Mathematics of Gerrymandering

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“The core principle of republican government [is] that the voters should choose their representatives, not the other way around.”

Acknowledgments

• Metric Geometry and Gerrymandering Group, Tufts University

• Students in my IB/AP Statistics Class

• Administration who lets me experiment

• My Family
Goals

• History of gerrymandering

• Mathematical modeling process

• Suggestions for high school math classes
History

• Massachusetts-1812
• Racial gerrymandering
• Partisan gerrymandering
Settled Law

• All districts in a state must have the same number of people, according to the census

• The district’s area must be contiguous

• The district must not violate Civil Rights Act
Partisan Gerrymandering

- **Wisconsin**
  - *Whitford v. Gill*—State Assembly Districts
  - 60/99 Seats Republican
  - Majority of the voters were Democratic

- **Maryland**
  - *Benisek v. Lomone*—6th Congressional District
  - Republican district split into two Democratic areas
  - 7/8 seats Democratic
  - About 40% of the voters are Republican
Why now?

“Workable standard for measuring a gerrymander’s burden on representational rights”

“Technology is both a threat and a promise.”

Why Teach?

• College, Career and Citizenship ready

• Develop confident students who can evaluate mathematical claims
Modeling Process

1. Identify the Issues
2. Define Variables
3. Do the Math
4. Expand Model
5. Evaluate Model
6. Report Results

Guidelines for Assessment and Instruction in Mathematical Modeling Education
Redistrict Squaritopia
The Issues

Compact Districts

Proportional Districts

For Gray

Against Gray
Modeling Process

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Guidelines for Assessment and Instruction in Mathematical Modeling Education
What is a fair shape?
Compact Districts
Perimeter = 12

Proportion District
Perimeter = 20

For Gray
Perimeter = 14

Against Gray
Perimeter = 20
Isoperimetric “Polsby-Popper”

- Normalize the ratio of the actual area to the maximum possible area of a circle for a given perimeter

\[ Iso = \frac{4\pi A}{P^2} \]

- “Squaretopia” Perimeter district to area of square with given perimeter.

\[ PP = \frac{16A}{P^2} \]
Polsby-Popper Ratios

Compact Districts

Perimeter = 12

\[ PP = \frac{16(9)}{P^2} \]

\[ PP = 1 \]

For Gray

Perimeter = 14

\[ PP = \frac{16(9)}{P^2} \]

\[ PP = 0.735 \]

Proportion District

Perimeter = 20

\[ PP = \frac{16(9)}{P^2} \]

\[ PP = 0.36 \]

Against Gray

Perimeter = 20

\[ PP = \frac{16(9)}{P^2} \]

\[ PP = 0.36 \]
Explore other shapes
Actual District

A = 678 sq. miles
P = 132.872 miles
PP = .44
Enclose district in a circle

• Ratio of the area of the district with the area the smallest circle that can enclose the district.

• “Squaretopia”- ratio of the area of the district with the area of the square that can enclose it.

\[ R = \frac{9}{L^2} \]

• Called Reock Measure
Enclosing Square
“Reock”

Compact Districts

\[ R = \frac{9}{L^2} \]
\[ R = \frac{9}{3 \times 3} \]
\[ R = 1 \]

Proportional Districts

\[ R = \frac{9}{9^2} \]
\[ R = 0.1111 \]

For Gray

\[ R = \frac{9}{4^2} \]
\[ R = 0.5625 \]

Against Gray

\[ R = \frac{9}{7^2} \]
\[ R = 0.1836 \]
Enclosing Circle
Convex Hull

- Compares the area of the district with the area the convex polygon that can enclose the district.

- Stretch a rubber band around the outside
Convex Hull Ratio

**Compact Districts**

For Gray

\[ R = \frac{9}{9 + 1.5} \]

R = 0.857

**Proportion District**

Against Gray

\[ R = \frac{9}{9 + 1.5 + 0.5 + 2.5 + 1.5} \]

R = 0.6
Geometric Models

• Can spot some suspicious districts

• Can help measure compactness which is a desirable measure on its own

• Doesn’t directly address voting behavior
Modeling Process

Identify the Issues
Define Variables
Do the Math
Expand Model
Evaluate Model
Report Results

Guidelines for Assessment and Instruction in Mathematical Modeling Education
What can statistics measure?

- Extreme partisan skew
- Entrenchment
- Symmetric Effects
- Receptivity to change in voter preferences
- Competitiveness
Efficiency Gap

- Measures “wasted votes” those not needed to actually elect the winner
- All votes for the losing candidate are “wasted”
- All votes above 50% for the winning candidate are wasted
Can the Efficiency Gap measure gerrymandering?

<table>
<thead>
<tr>
<th>District</th>
<th>A votes</th>
<th>B votes</th>
<th>Winner</th>
<th>Wasted A “$W_a$”</th>
<th>Wasted B “$W_b$”</th>
<th>$W_a + W_b$</th>
<th>$W_a - W_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>95</td>
<td>5</td>
<td>A</td>
<td>45</td>
<td>5</td>
<td>50</td>
<td>40</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>60</td>
<td>B</td>
<td>40</td>
<td>10</td>
<td>50</td>
<td>30</td>
</tr>
<tr>
<td>3</td>
<td>75</td>
<td>25</td>
<td>A</td>
<td>25</td>
<td>25</td>
<td>50</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>45</td>
<td>55</td>
<td>B</td>
<td>45</td>
<td>5</td>
<td>50</td>
<td>40</td>
</tr>
<tr>
<td>5</td>
<td>45</td>
<td>55</td>
<td>B</td>
<td>45</td>
<td>5</td>
<td>50</td>
<td>40</td>
</tr>
<tr>
<td>Totals</td>
<td>300</td>
<td>200</td>
<td>A:B 2:3</td>
<td>200</td>
<td>50</td>
<td>250</td>
<td>150</td>
</tr>
</tbody>
</table>

$$EG = \frac{W_a - W_b}{V} = \frac{200 - 50}{500} = .3$$
Analysis

- Measures well the packing of votes as well as cracking
- But, can’t measure well what happens in uncontested districts
- Could have

\[ EG2 = \frac{W_a}{V_a} - \frac{W_b}{V_b} = \frac{200}{300} - \frac{50}{200} = 0.42 \]
Can E.G. validate Proportionality?

<table>
<thead>
<tr>
<th>District</th>
<th>A votes</th>
<th>B votes</th>
<th>Winner</th>
<th>Wasted A “Wa”</th>
<th>Wasted B “Wb”</th>
<th>Wa + Wb</th>
<th>Wa – Wb</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>75</td>
<td>25</td>
<td>A</td>
<td>25</td>
<td>25</td>
<td>50</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>65</td>
<td>35</td>
<td>A</td>
<td>15</td>
<td>35</td>
<td>50</td>
<td>-20</td>
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<tr>
<td>3</td>
<td>70</td>
<td>30</td>
<td>A</td>
<td>20</td>
<td>30</td>
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<td>55</td>
<td>B</td>
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<td>5</td>
<td>50</td>
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<tr>
<td>5</td>
<td>45</td>
<td>55</td>
<td>B</td>
<td>45</td>
<td>5</td>
<td>50</td>
<td>40</td>
</tr>
<tr>
<td>Totals</td>
<td>300</td>
<td>200</td>
<td>A:B 3:2</td>
<td>150</td>
<td>100</td>
<td>250</td>
<td>50</td>
</tr>
</tbody>
</table>

\[ EG = \frac{50}{500} = 0.10 \]

\[ EG2 = \frac{Wa}{Va} - \frac{Wb}{Vb} = \frac{150}{300} - \frac{100}{200} = 0 \]
Symmetric Treatment

- Are the parties treated similarly?

- Do both parties need a big swing in votes to gain a seat?

- Do both parties need just a small swing to gain a seat?
Compare Mean vs. Median

• Well-established measure of skewed data

• Median of party percentages of the vote minus mean of district percentages

• Targeted party wins a few seats by lopsided victories
How does the Mean-Median measure asymmetry?

- Democratic share of the total vote in the 18 districts
  - Democrats win 5/18 seats with 45% of the vote
  - Democrats won their seats by an average of 80%, Republicans won by an average of 60%

<table>
<thead>
<tr>
<th>Vote Percentage</th>
<th>Democratic</th>
<th>Republican</th>
</tr>
</thead>
<tbody>
<tr>
<td>29.7</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>32.7</td>
<td>0</td>
<td>0</td>
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<tr>
<td>33.9</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>36.3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>36.6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>38</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>38.1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>40.3</td>
<td>0</td>
<td>0</td>
</tr>
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<td>42.7</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>42.7</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>45.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>53.8</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>74.3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>82.2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>90.2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Median = 39.2
Mean = 45.4
California 2016

Median = 66.3
Mean = 65.1
Outlier test

• Find the distribution of all possible maps of districts that meet established requirements

• Make a histogram of expected voting results

• If the proposed plan seems unlikely to occur by chance, then the plan could be replaced with one that seems likely
Is the Wisconsin map an outliner?

• Compare possible plans to current plan of predicted number of seats

• About 19,000 different maps compared
Outliner in Mean-Median?

- 1,000,000,000 (1 Billion) plans generated
- Histogram of voting outcomes based on Mean-Median analysis
- Proposed plans marked in red
Efficiency Gap

The separate groupings occur as the number of seats for each party changes recalculating the efficiency gap.
Modeling Process

1. Identify the Issues
2. Define Variables
3. Do the Math
4. Evaluate Model
5. Expand/Refine Model
6. Report Results

Guidelines for Assessment and Instruction in Mathematical Modeling Education
The People Chose

- Empower students with the mathematical background and confidence to understand contemporary issues
Thank you

• Background information on the link

• Follow me on twitter @LindaSaeta
Student Reactions

- Enjoyed how it was relevant to “real life” and related to their government classes

- Many mentioned they liked learning what modeling meant

- As the week went on, I noticed students using better vocabulary such as competitive districts, compact, proportional, motivation to vote

- They felt the unit was very interactive

- I had students volunteer in discussion who normally don’t.

- Surprised how well “Squaretopia” also made the point about wasted votes by comparing either the compact or proportional district to a “For Gray” or Against.