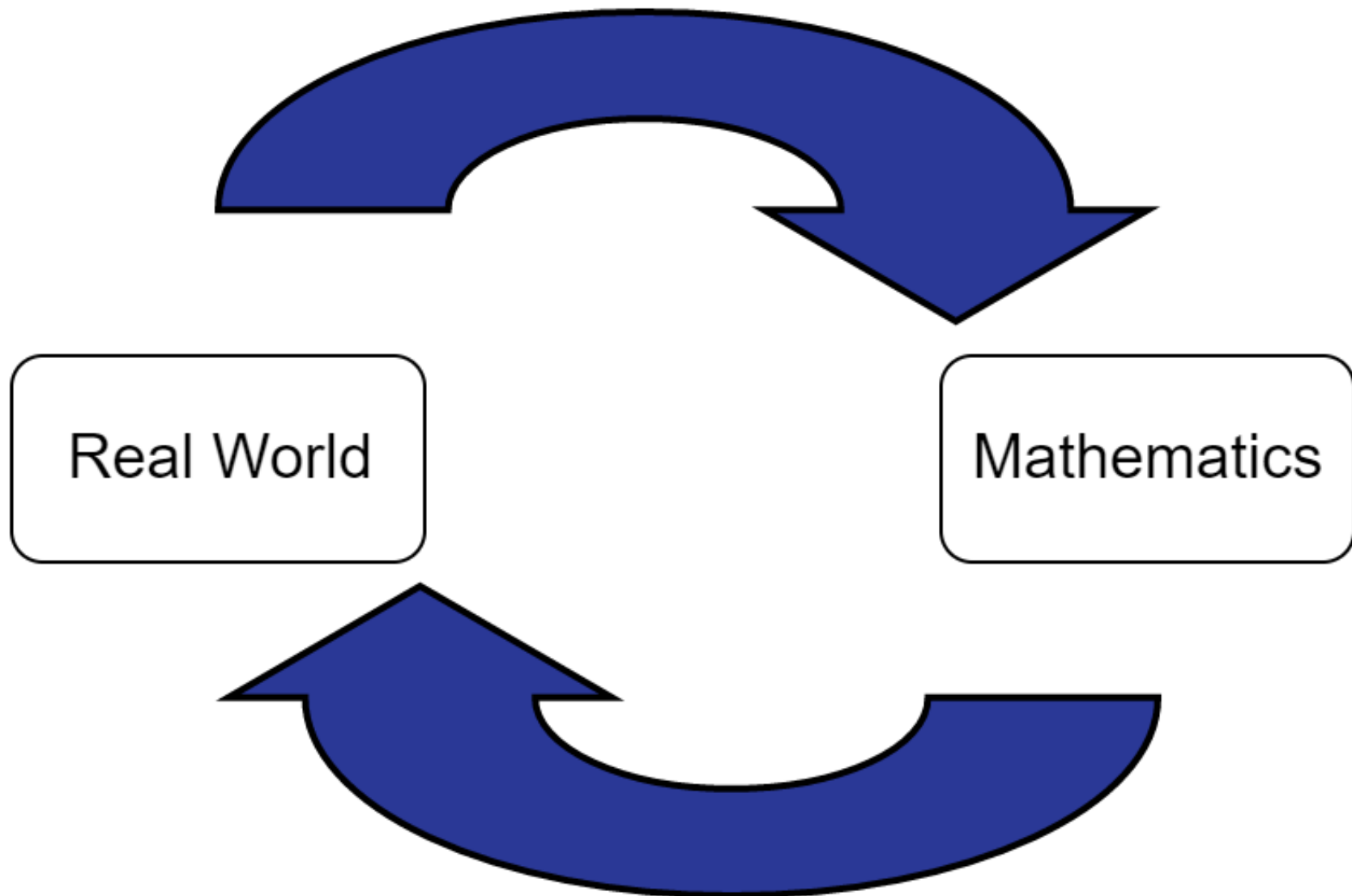


What the Common Core says about modeling

“Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions.” (CCSSI, p. 72)

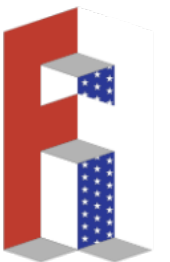


Modeling as a form of translation

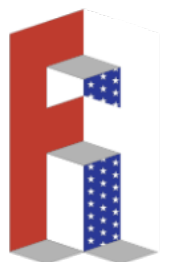
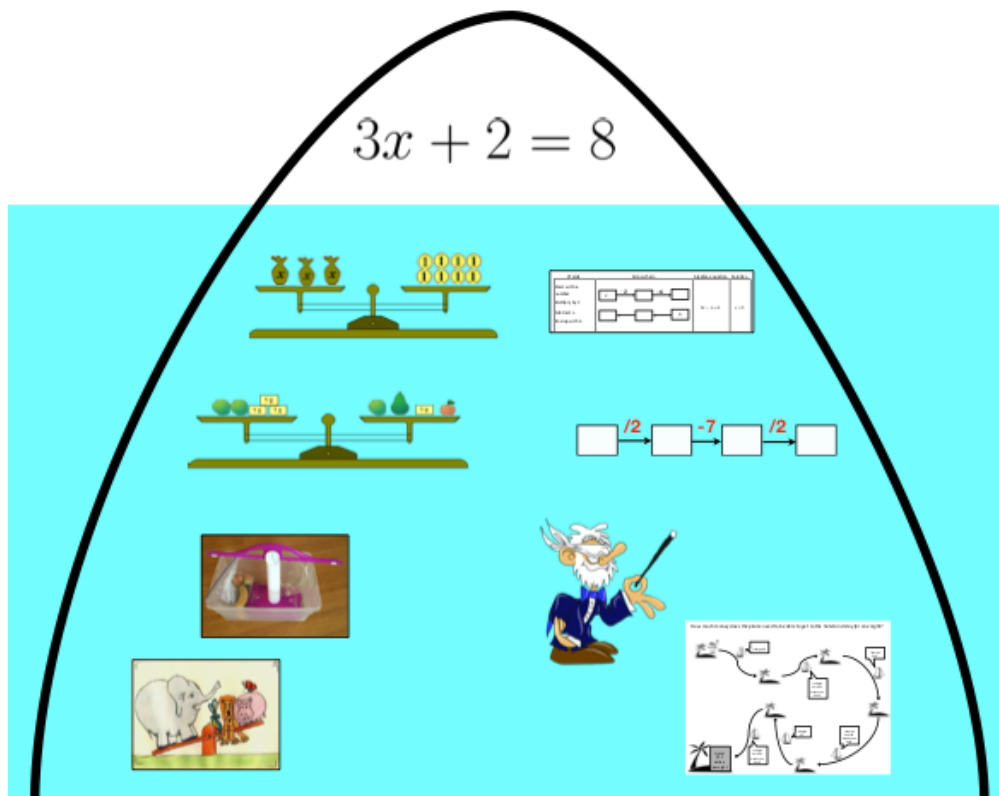


RME: Modeling as a form of *organizing*

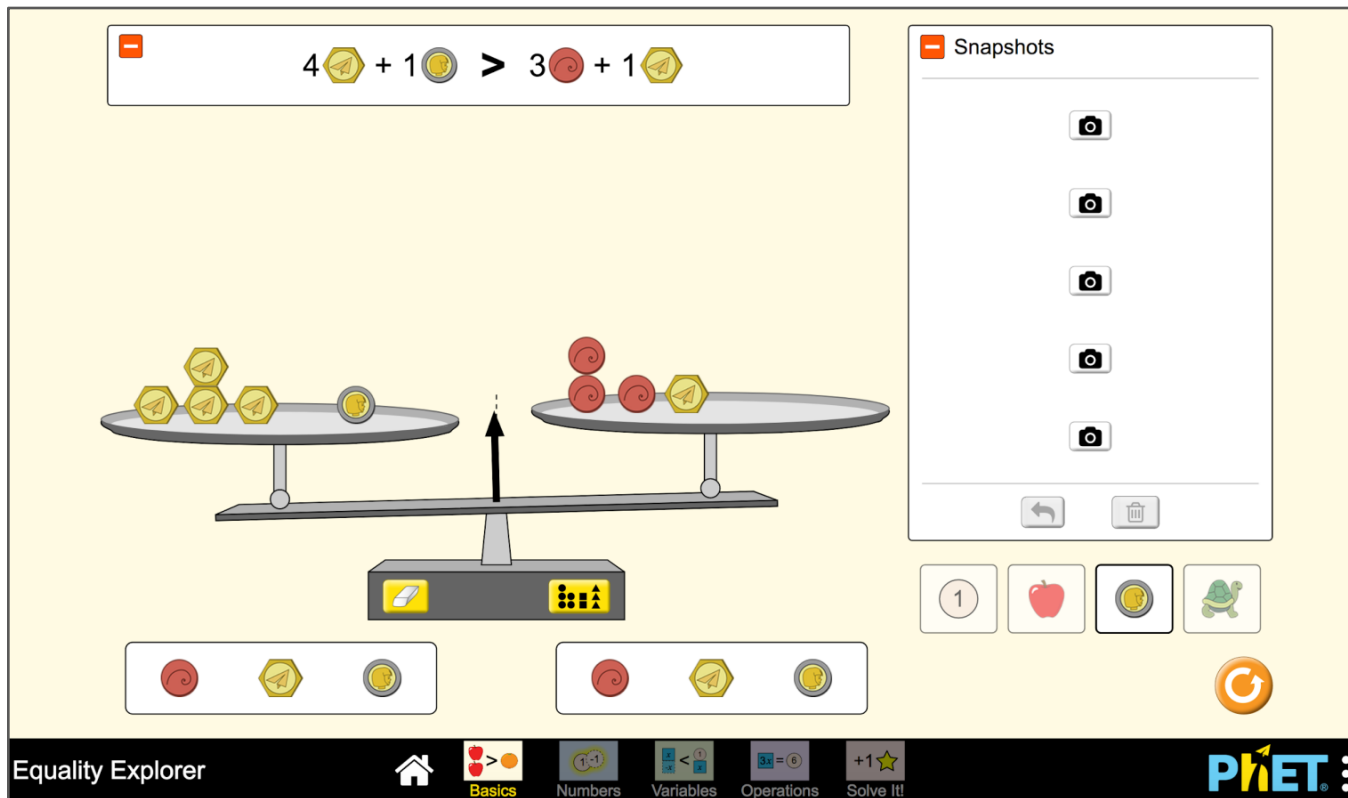
“The idea is that informal ways of modelling emerge when the students are organizing contextual problems. Later, these ways of modelling serve as a basis for developing formal mathematical knowledge. To be more precise, at first a model is constituted as a context-specific model *of* acting in a situation, then the model is generalized over situations. Thus, the model changes character, it becomes an entity of its own, and in this new shape it can function as a model *for* more formal mathematical reasoning.” (Gravemeijer, 1997, p. 394)



Progressive formalization & iceberg metaphor

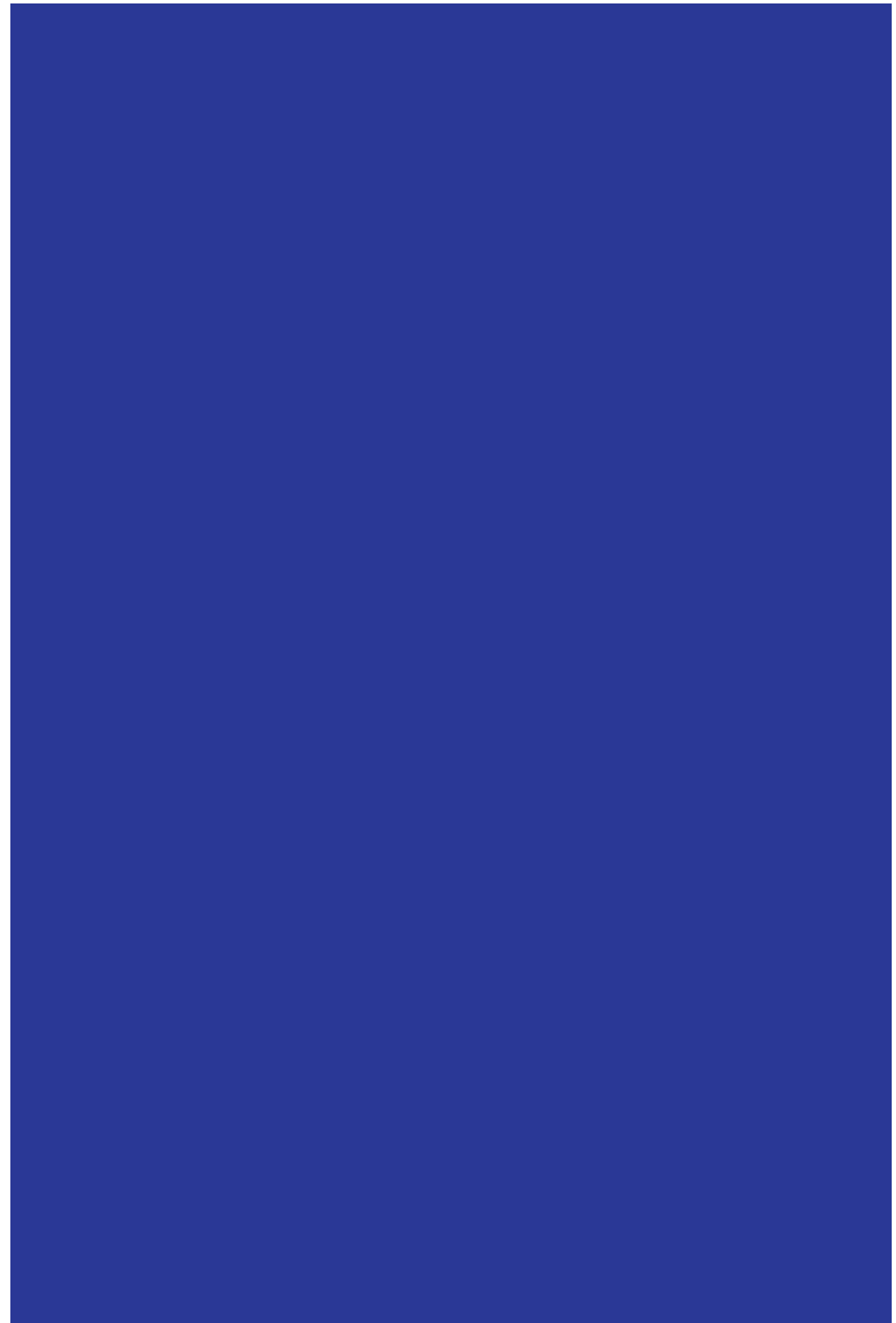


<https://tinyurl.com/PhETEquality>



DRL #1503510

The arrow chain sequence



The arrow chain model

What does the arrow chain model do for students?

$$\frac{3}{4}x - 9 = -3$$

- What kinds of **structure** and **meaning** does the arrow chain model help students see in these symbols?
- What kinds of **meaningful strategies** for symbol manipulation does it help students develop?

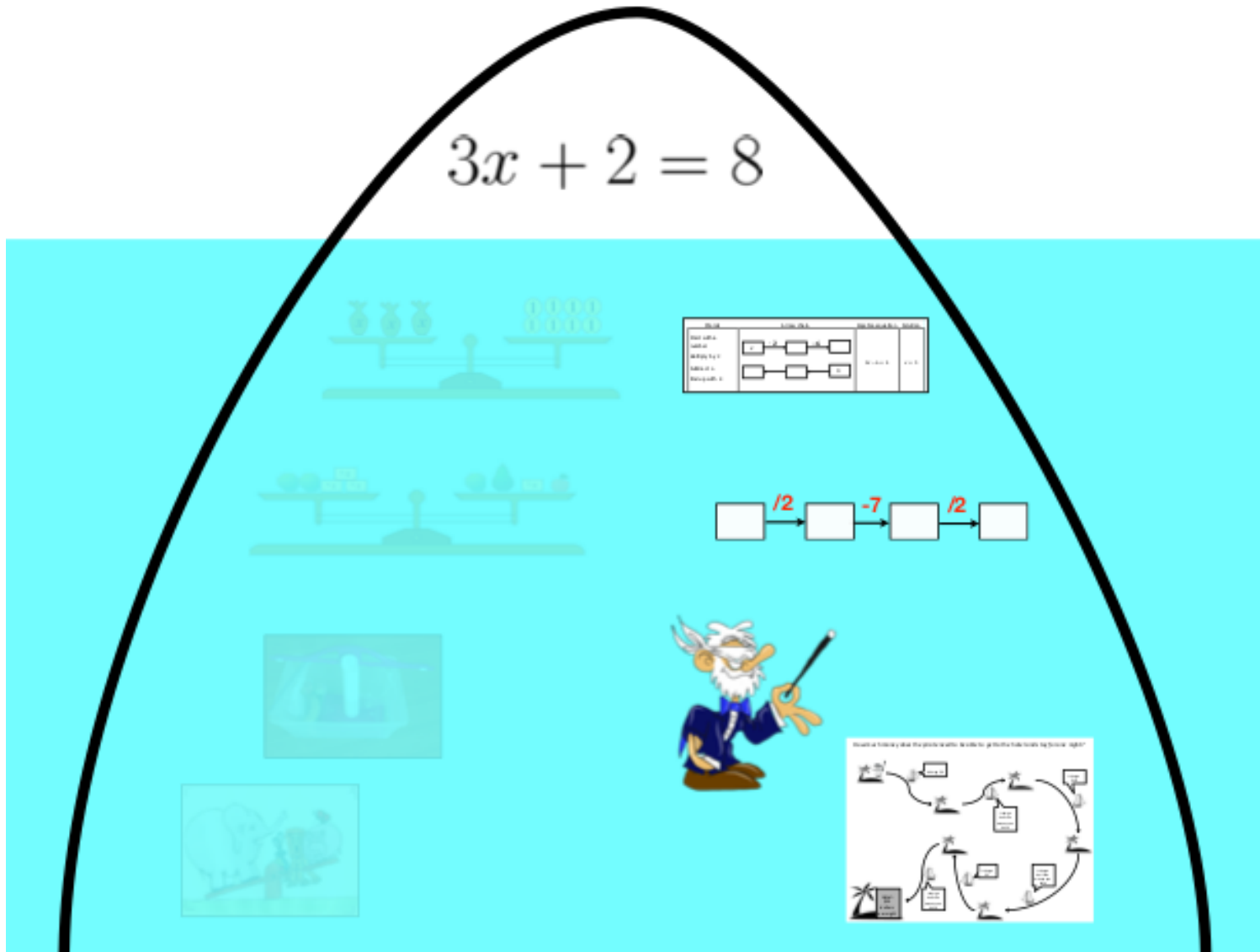
The arrow chain model

$$\frac{3}{4}x - 9 = -3$$

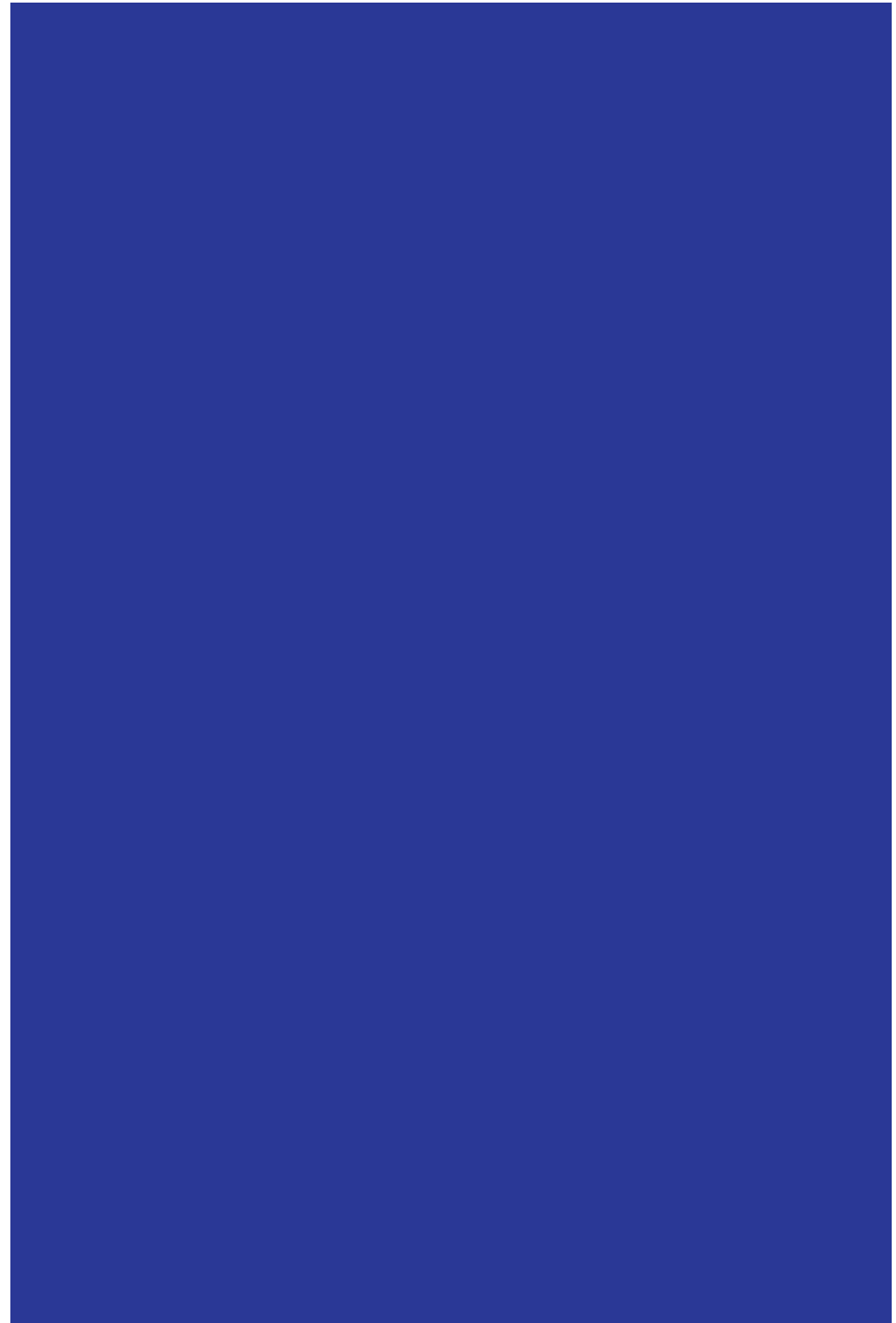
- **Structure and meaning?**
 - The equation describes a *process*: There is an unknown number and something happens to it
 - The equals sign means “the result”
- **Strategies** for symbol manipulation
 - Backtracking: Undo by doing the opposite operation in the opposite order

Progressive formalization for the arrow chain

$$3x + 2 = 8$$



In the classroom



What do models do for students?

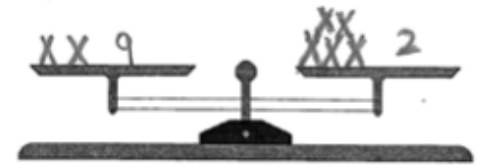
Models

Structure &
meaning

1. Use the balance scale on the right to model the equation,

$$2x + 9 = 5x + 2$$

$x = x$
(x is displayed as an object)



What do models do for students?

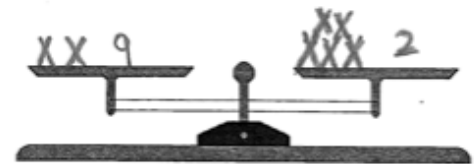
Models

Structure & meaning

Strategies

1. Use the balance scale on the right to model the equation,

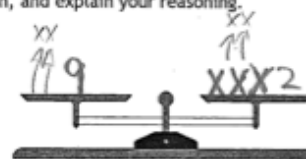
$$2x + 9 = 5x + 2$$



$x = x$
(x is displayed as an object)

2. Then, use balance strategies to solve for x . For each step, draw the balance scale, write the equation, and explain your reasoning.

Step 1:



Equation: $9 = 3x + 2$

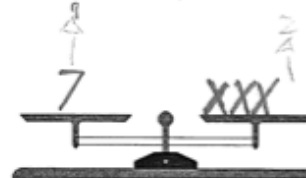
What did you do?

$-2x$ on both sides

Why will the scale remain balanced?

I did the same to both sides.

Step 2:



Equation: $7 = 3x$

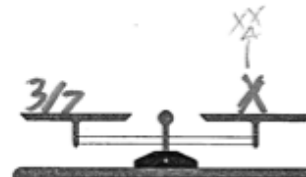
What did you do?

-2 from both sides

Why will the scale remain balanced?

I did the same to both sides.

Step 3:



Equation: $3/7 = x$

What did you do?

$\div 3$ to both sides

Why will the scale remain balanced?

I did the same problem to both sides evenly.

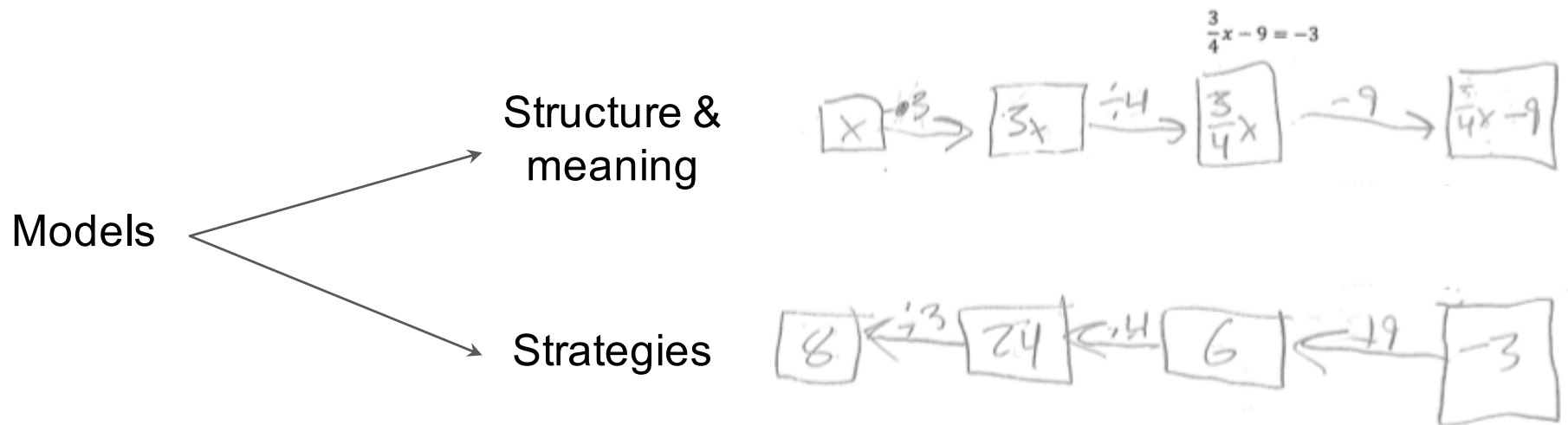
What do models do for students?

Models

Structure &
meaning

A sequence of four boxes connected by arrows, showing the steps to solve the equation $\frac{3}{4}x - 9 = -3$. The first box contains x . An arrow with $\times 3$ above it points to the second box, which contains $3x$. An arrow with $\div 4$ above it points to the third box, which contains $\frac{3}{4}x$. An arrow with -9 above it points to the fourth box, which contains $\frac{3}{4}x - 9$. Above the third box, the original equation $\frac{3}{4}x - 9 = -3$ is written.

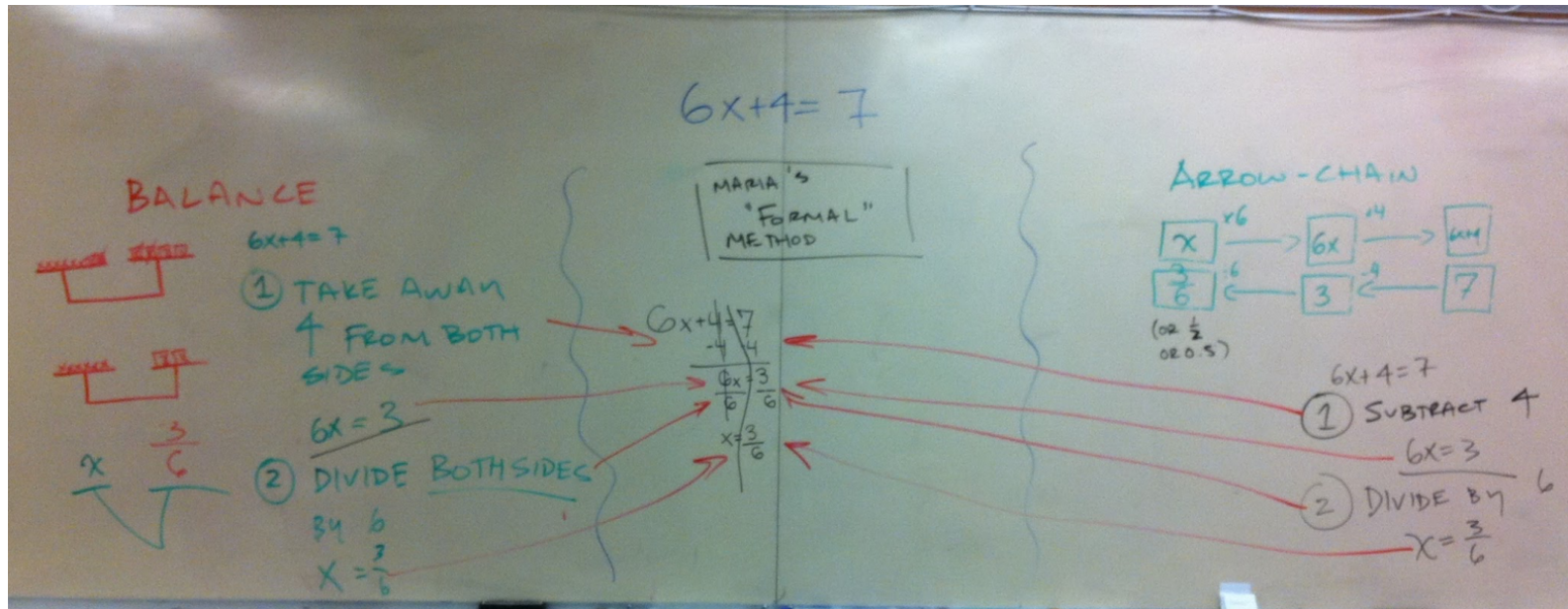
What do models do for students?



Models help students *learn* math

Models are tools that students can use to *do* math

What do models do for students?



Models help students *learn* math

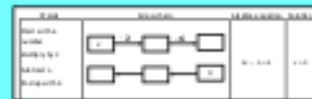
Models are tools that students can use to *do* math

Instructional sequences



Focus on the middle layer

$$3x + 2 = 8$$

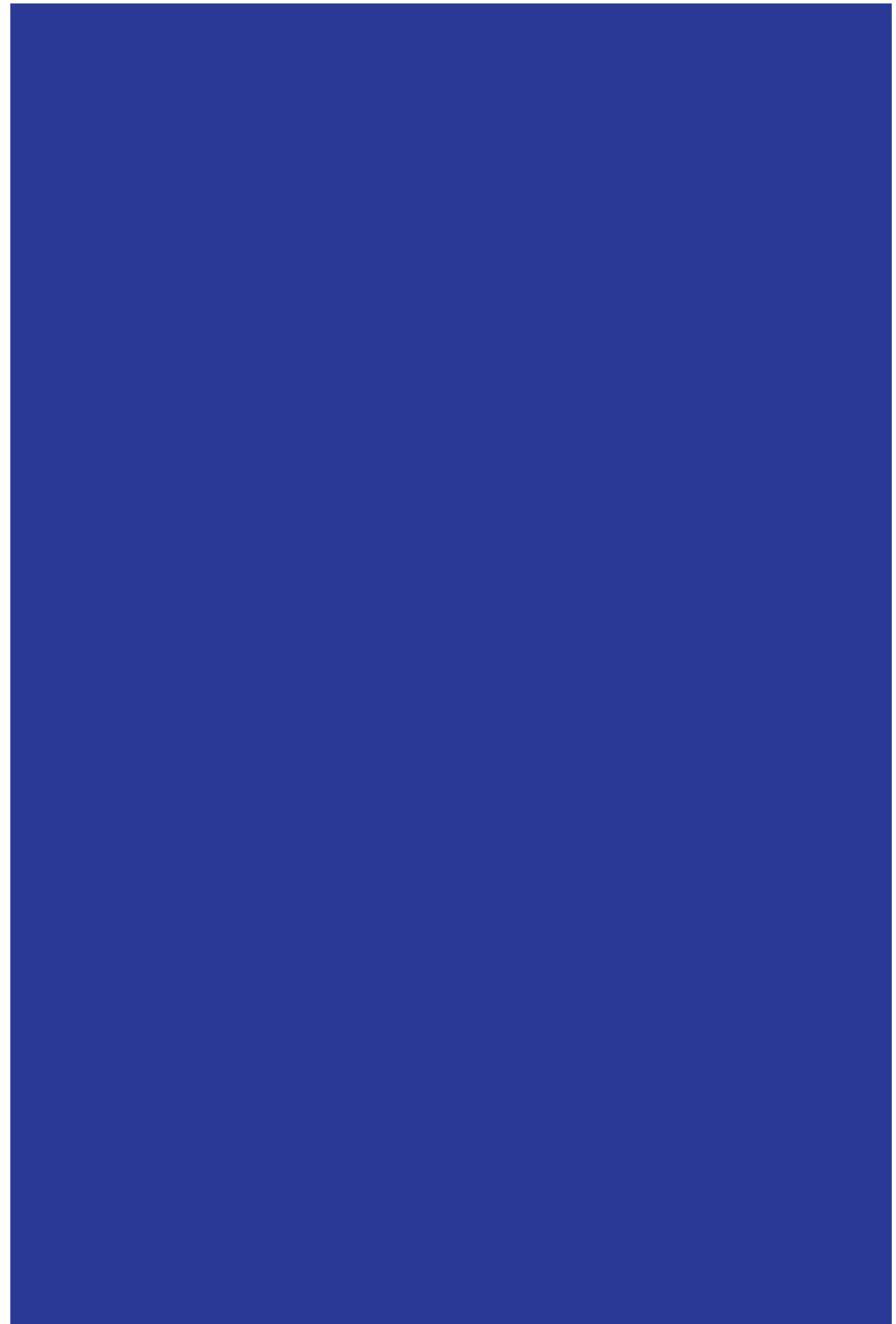


Reflection questions to guide design

1. What is the **key mathematical structure** or idea?
2. What is a pre-formal **model** that can reveal this structure, and be used as a tool to do mathematics?
3. What **context** is (a) “begging to be organized” and (b) is such that a model *of* activity that can become a model *for* activity?
4. Sketch an **iceberg** that captures the experiential activity in context (lower level), the model(s) (middle level), and the formal mathematics (upper level).



Extensions





Which equation can you solve using an arrow chain?

$$16x^2 - 8x + 11 = 9$$

$$(4x - 2)^2 + 7 = 9$$

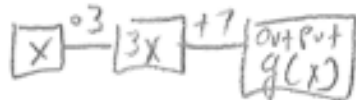
Arrow chains as representation of a function

3. Consider the function, $g(x) = 3x + 7$

a. Write a real-world situation for this function.

You started with \$7 & get \$3 every day.

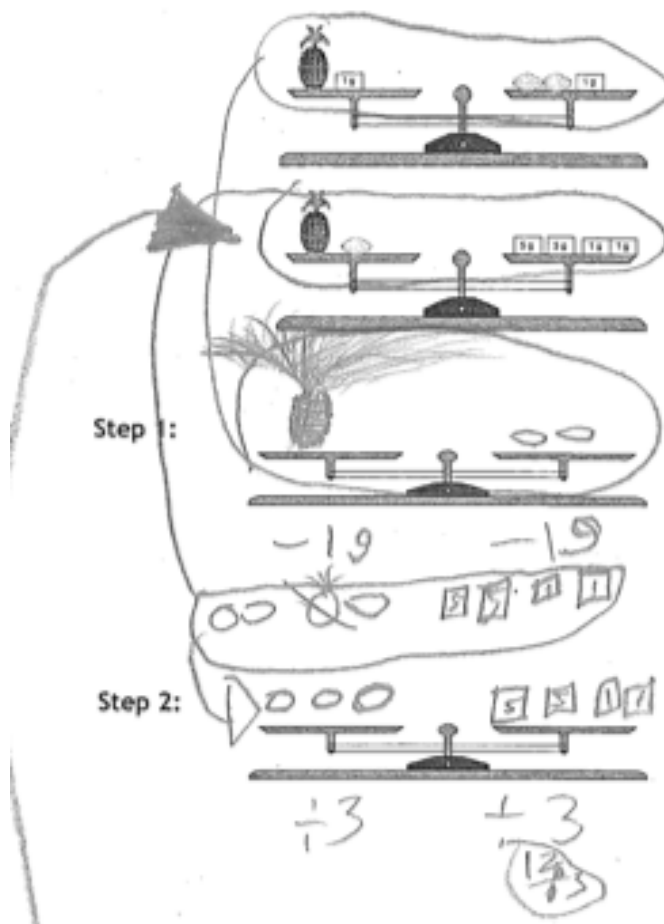
b. Make an arrow chain for this function.



c. Complete the table below

Input x	Output $g(x)$
-3	-2
-1	4
2	13
5	22

Balance model for substitution



Consider the balance scales on the left.
Find the weight of one lemon and one
pineapple using a *substitution strategy*.

For each step, draw the balance scales
and explain your method.

What did you do?

I subtracted 19 from
both sides of top scale.

Why will the scale remain balanced?

I made even trades.

What did you do?

I felt as substituted
a pineapple for 2 lemons.

Why will the scale remain balanced?

they were equal.

Learn more about progressive formalization

Webb, D. C., Boswinkel, N., & Dekker, T. (2008). Beneath the tip of the iceberg: Using representations to support student understanding. *Mathematics Teaching in the Middle School*, 14(2), 110–113.

Webb, D.C. (2017). [The Iceberg Model: Rethinking Mathematics Instruction from a Student Perspective](#). In L. West & M. Boston (Eds.), *Annual Perspectives in Mathematics Education: Reflective and Collaborative Processes to Improve Mathematics Teaching* (pp. 201-209). Reston, VA: NCTM.

<http://www.RMEintheclassroom.com>

http://www.staff.science.uu.nl/~heuve108/download-rme/vdHeuvel-2000_rme-guided-tour.pdf

<https://www.uu.nl/en/research/freudenthal-institute>

<https://www.icrme.net>

[Presentation: Making Sense of Algebra with Realistic Mathematics Education](#)

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