1 to 1 Million: Number Sense Progressions for K-5

NCTM 2018

Arjan Khalsa

akhalsa@activatelearning.com

@arjankhalsa

www.arjankhalsa.com

www.conceptuamath.com

Free Resources on Progressions

Progressions for the Common Core State Standards in Mathematics (draft)

©The Common Core Standards Writing Team

29 May 2011

K, Counting and Cardinality; K–5, Operations and Algebraic Thinking

Grade 1

Students extend their work in three major and interrelated ways, by:

- Representing and solving a new type of problem situation (Compare);
- Representing and solving the subtypes for all unknowns in all three types;
- Using Level 2 and Level 3 methods to extend addition and subtraction problem solving beyond 10, to problems within 20. In particular, the OA progression in Grade 1 deals with adding two single-digit addends, and related subtractions.

Representing and solving a new type of problem situation (Compare) In a Compare situation, two quantities are compared to find "How many more" or "How many less."[K.CC.6, K.CC.7] One reason Compare problems are more advanced than the other two major types is that in Compare problems, one of the quantities (the difference) is not present in the situation physically, and must be conceptualized and constructed in a representation, by showing the "extra" that when added to the smaller unknown makes the total equal to the bigger unknown or by finding this quantity embedded within the bigger unknown.

The language of comparisons is also difficult. For example, "Julie has three more apples than Lucy" tells both that Julie has more apples and that the difference is three. Many students "hear" the part of the sentence about who has more, but do not initially hear the part about how many more; they need experience hearing and saying a separate sentence for each of the two parts in order to comprehend and say the one-sentence form. Another language issue is that the comparing sentence might be stated in either of two related ways, using "more" or "less." Students need considerable experience with "less" to differentiate it from "more"; some children think that "less" means "more." Finally, as well as the basic "How many more/less" question form, the comparing sentence might take an active, equalizing and counterfactual form (e.g., "How many more apples does Lucy need to have as many as Julie?") or might be stated in a static and factual way as a question about how many things are unmatched (e.g., "If there are 8 trucks and 5 drivers, how many trucks do not have a driver?"). Extensive experience with a variety of contexts is needed to master these linguistic and situational complexities. Matching with objects and with drawings, and labeling each quantity (e.g., J or Julie and L or Lucy) is helpful. Later in Grade 1, a tape diagram can be used. These comparing diagrams can continue to be used for multi-digit numbers, fractions, decimals, and variables, thus connecting understandings of these numbers in

- Other Grade 1 problems within 20, such as 14 + 5, are best viewed in the context of place value, i.e., associated with 1.NBT.4. See the NBT Progression.
- Compare problems build upon Kindergarten comparisons, in which students identified "Which is more?" or "Which is less?" without ascertaining the difference between the numbers.

K.CC.6 Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.

K.CC.7 Compare two numbers between 1 and 10 presented as written numerals.

Representing the difference in a Compare problem

Compare problem solved by matching

Compare problem represented in tape diagram

Draft, 5/29/2011, comment at commoncoretools.wordpress.com.

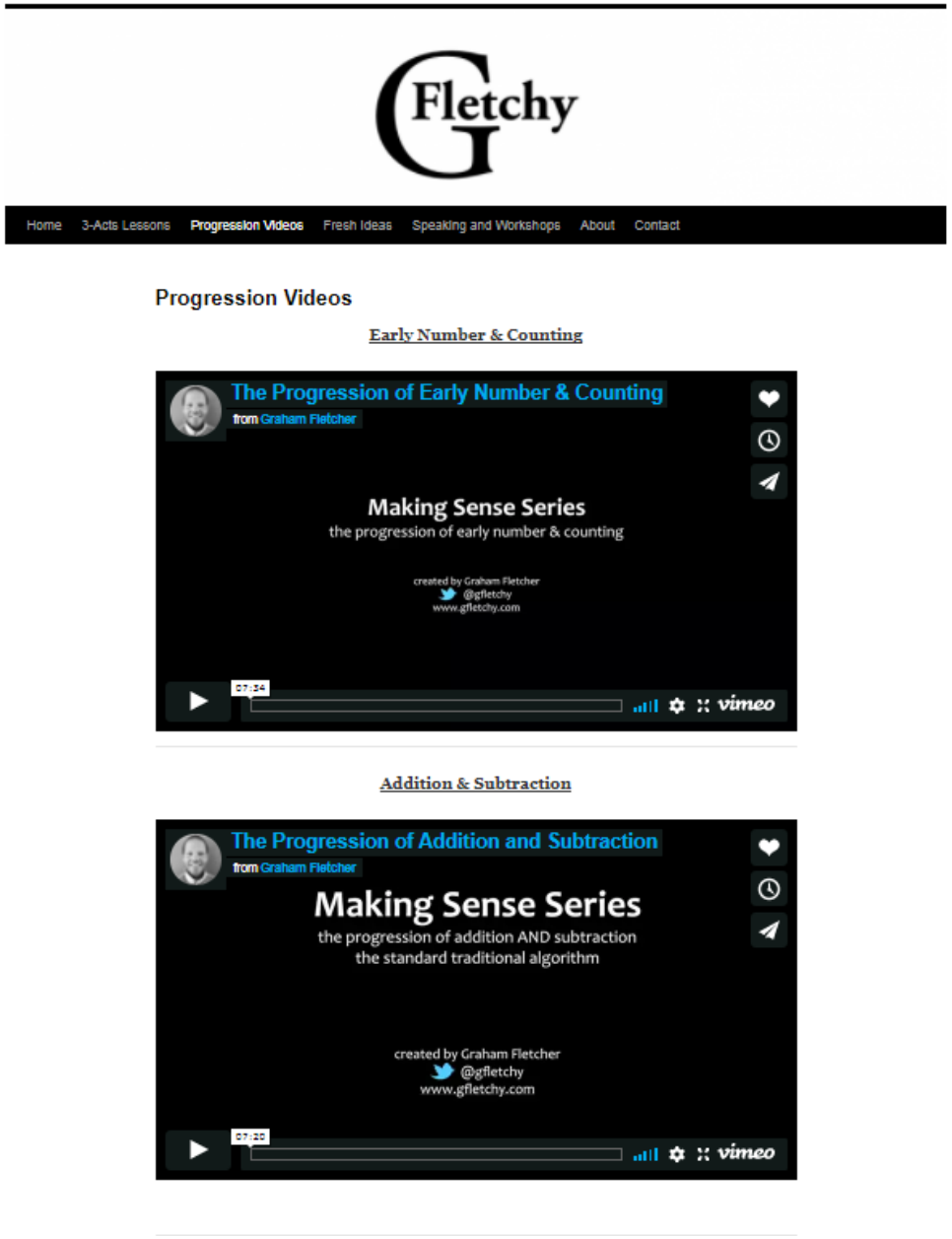
Free, online PDFs

https://commoncoretools.files.wordpress.com/2011/05/ccss_progression_cc_oa_k5_2011_05_302.pdf

http://commoncoretools.me/wp-content/uploads/2015/03/ccss_progression_nbp_k5_2015_03_16.pdf

Videos by Graham Fletcher

https://gfletchy.com/progression-videos/

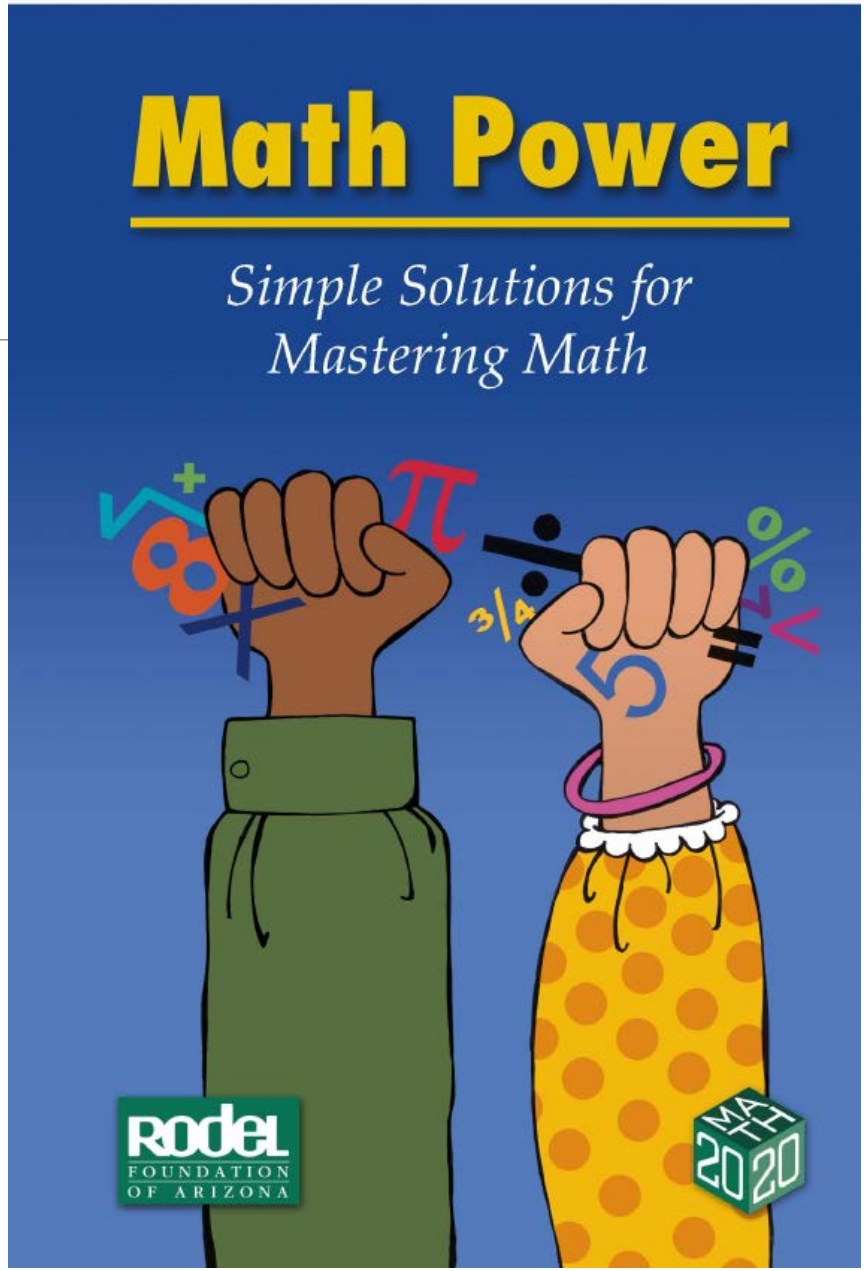


A Book from the Rodel Foundation of Arizona

Includes excellent graphics used in this presentation.

Free, online PDF

www.bit.ly/mathpowerbook

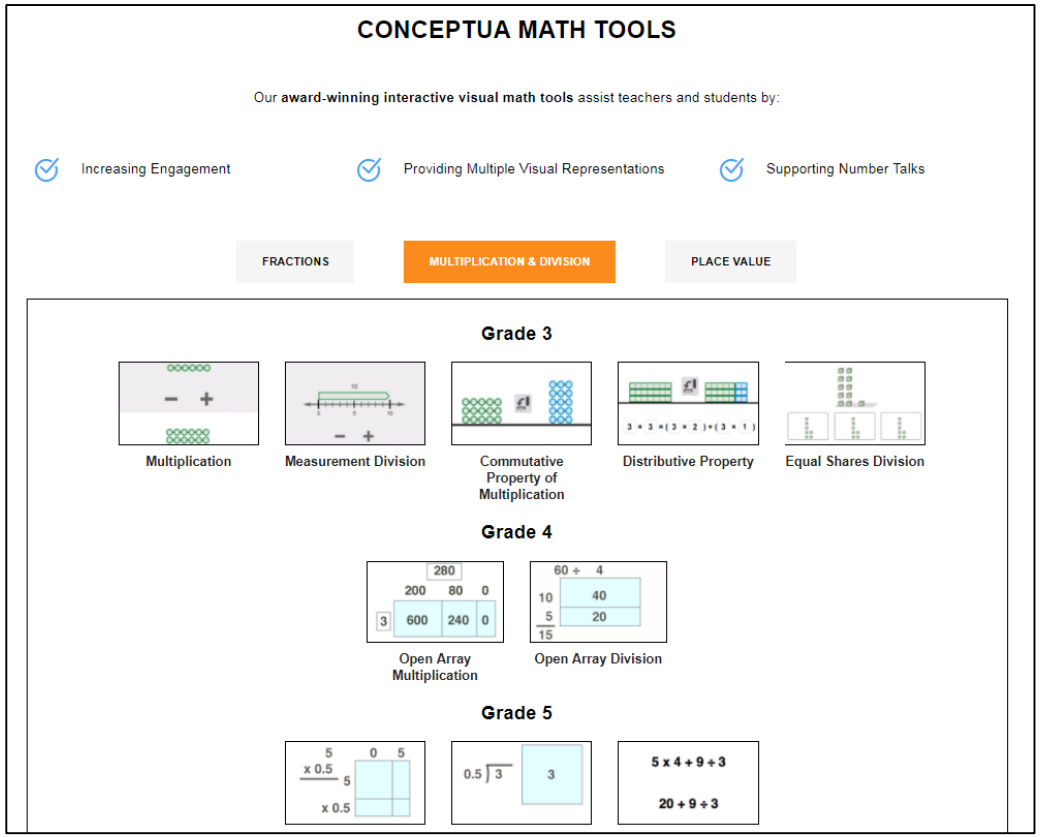


Math Tools from Conceptua Math

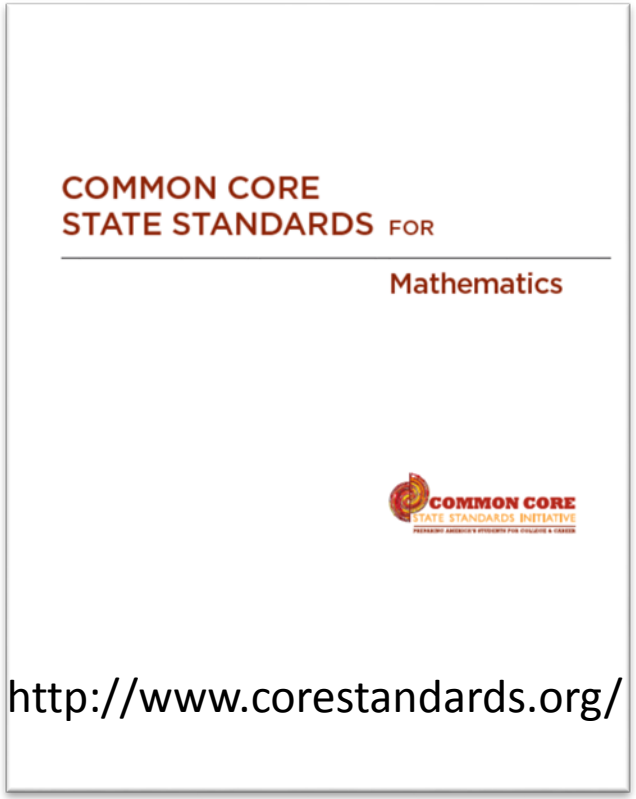
Excellent graphics used in this presentation came from these tools and their authors.

Free, online, and interactive tool

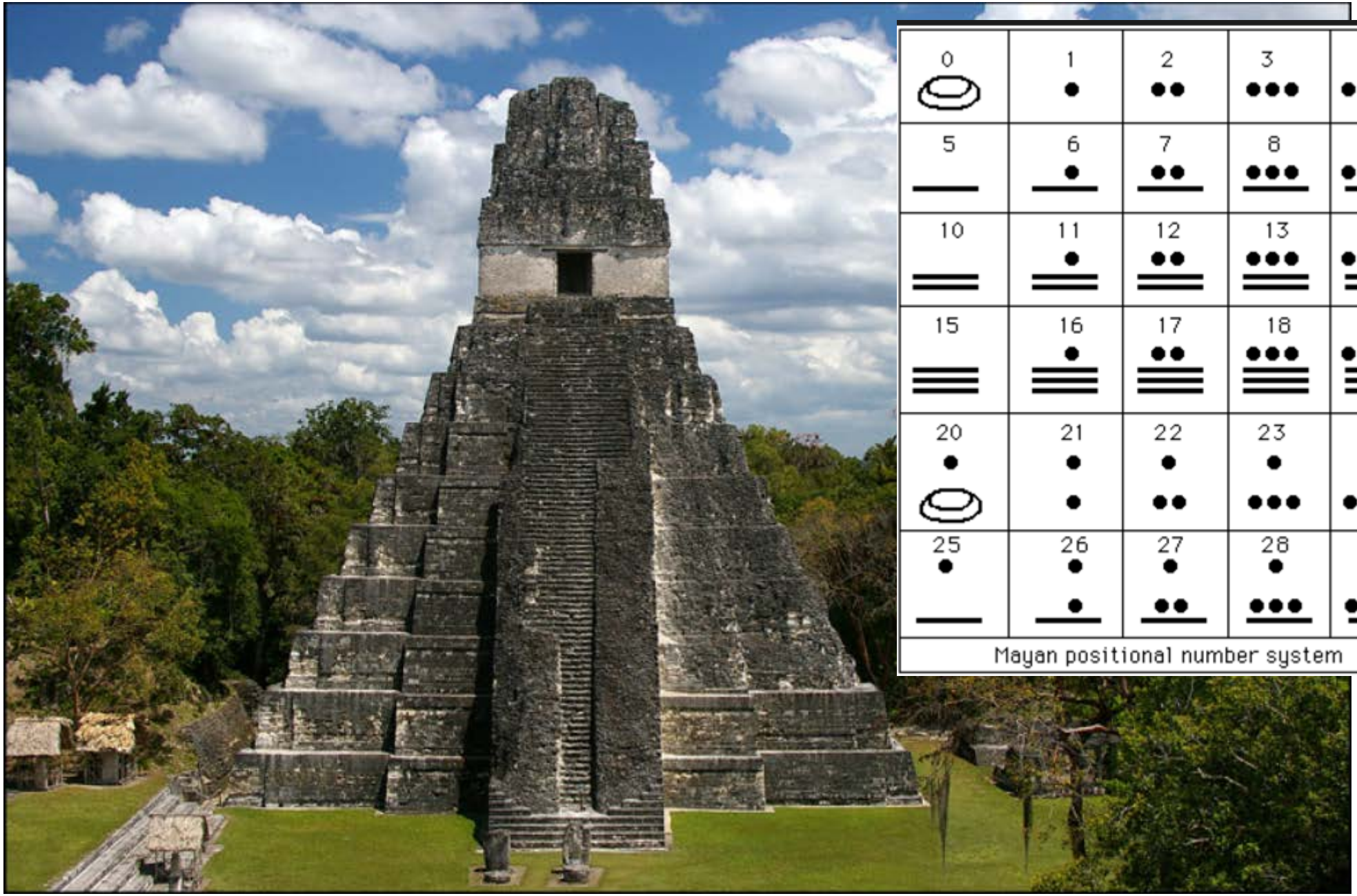
www.conceptuamath.com/math-tools/



Common Core State Standards for Math



http://www.corestandards.org/



0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	29

Mayan positional number system

Working in Base 4

Represent the value that is equivalent to 100.

What name would you give this amount?

Understanding Base 4

256’s	64’s	16’s	4’s	1’s

Finding an Equivalent for 100

1	2	1	0
64	16	4	1
1 x 64	2 x 16	1 x 4	0 x 1

64 + 32 + 4 + 0 = 100

Finding an Equivalent for 100

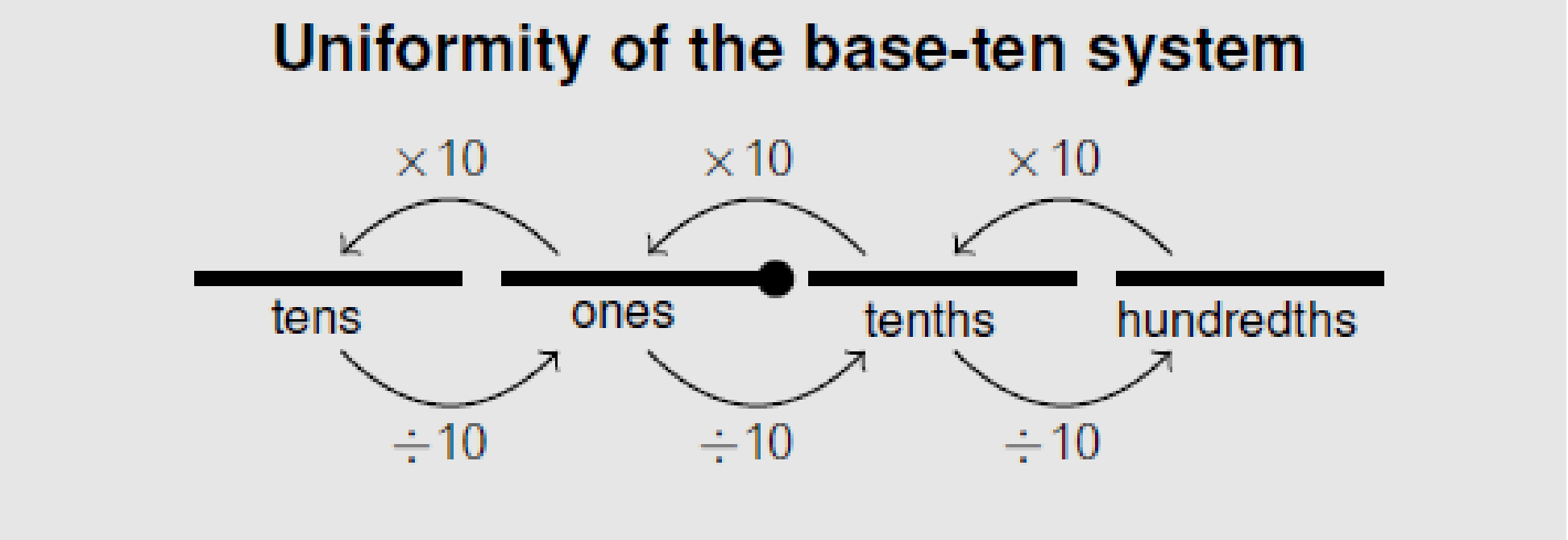
1	2	1	0
64	16	4	1
1 x 64	2 x 16	1 x 4	0 x 1

64 + 32 + 4 + 0 = 100

What Do We Call This?

1210

We have a GREAT number system!



It works, really well!

1

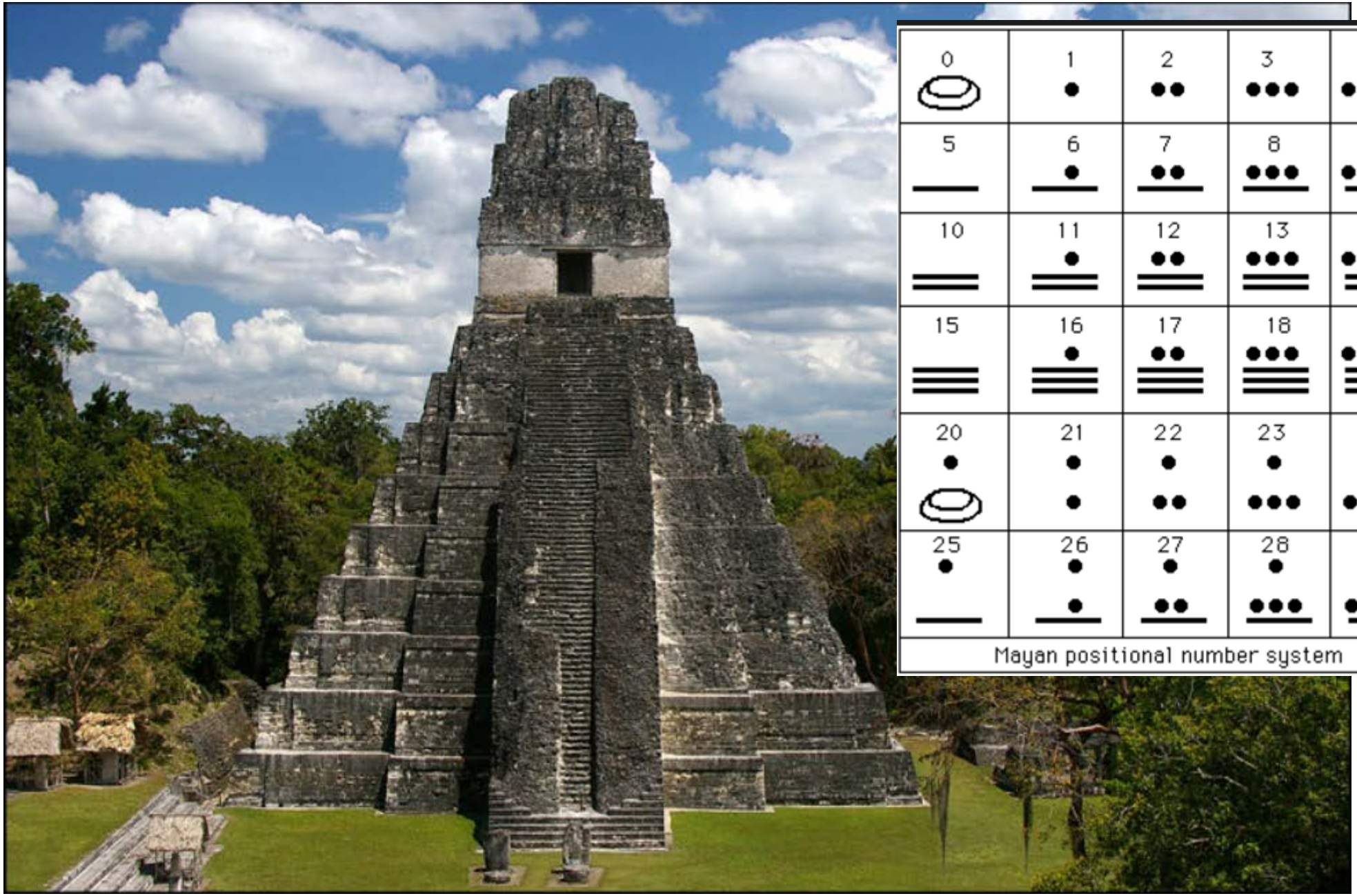
21

673

8,463

924,576

3,476,645



0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	29

Mayan positional number system

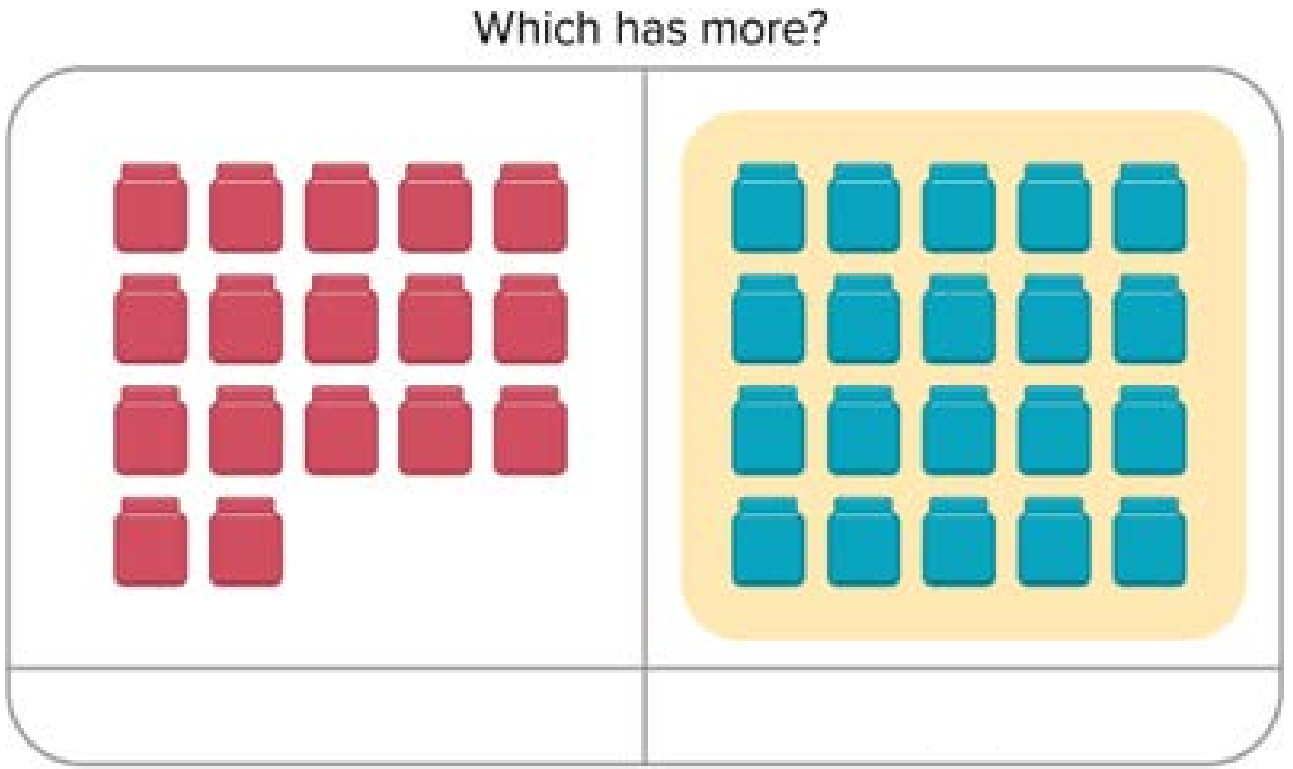
Counting and Cardinality

Recognizing and Building Quantities

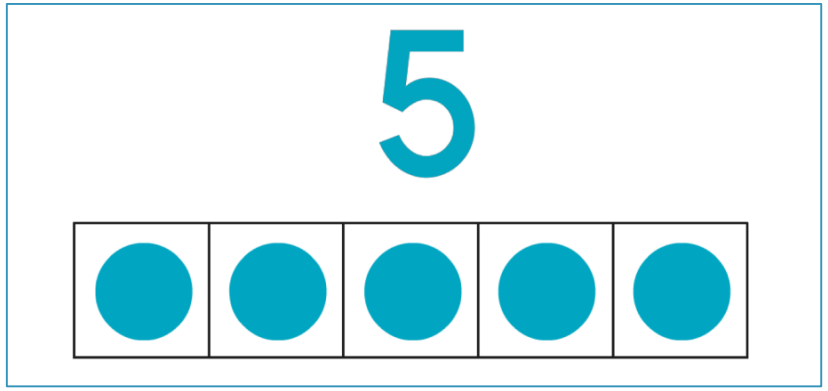
Recognizing and Building Quantities

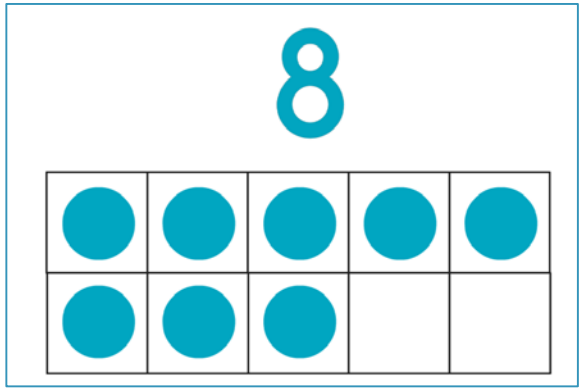
Counting Collections

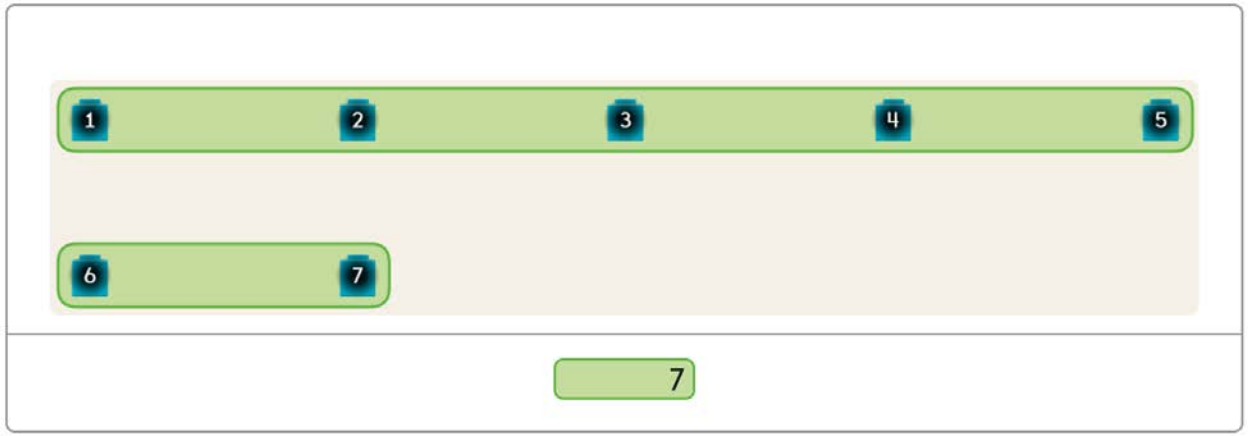
Comparing Numbers



Number Names, Numerals, and Cardinality

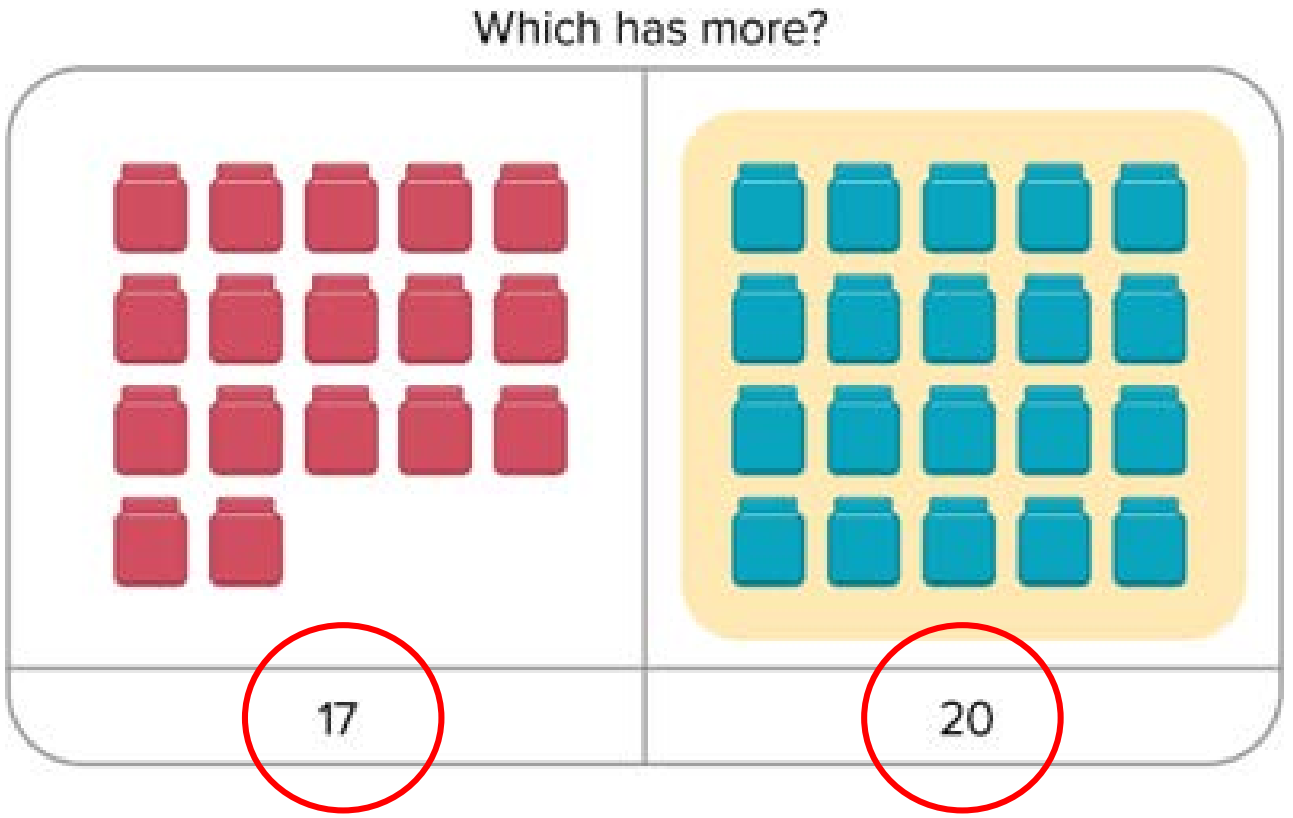




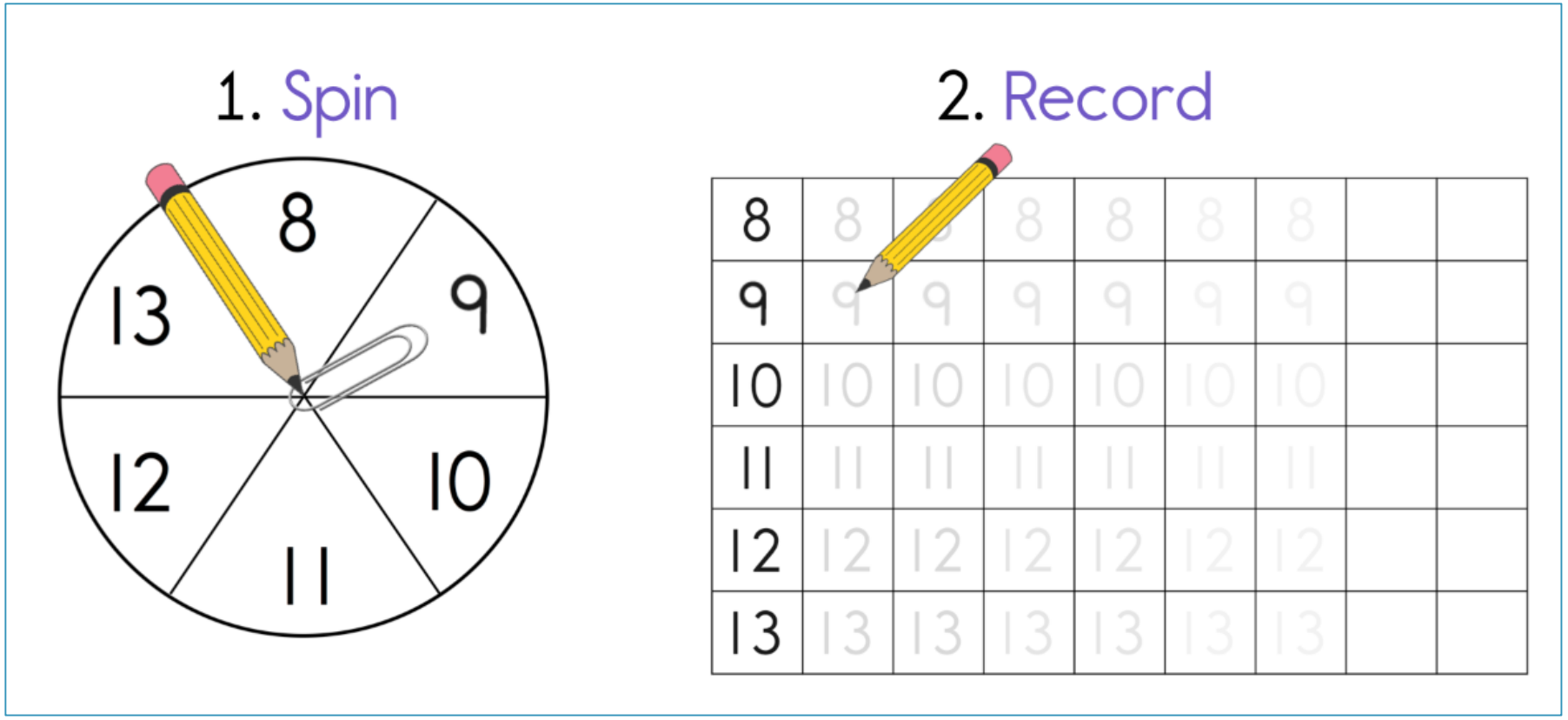


One-to-one correspondence

Comparing Numbers

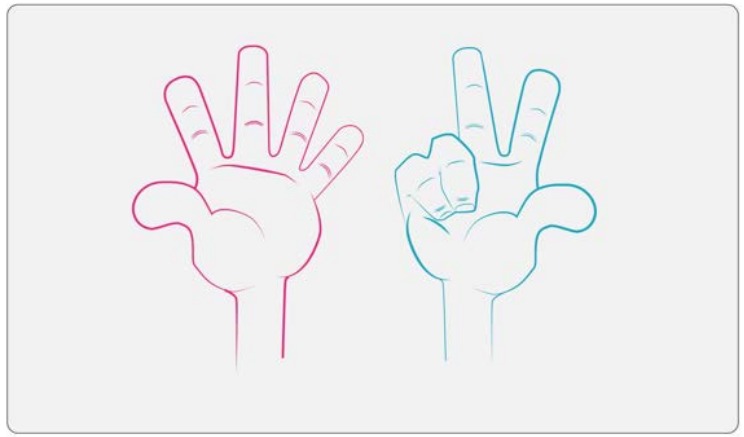


Writing Numbers

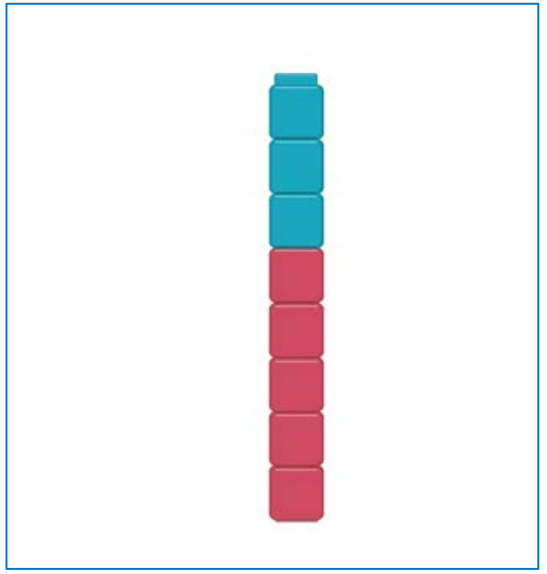


Compose Numbers

Ten Frames - Fingers

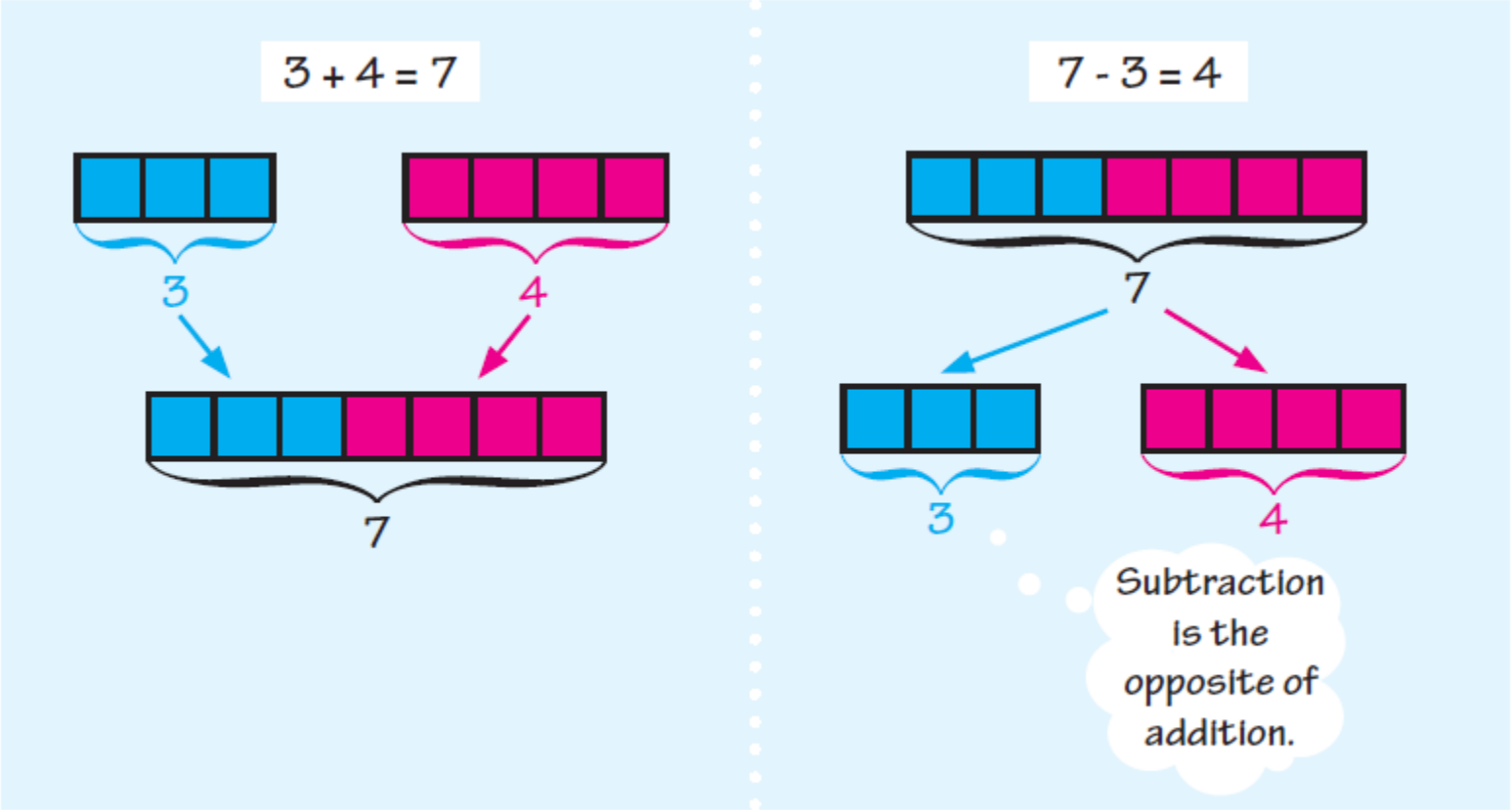


Linking Cubes



Hierarchical inclusion

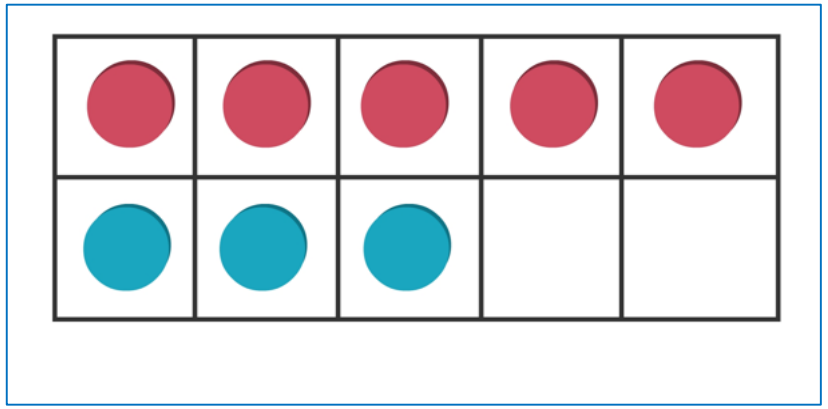
Composing and Decomposing Numbers



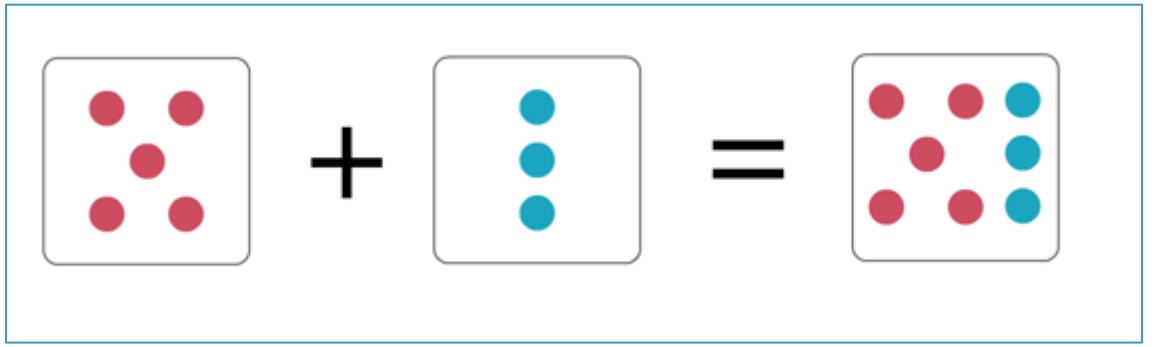
Number Conservation

Seeing a Number

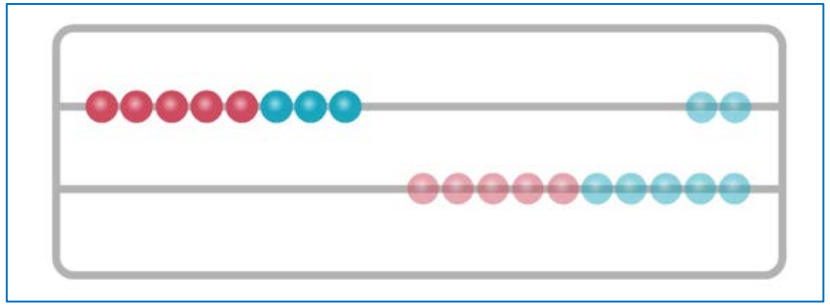
Ten Frames



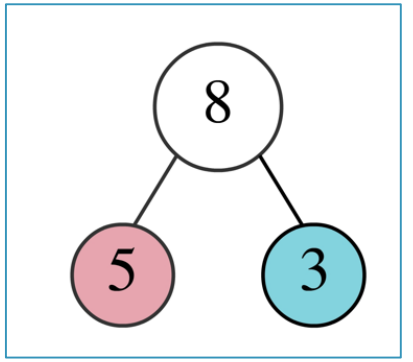
Dot Cards



Rekenrek



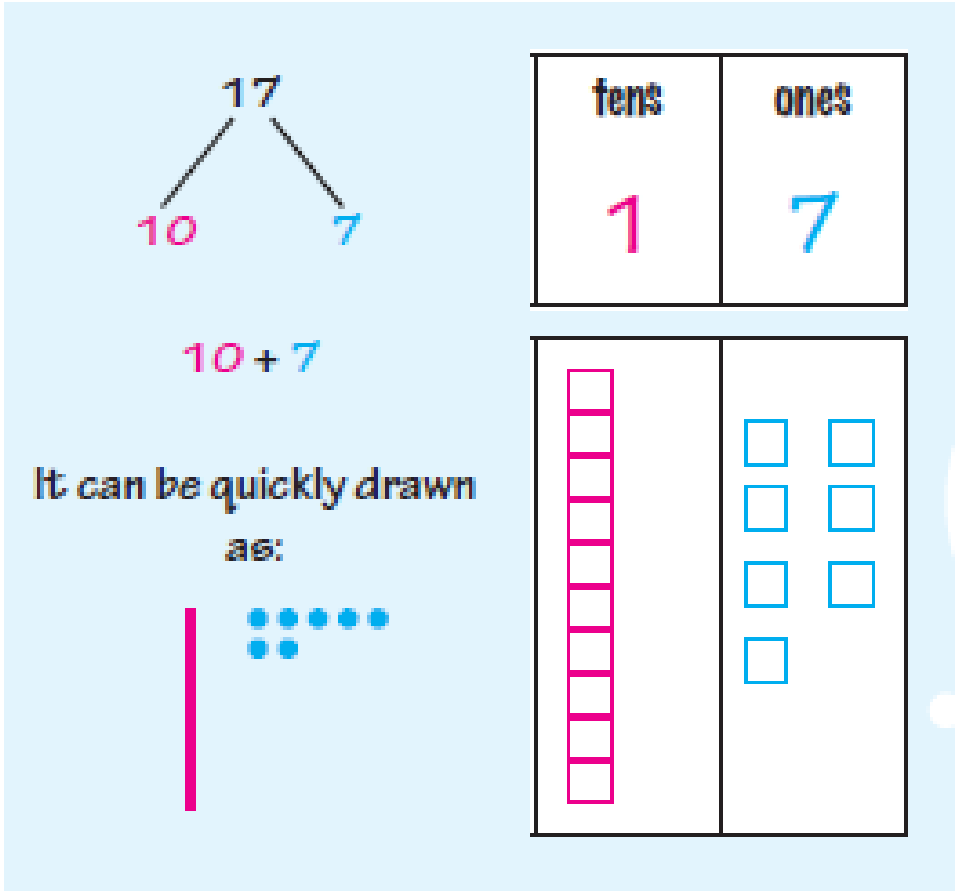
Number Bonds – Numerals!



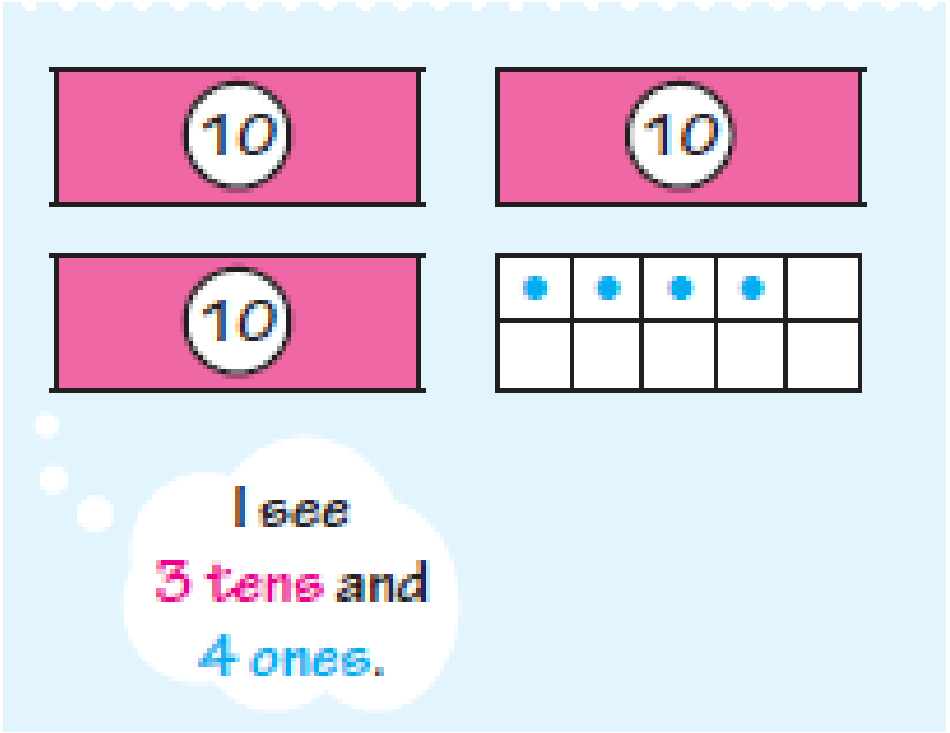
Numbers and Operations in Base 10

Kindergarten – It’s all about 10 Units

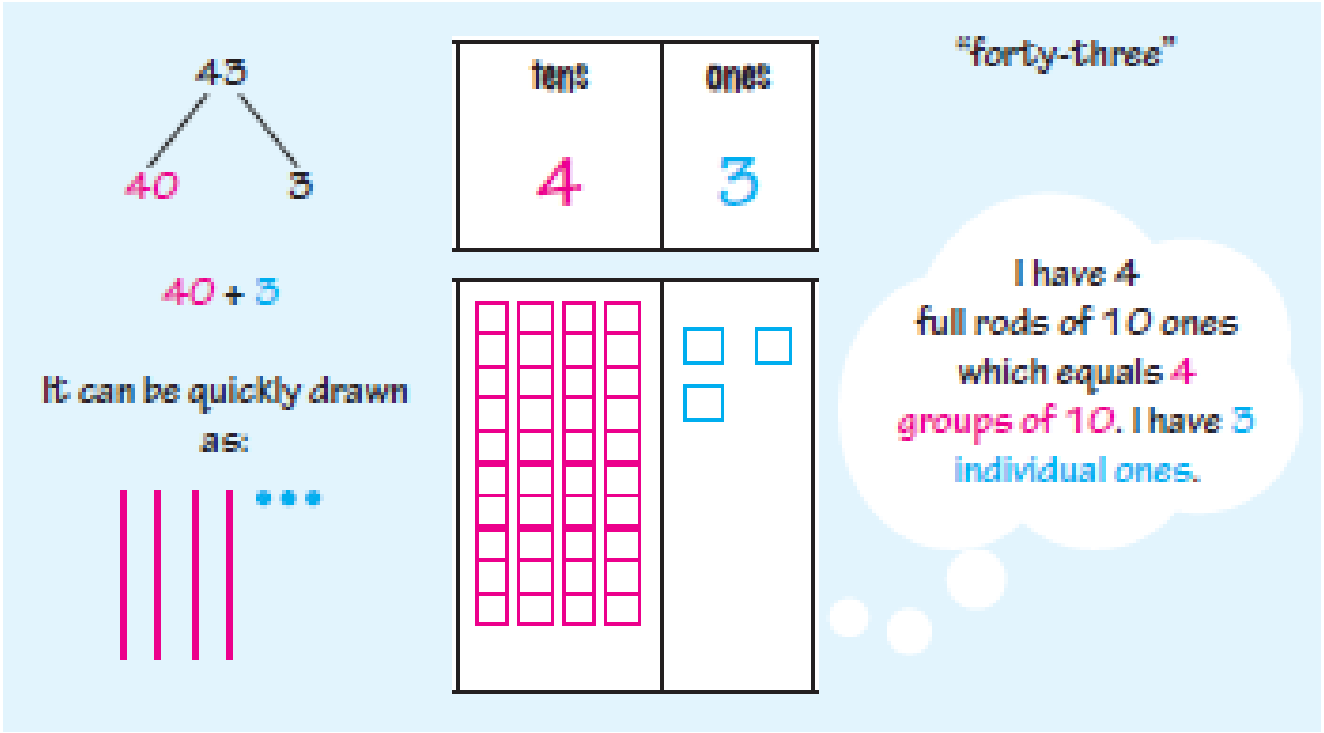
First Grade – It's All About a Unit Called 10



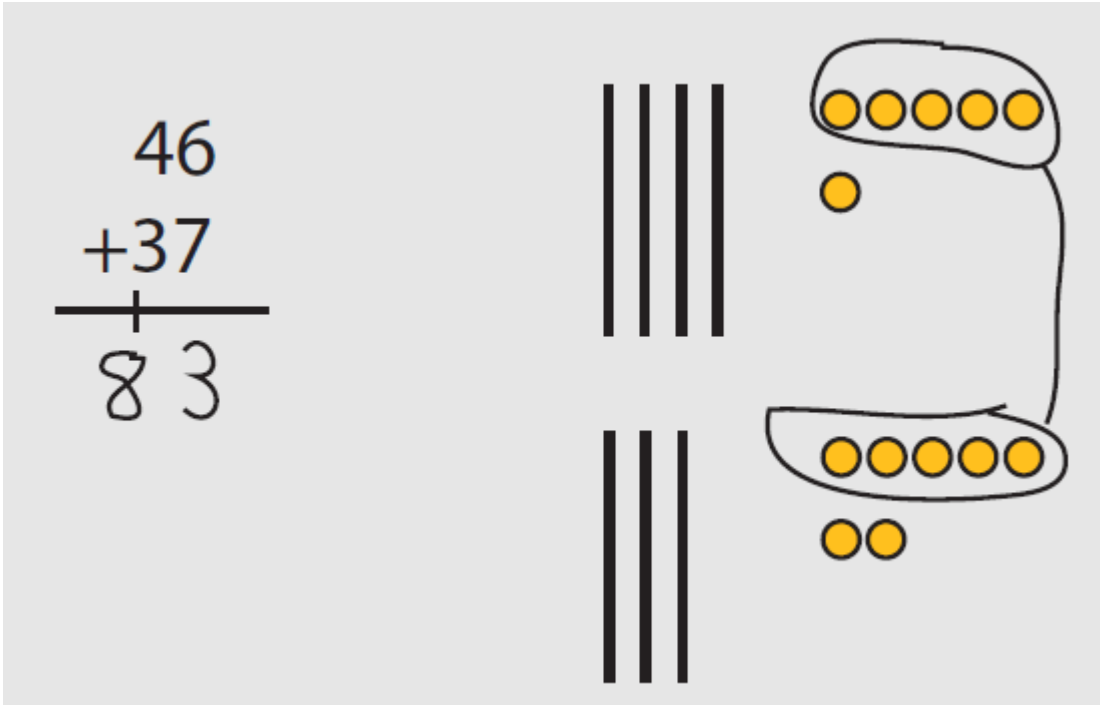
Ten Frames

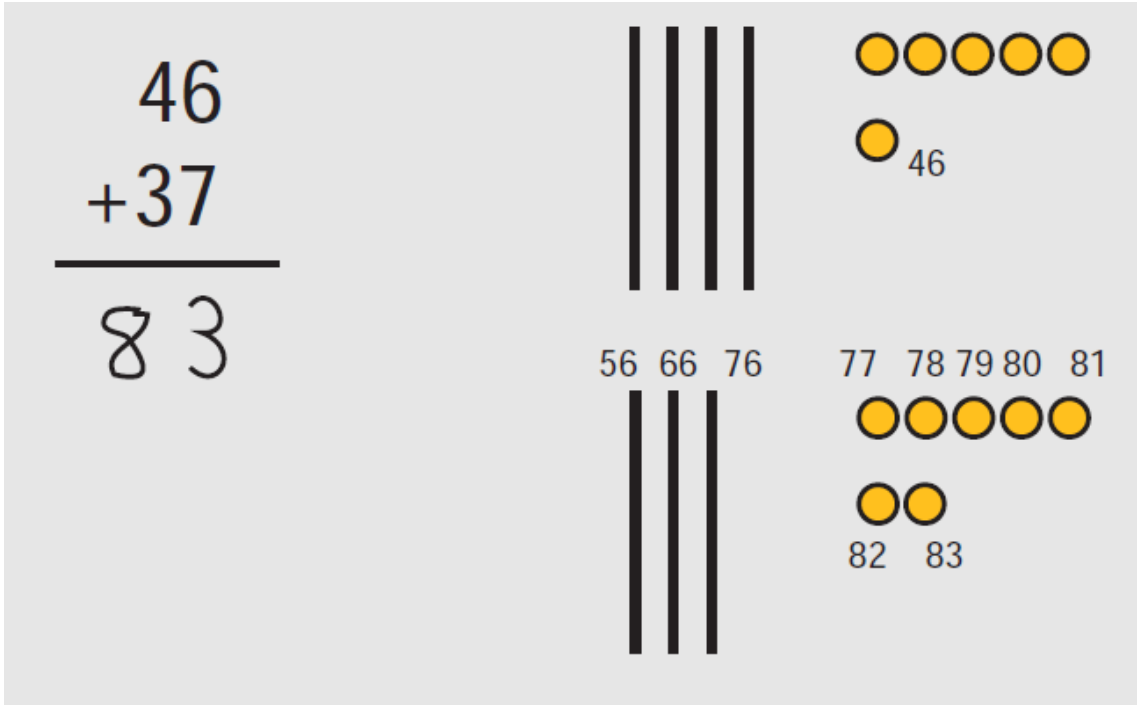


Ten Frames, Ten Rods, and Visual Representations

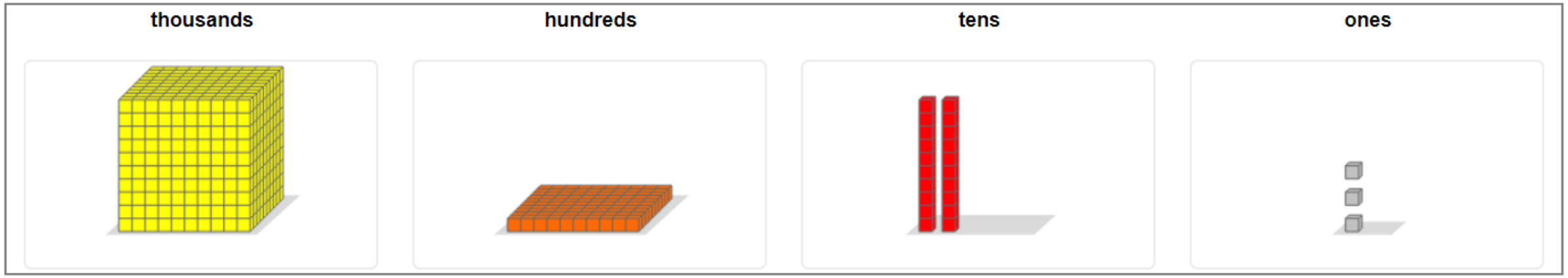


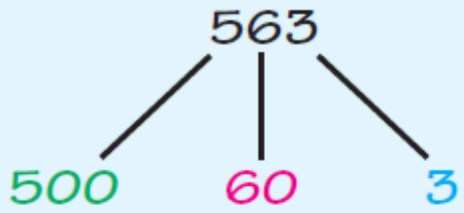
Grade 1 – Adding Within 100, Strategies around the Units of 10





Grade 2 – Fluent within 100, and Working with Hundreds



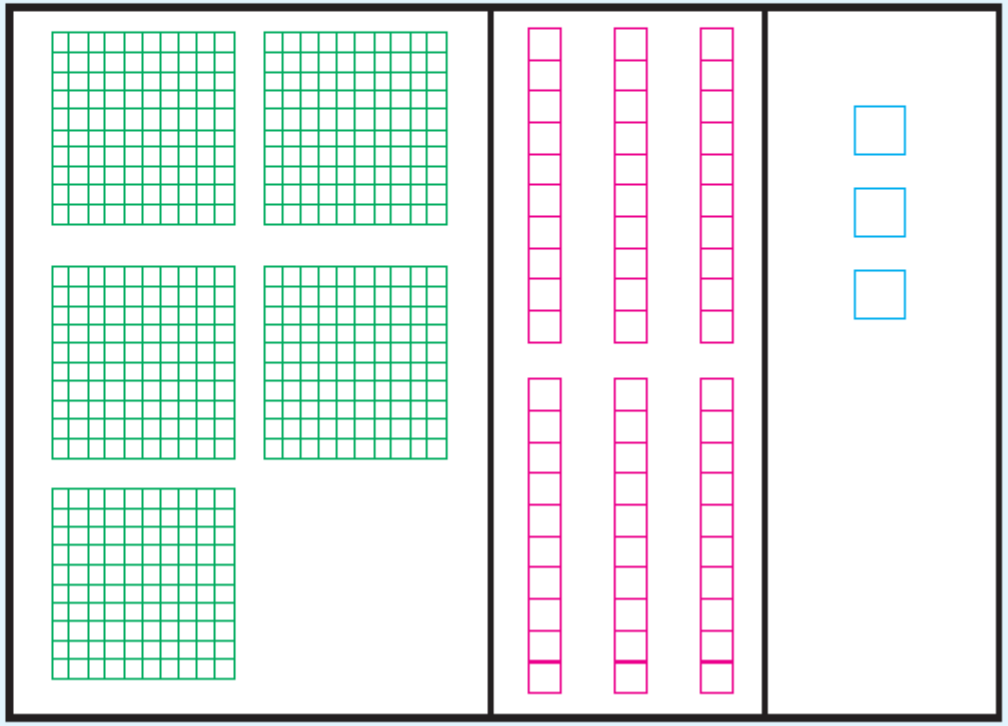


500 + 60 + 3

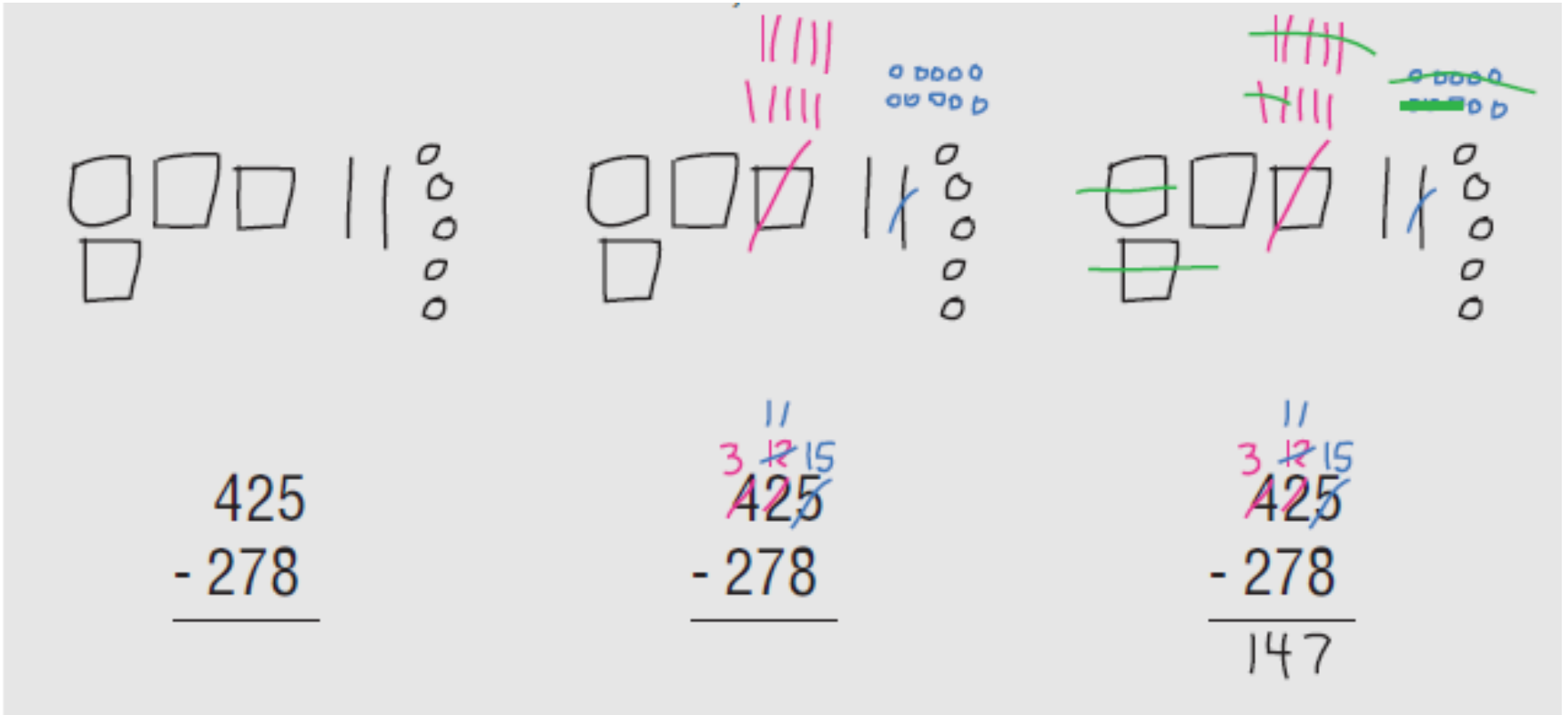
I have 5 full flats of 100 ones which equals 5 groups of 100.

"five hundred sixty-three"

hundreds	tens	ones
5	6	3

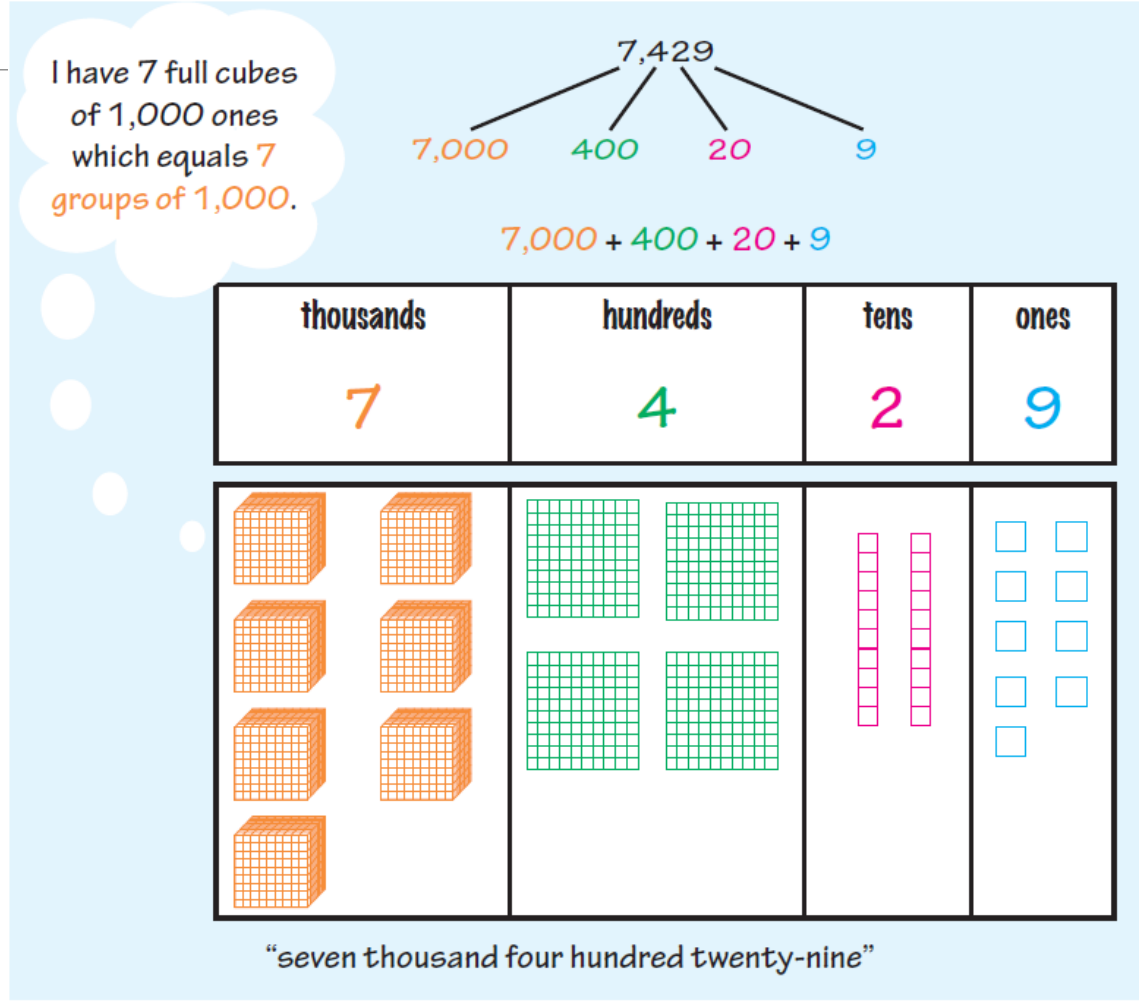


Example of Efficient Decomposition for a Subtraction Problem

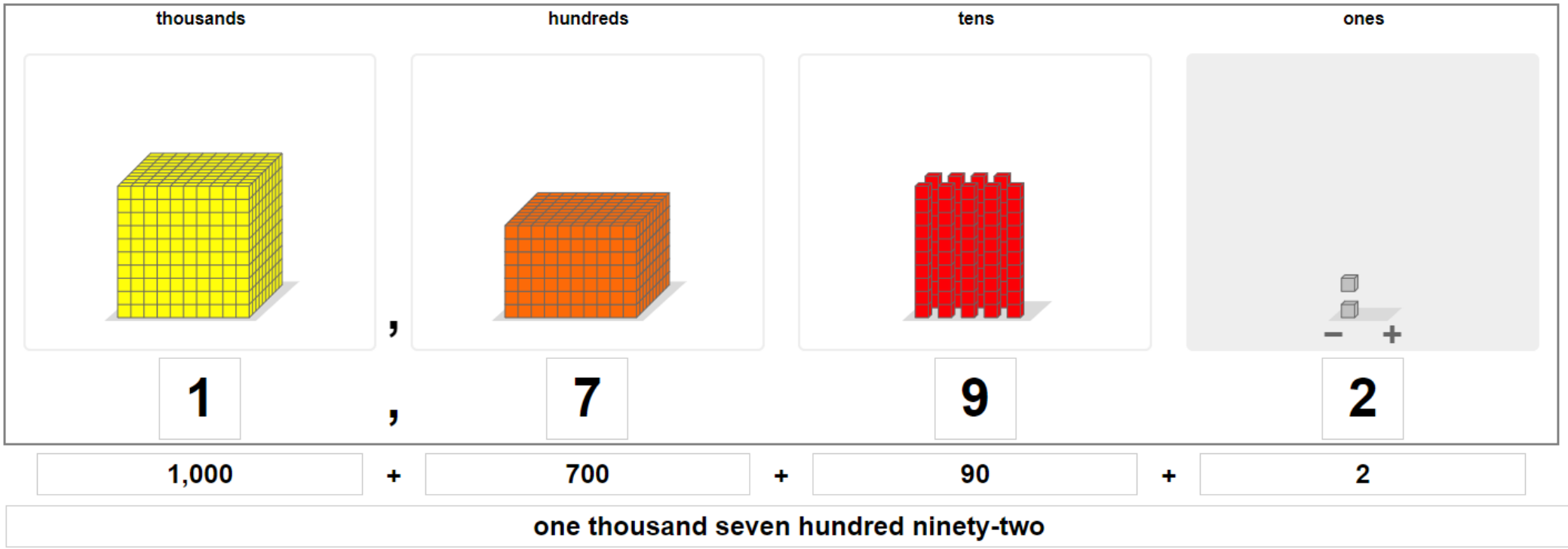


Grade 3 – Fluency with 1000, Working with Larger Numbers

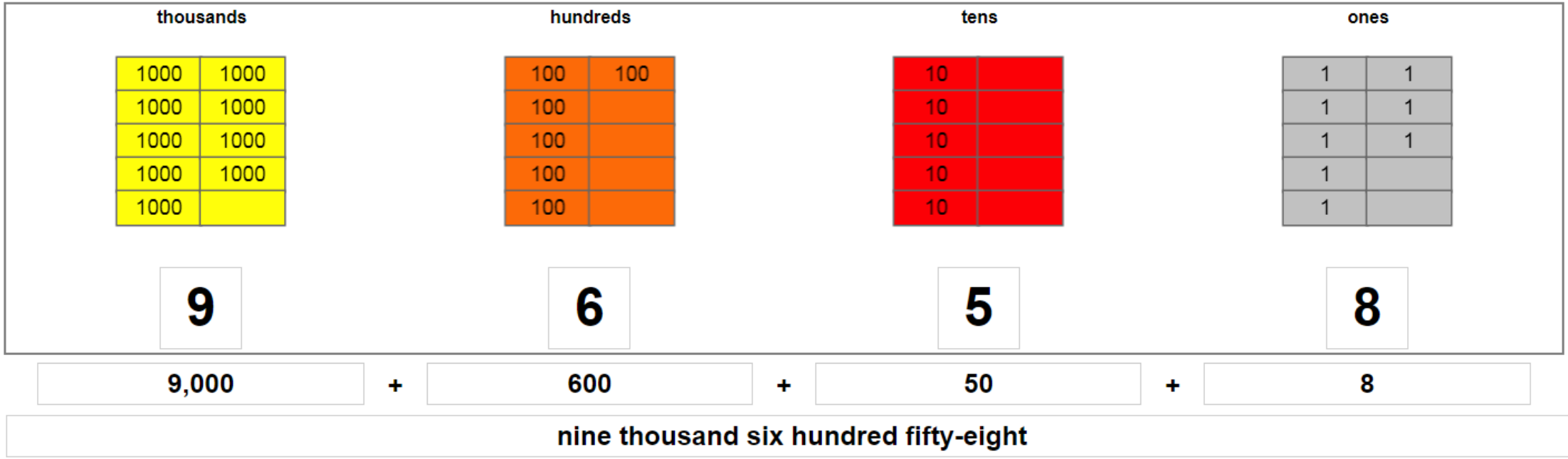
The major focus is multiplication



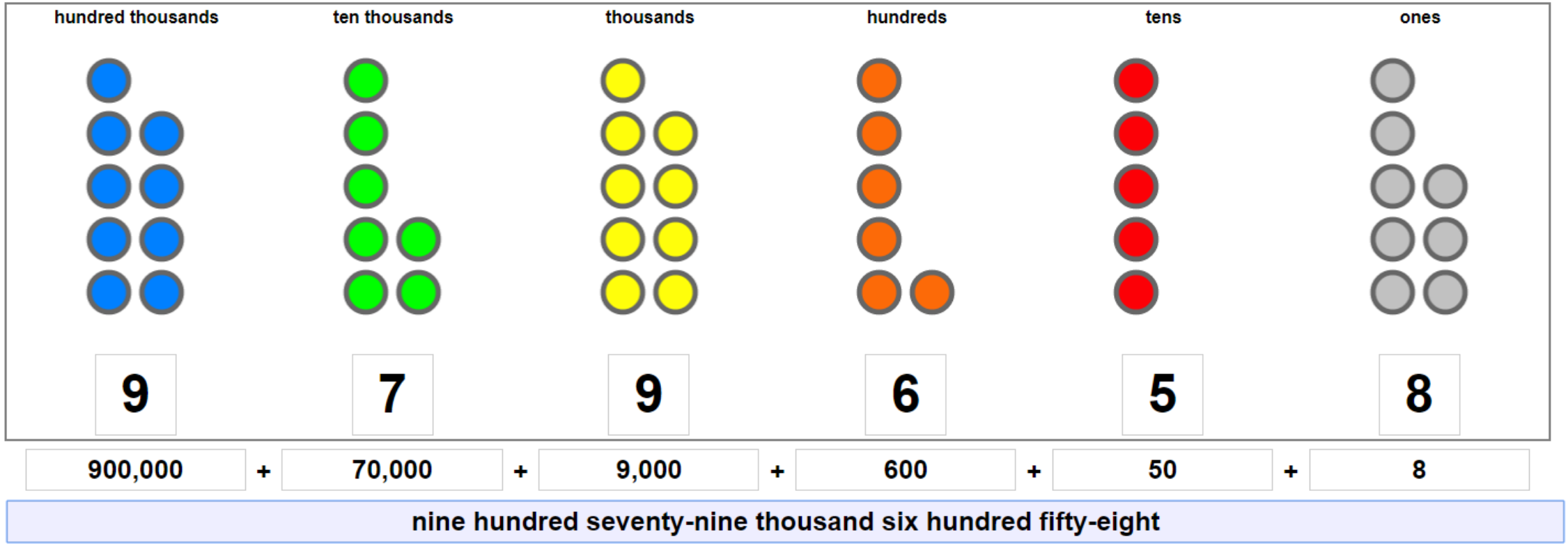
Grade 4 – Values to 1,000,000 with all Forms of Notation



Building Towards Abstraction

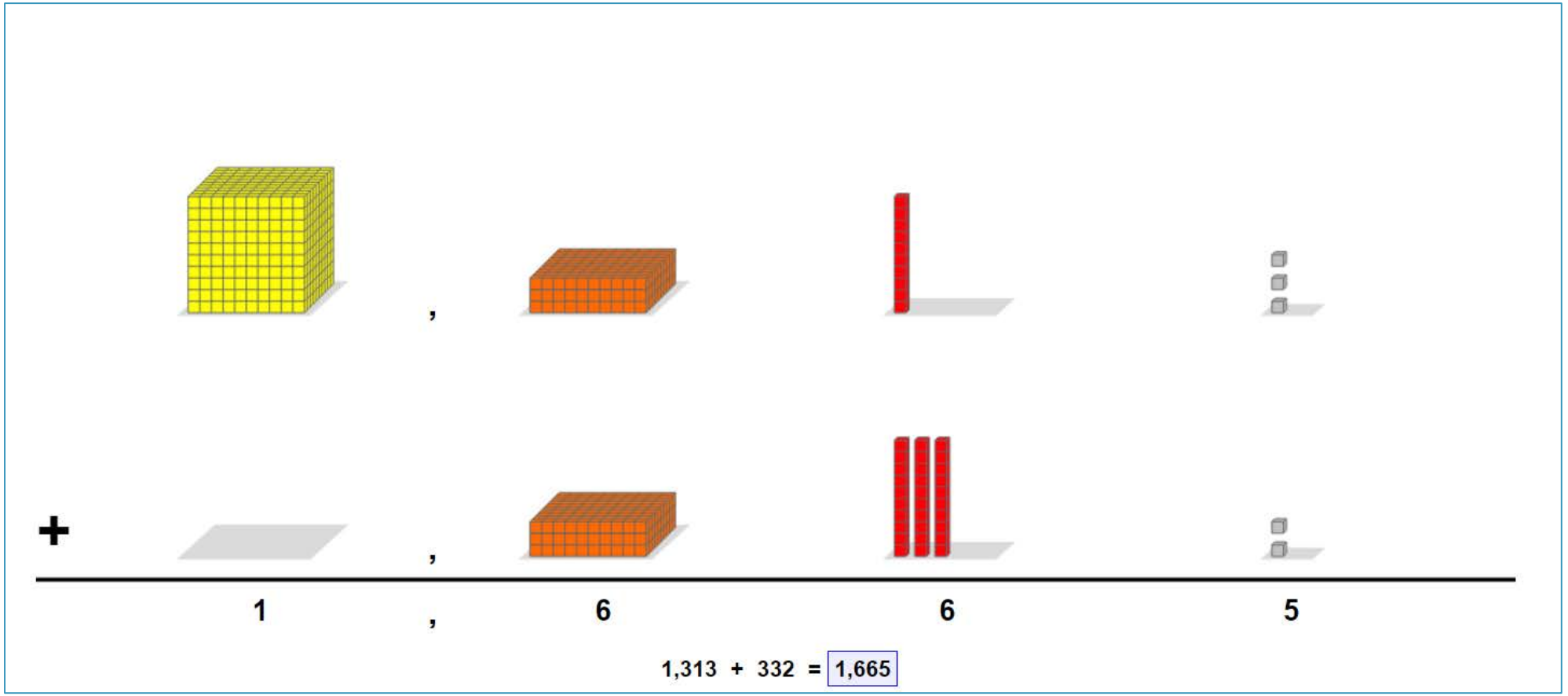


Heading Towards 1,000,000

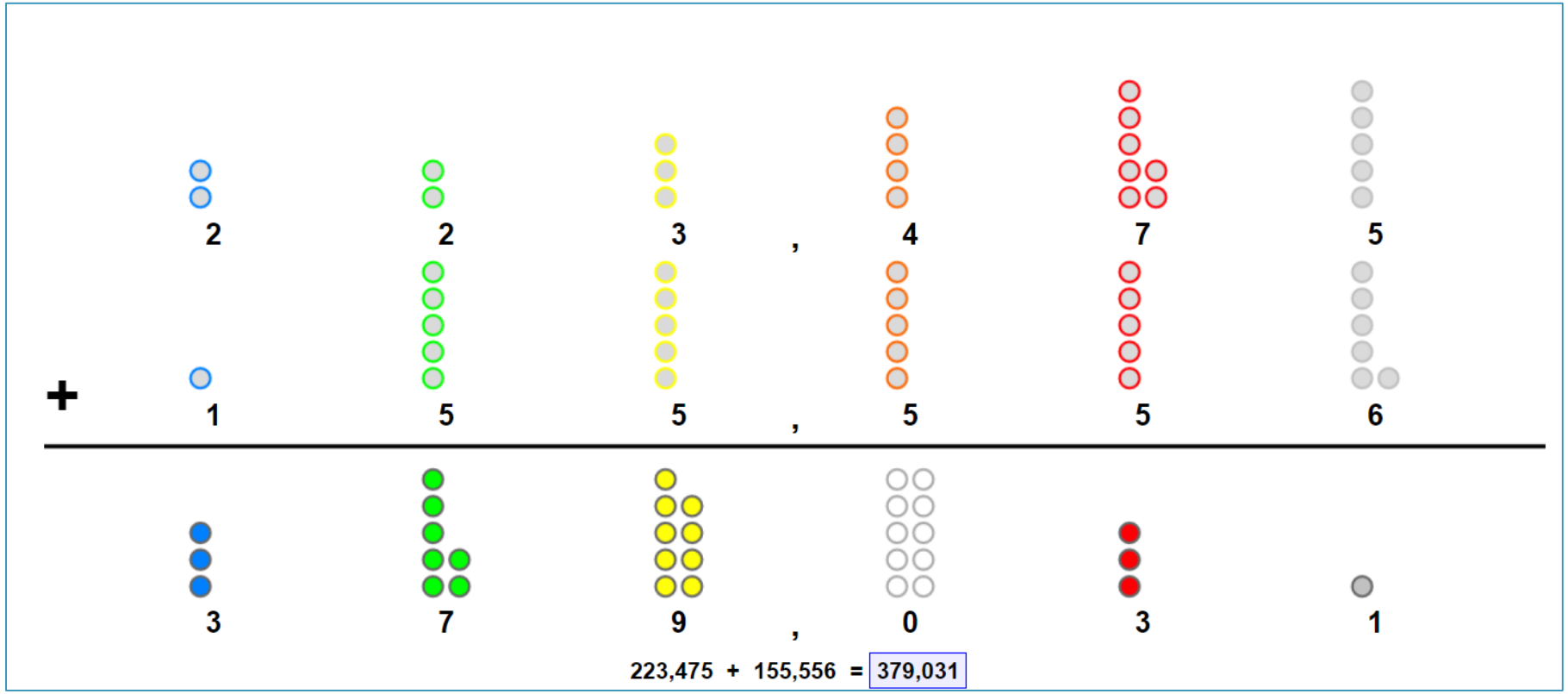


Nine hundred seventy-nine thousand six hundred fifty-eight

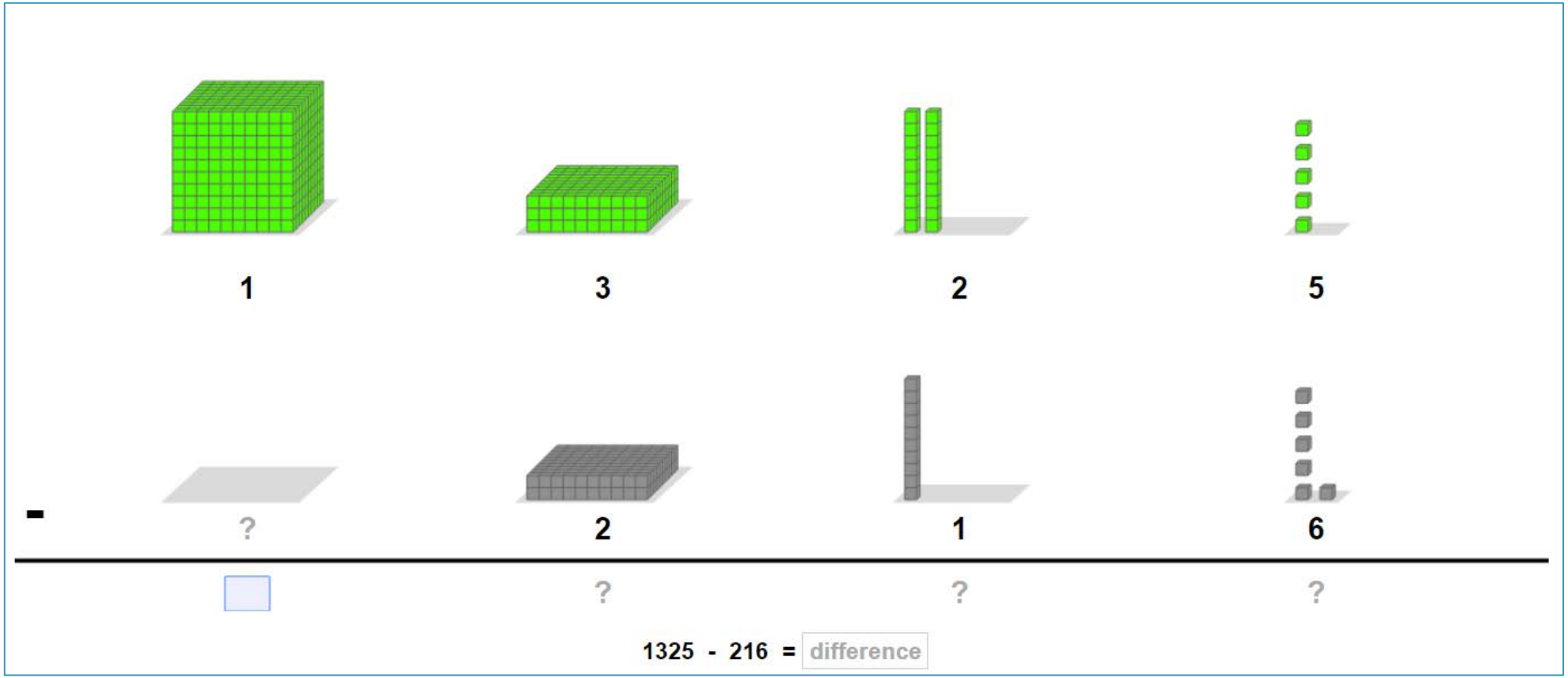
Place Value Addition



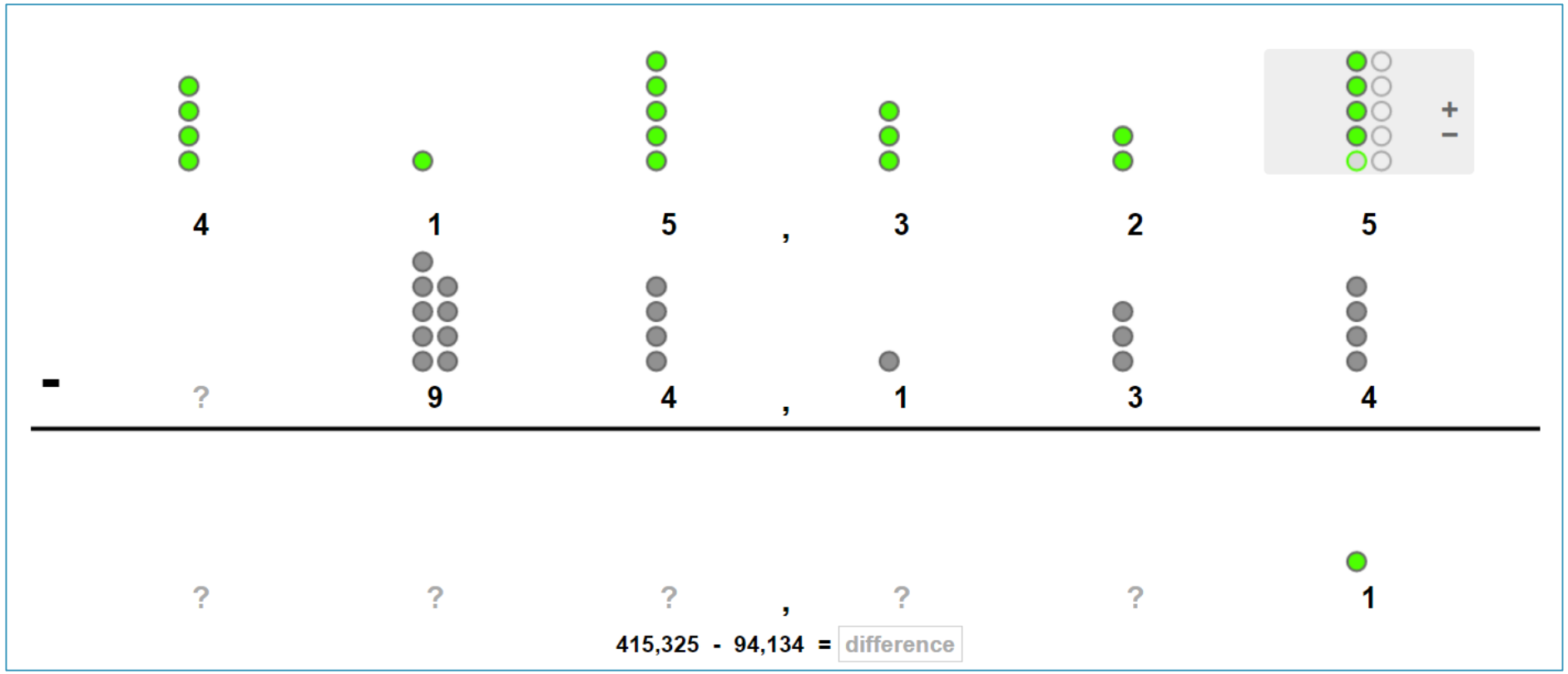
Place Value Addition with Large Numbers



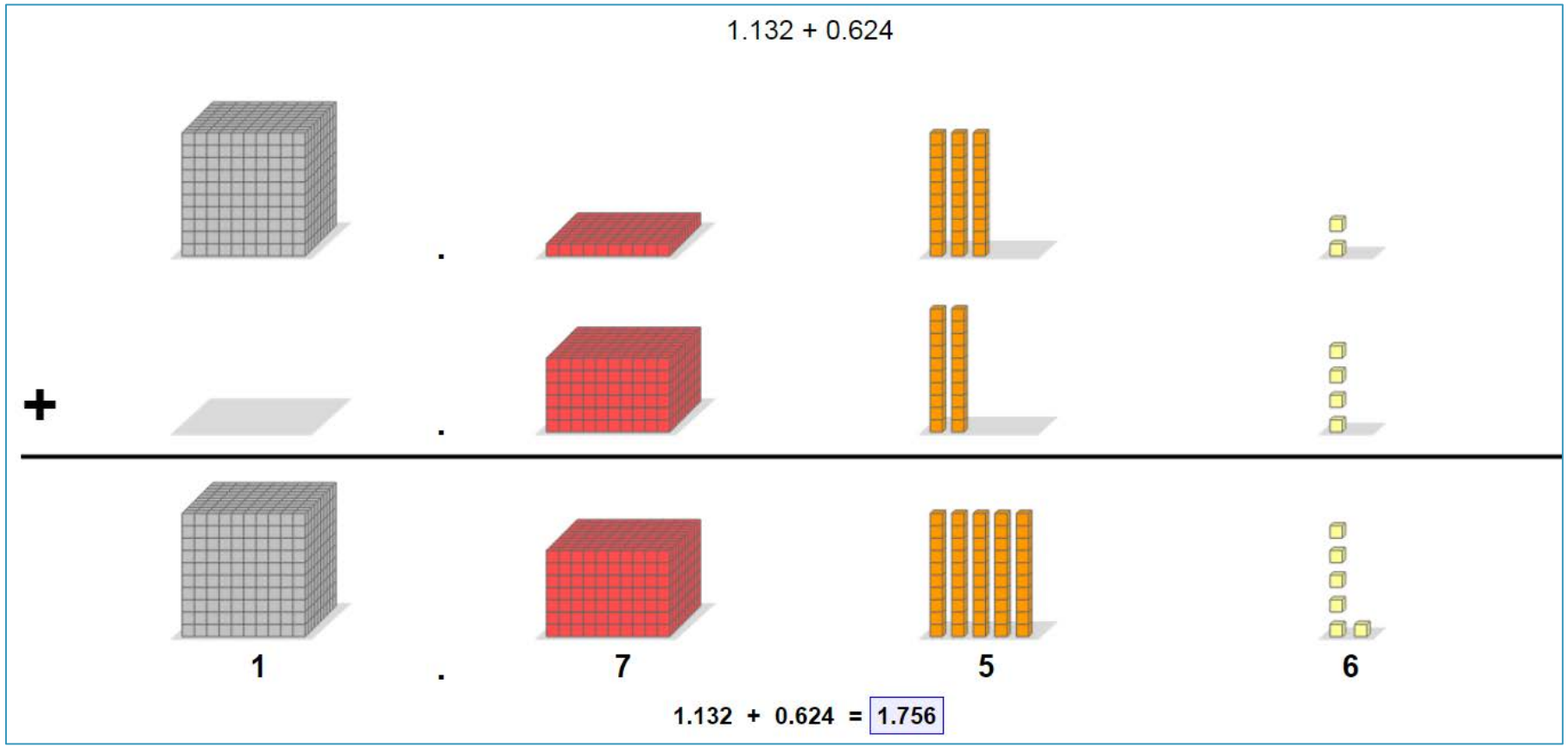
Place Value Subtraction



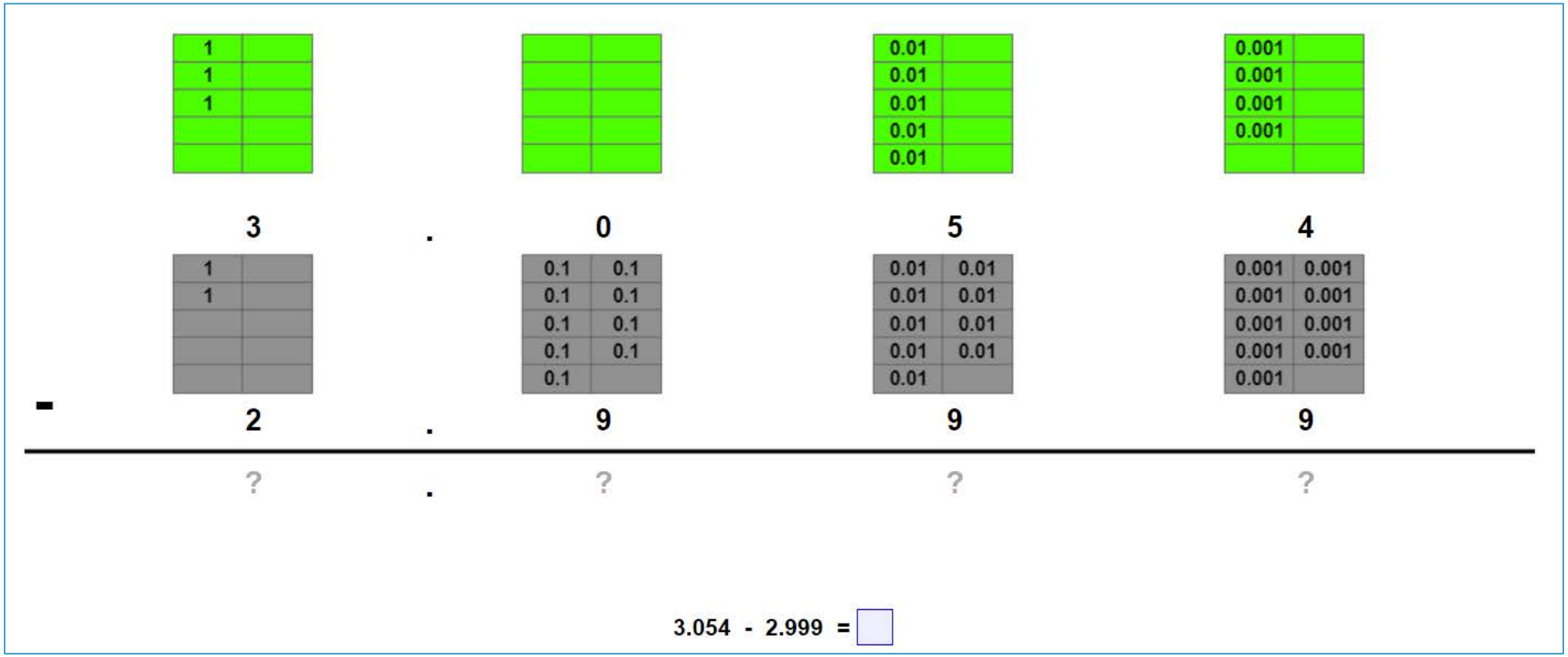
Place Value Subtraction with Large Numbers



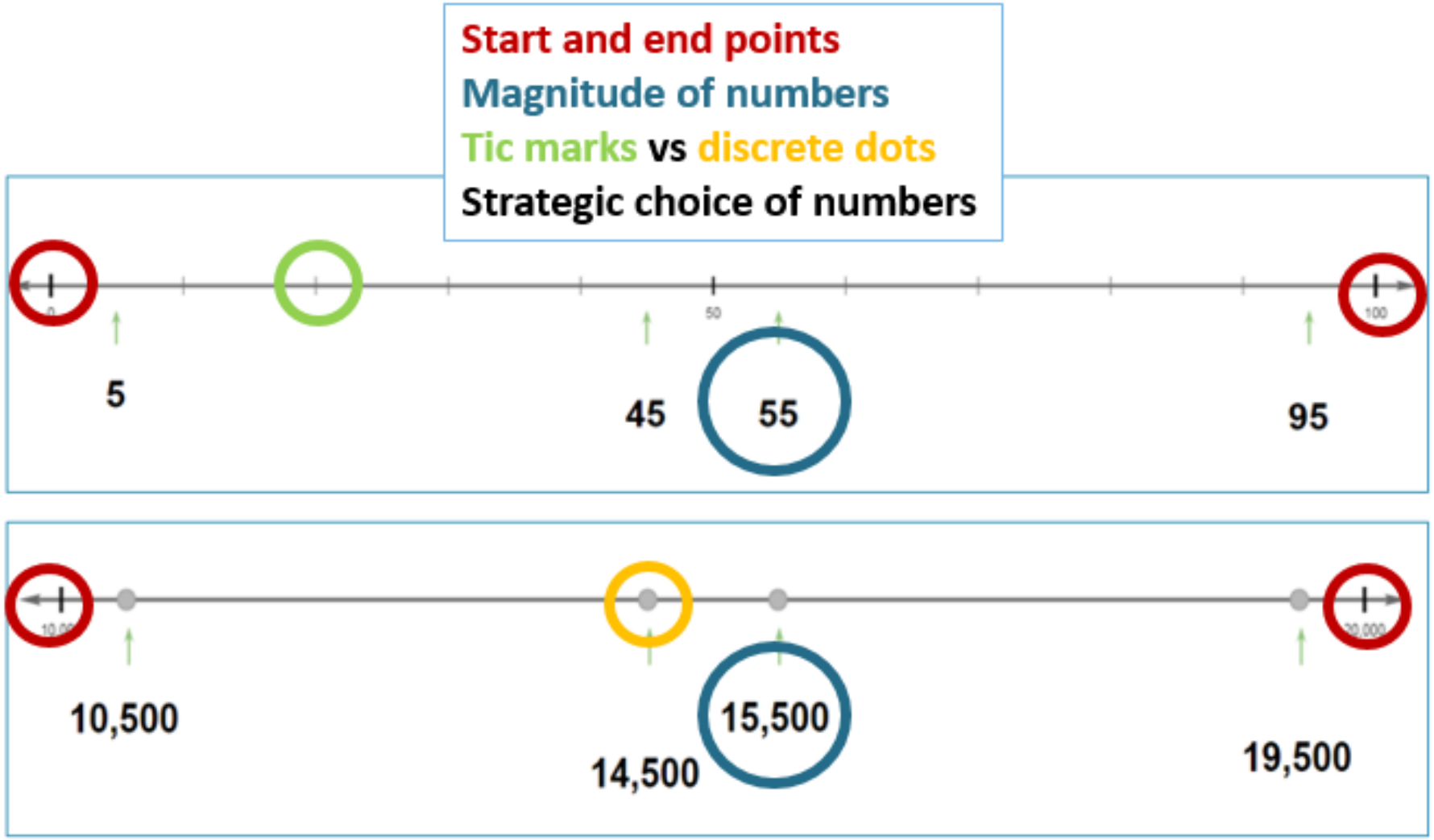
Grade 5 - Place Value Addition and Subtraction with Decimals



Grade 5 - Place Value Addition and Subtraction with Decimals



The Number Line


Start and end points
Magnitude of numbers
Tic marks vs discrete dots
Strategic choice of numbers
0
50
100
5
45
55
95
10,500
14,500
15,500
19,500

Number Line Considerations

Tic Marks: What is their purpose? How many? Labeled?

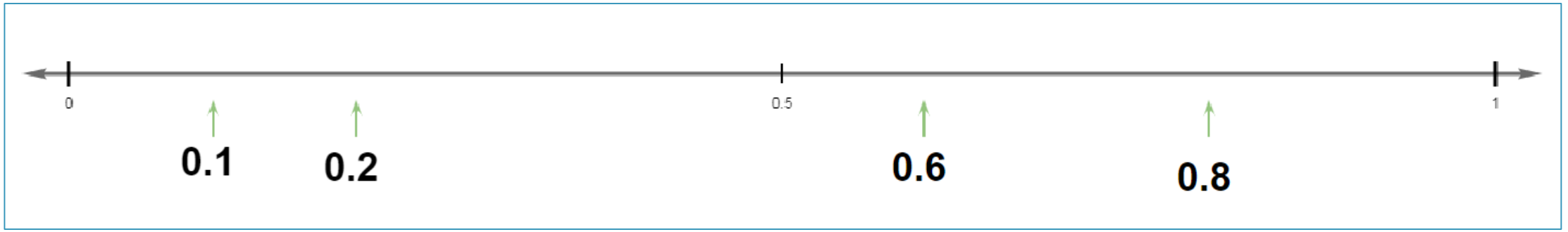
Dots: Consider discrete locations instead of tic marks.

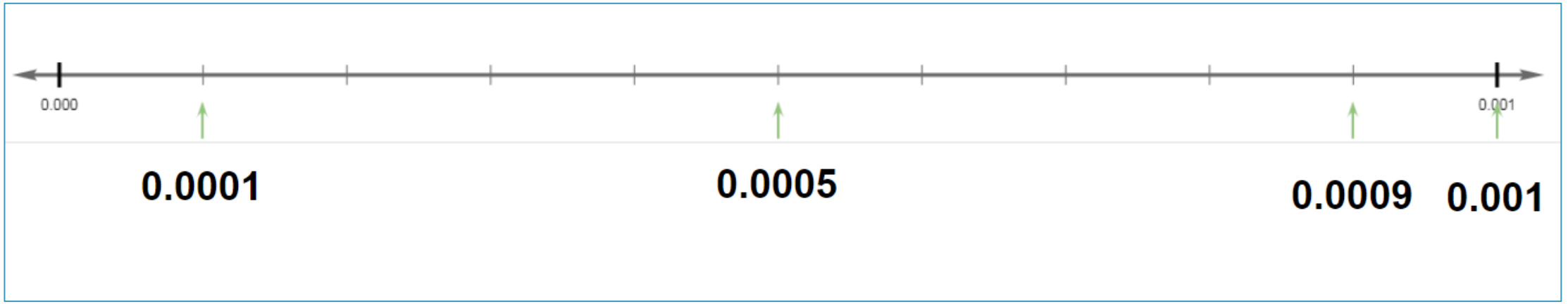
Endpoints: Where does the number line segment start? End?

Magnitude: What place value are you teaching?

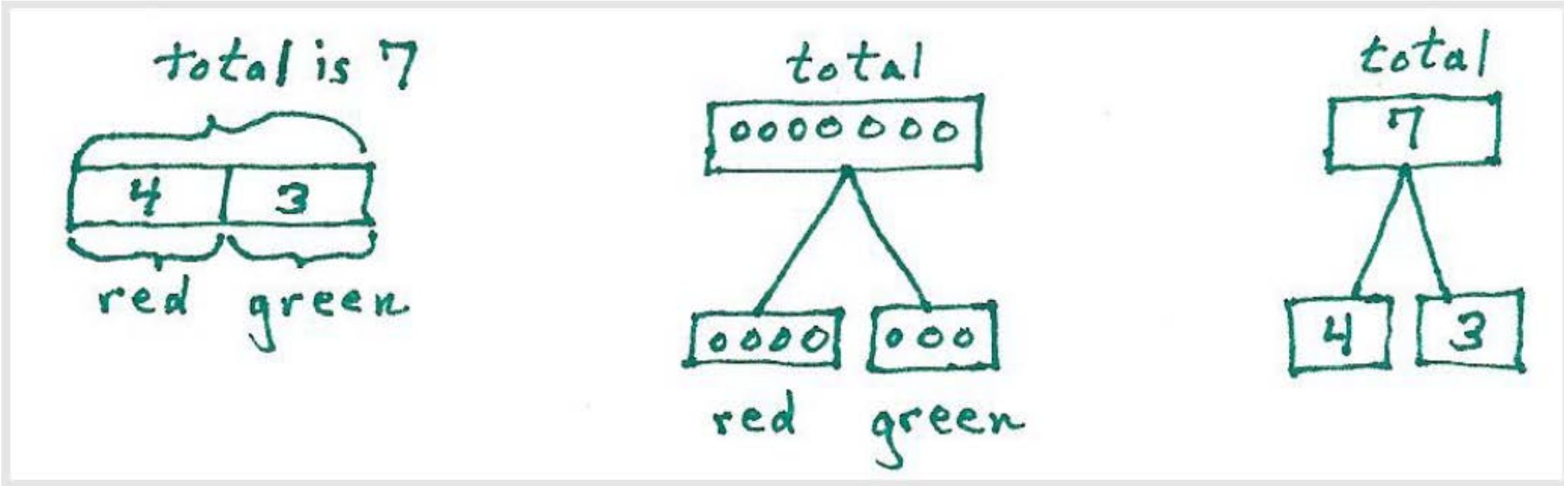
Numbers for Placement: What numbers? Why? How Many?

How Can You Apply this to Decimals?

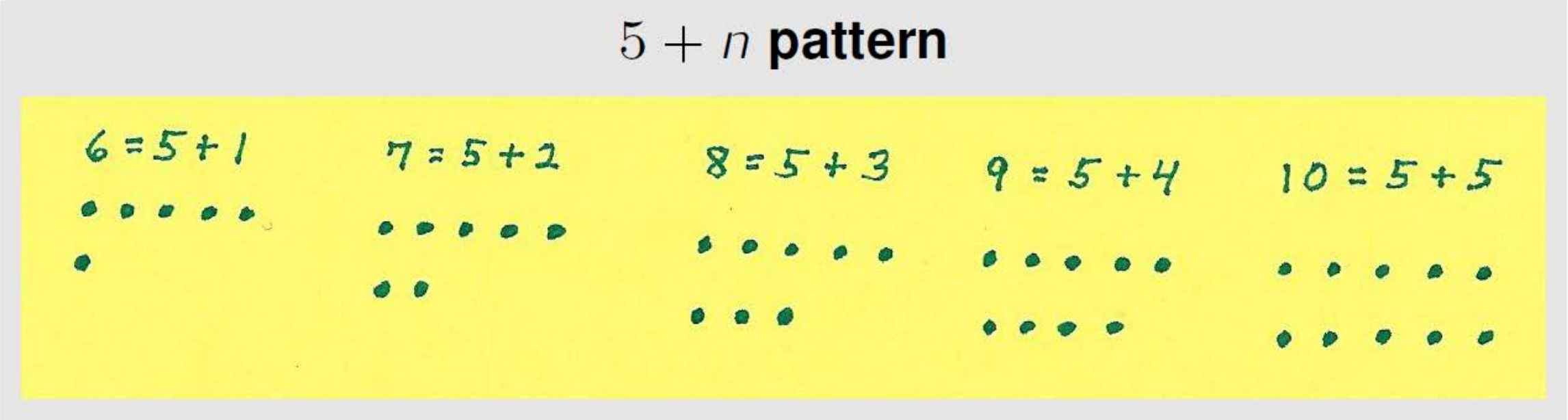




Operations and Algebraic Thinking

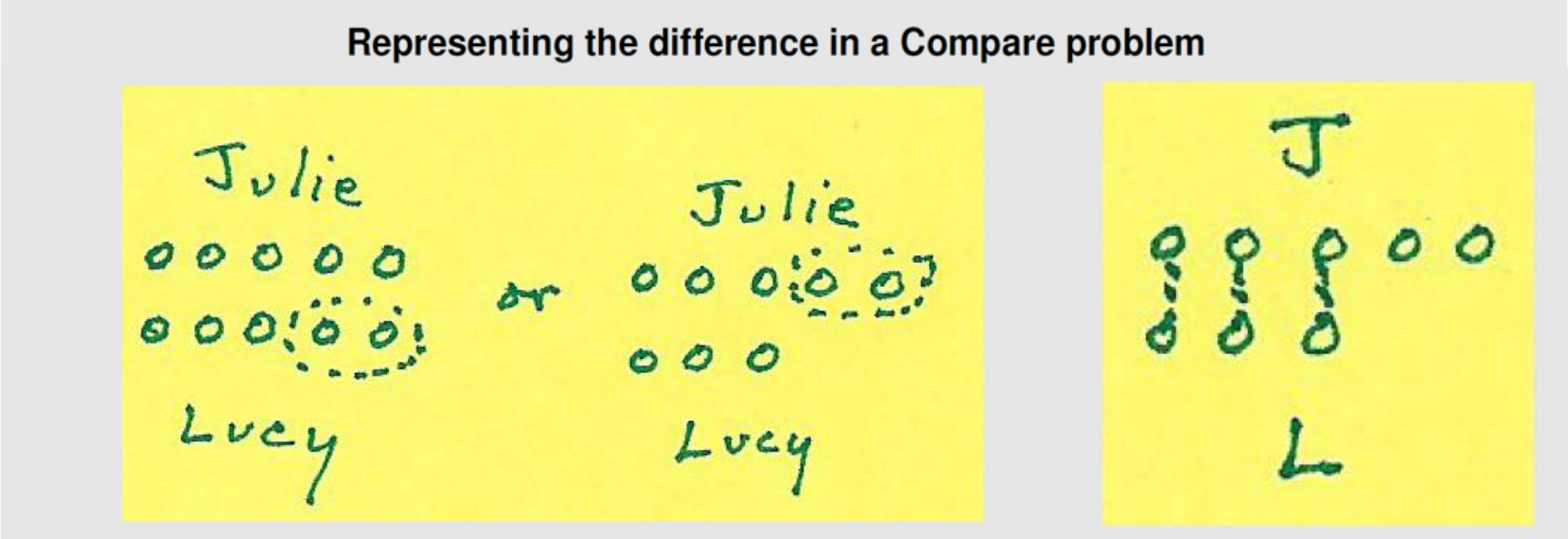

total is 7
4
3
red
green
total
red
green
total
7
4
3

Kindergarten

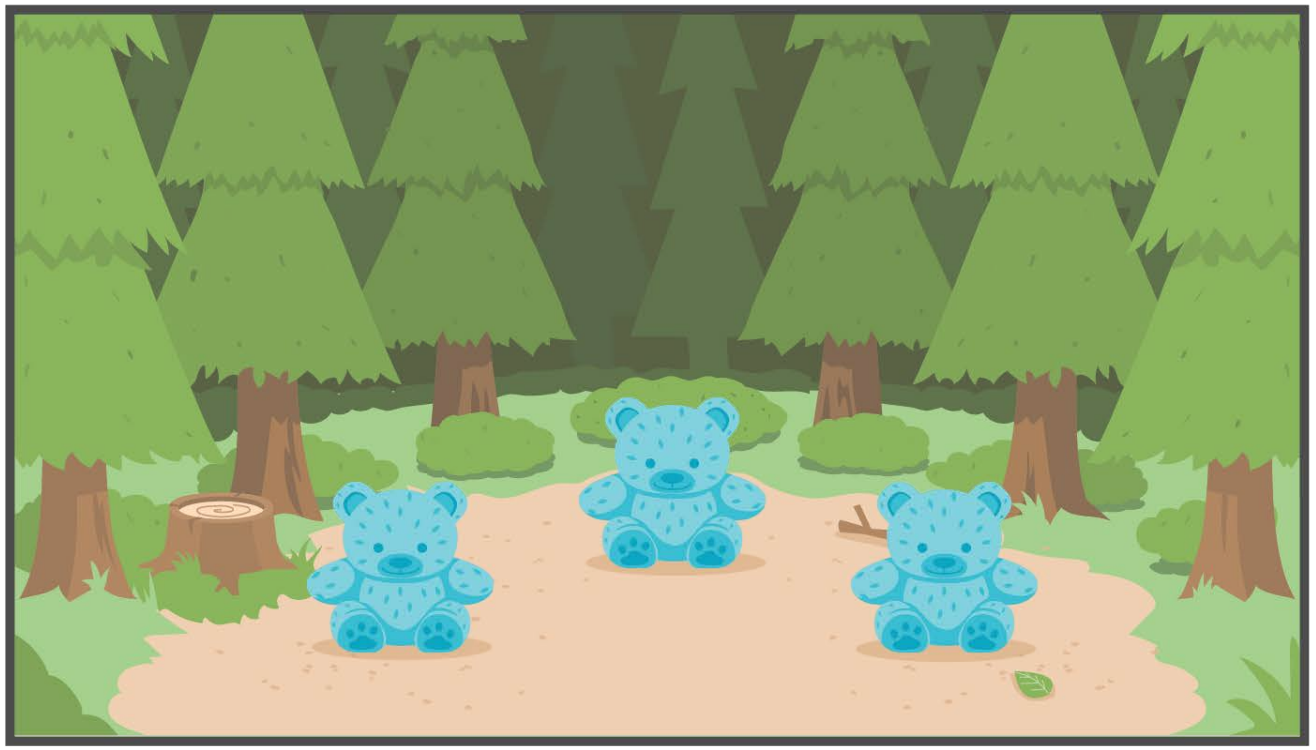


Composing and decomposing within 10

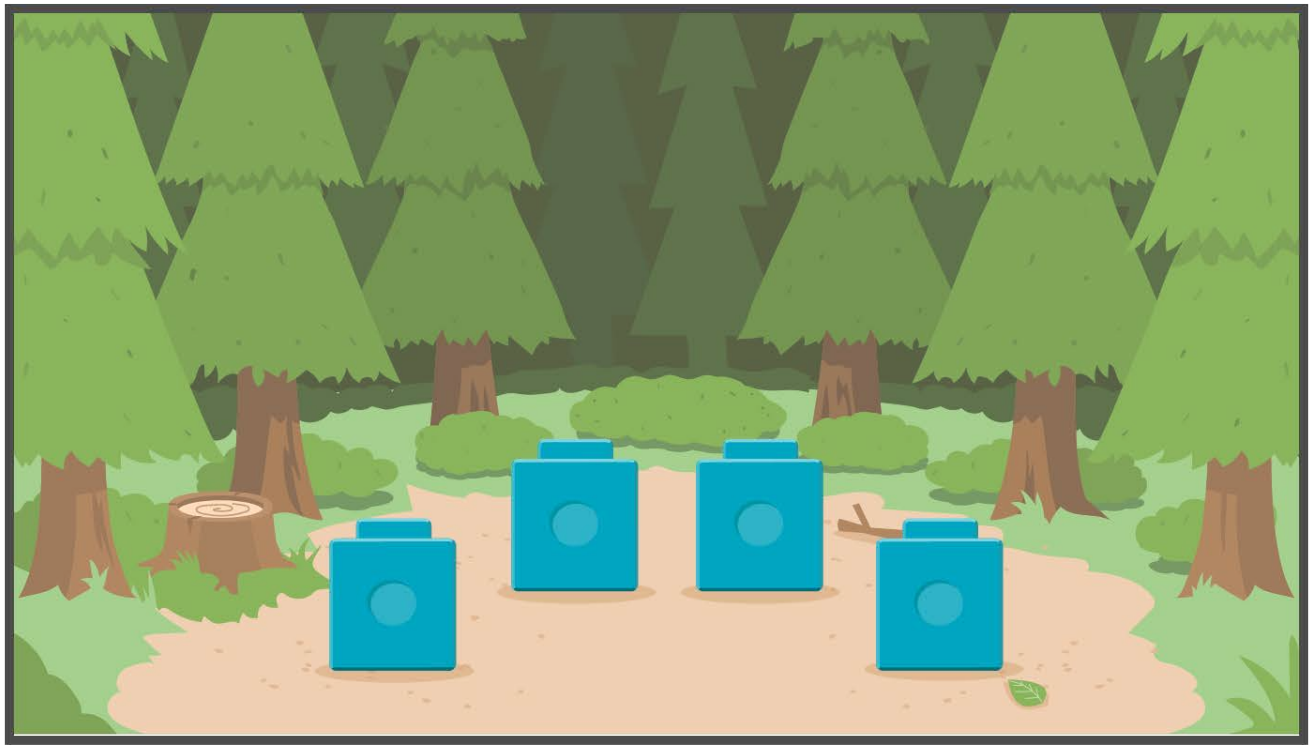
First Grade


Representing the difference in a Compare problem
Julie
Lucy
or
Julie
Lucy
J
L

Three bears are sitting in the forest. Two more bears join them. How many bears are sitting in the forest now?



4 bears are playing in the forest. 2 bears leave.
How many bears are playing in the forest now?

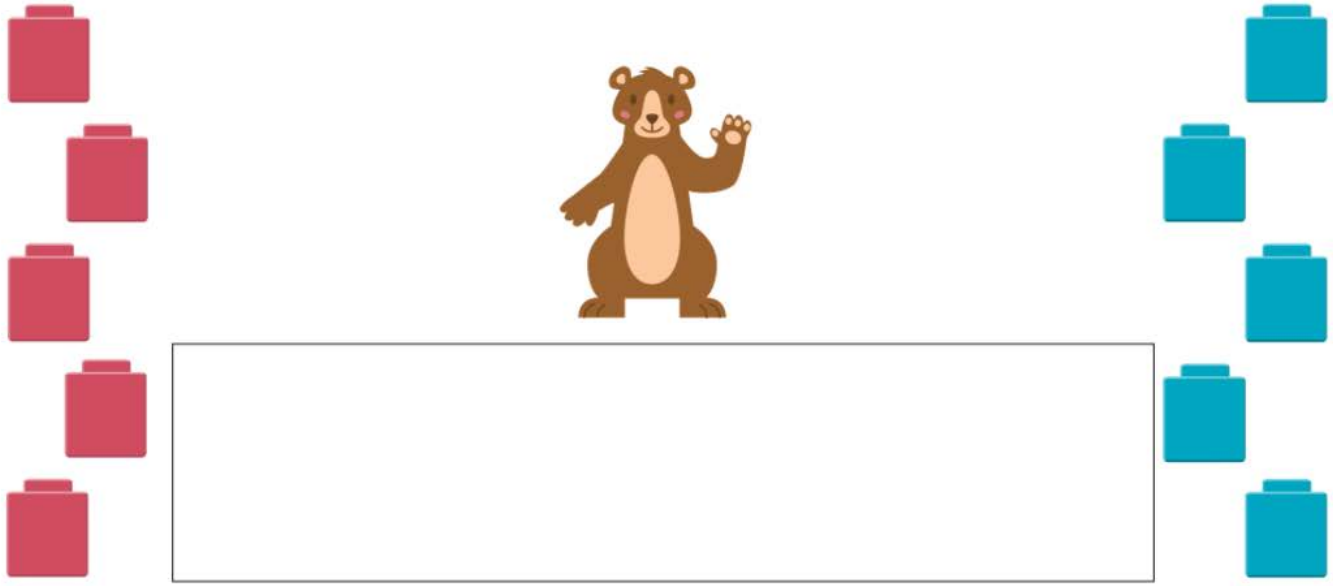


	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ?	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? 2 + ? = 5	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? ? + 3 = 5
Take from	Five apples were on the table. I ate two apples. How many apples are on the table now? 5 – 2 = ?	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? 5 – ? = 3	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? ? – 2 = 3

	Total Unknown	Addend Unknown	Both Addends Unknown[1]
Put Together/ Take Apart[2]	Three red apples and two green apples are on the table. How many apples are on the table? 3 + 2 = ?	Five apples are on the table. Three are red and the rest are green. How many apples are green? 3 + ? = 5, 5 – 3 = ?	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0 5 = 1 + 4, 5 = 4 + 1 5 = 2 + 3, 5 = 3 + 2

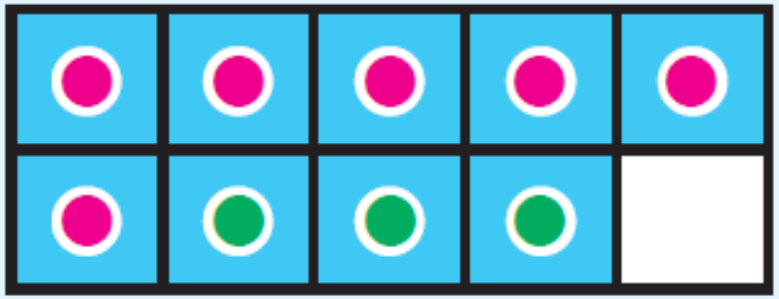
	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare[3]	("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? 2 + ? = 5, 5 – 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? 2 + 3 = ?, 3 + 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? 5 – 3 = ?, ? + 3 = 5

Blair ate 5 red fish and 2 blue fish. How many more red fish did Blair eat than blue fish?



Put Together / Take Apart

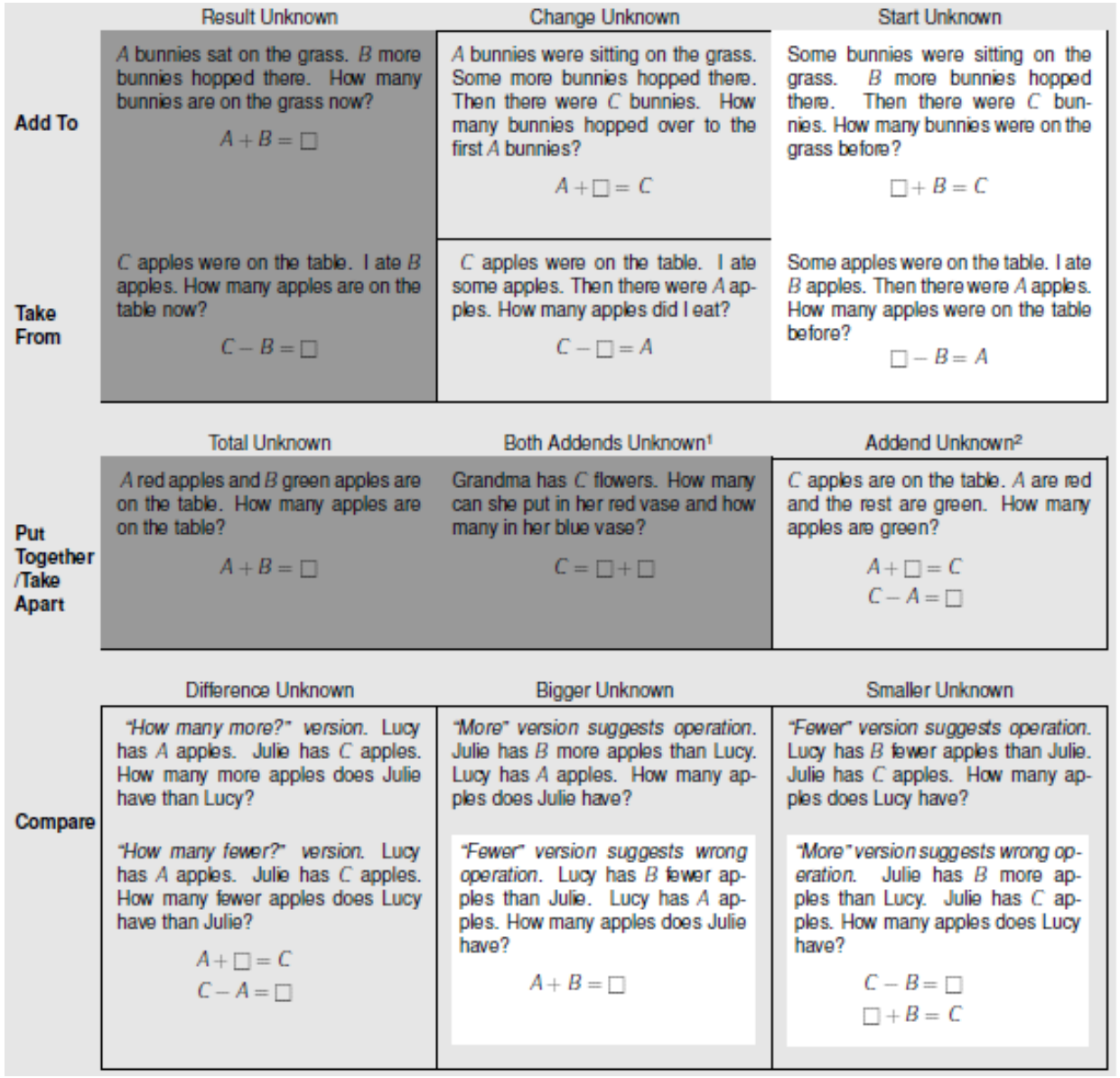
9 dinner rolls are on the table, 6 are wheat and the rest are white. How many dinner rolls are white?



6 + ? = 9,
9 - 6 = ?

6 + 3 = 9

9 - 6 = 3



	Result Unknown	Change Unknown	Start Unknown
Add To	A bunnies sat on the grass. B more bunnies hopped there. How many bunnies are on the grass now? $A + B = \square$	A bunnies were sitting on the grass. Some more bunnies hopped there. Then there were C bunnies. How many bunnies hopped over to the first A bunnies? $A + \square = C$	Some bunnies were sitting on the grass. B more bunnies hopped there. Then there were C bunnies. How many bunnies were on the grass before? $\square + B = C$
Take From	C apples were on the table. I ate B apples. How many apples are on the table now? $C - B = \square$	C apples were on the table. I ate some apples. Then there were A apples. How many apples did I eat? $C - \square = A$	Some apples were on the table. I ate B apples. Then there were A apples. How many apples were on the table before? $\square - B = A$

	Total Unknown	Both Addends Unknown[1]	Addend Unknown[2]
Put Together /Take Apart	A red apples and B green apples are on the table. How many apples are on the table? $A + B = \square$	Grandma has C flowers. How many can she put in her red vase and how many in her blue vase? $C = \square + \square$	C apples are on the table. A are red and the rest are green. How many apples are green? $A + \square = C$ $C - A = \square$

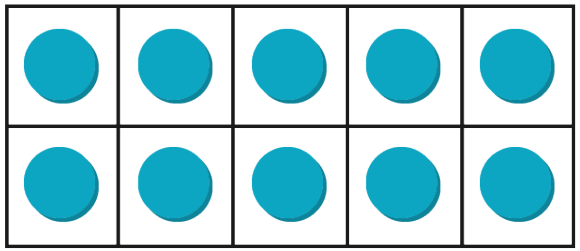
	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	*"How many more?" version.* Lucy has A apples. Julie has C apples. How many more apples does Julie have than Lucy? *"How many fewer?" version.* Lucy has A apples. Julie has C apples. How many fewer apples does Lucy have than Julie? $A + \square = C$ $C - A = \square$	*"More" version suggests operation.* Julie has B more apples than Lucy. Lucy has A apples. How many apples does Julie have? *"Fewer" version suggests wrong operation.* Lucy has B fewer apples than Julie. Lucy has A apples. How many apples does Julie have? $A + B = \square$	*"Fewer" version suggests operation.* Lucy has B fewer apples than Julie. Julie has C apples. How many apples does Lucy have? *"More" version suggests wrong operation.* Julie has B more apples than Lucy. Julie has C apples. How many apples does Lucy have? $C - B = \square$ $\square + B = C$

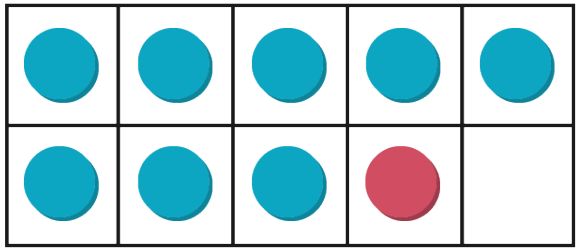
k

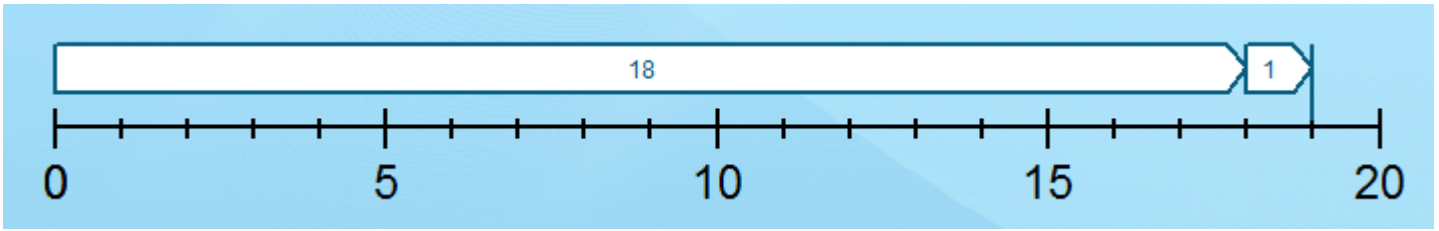
1

2

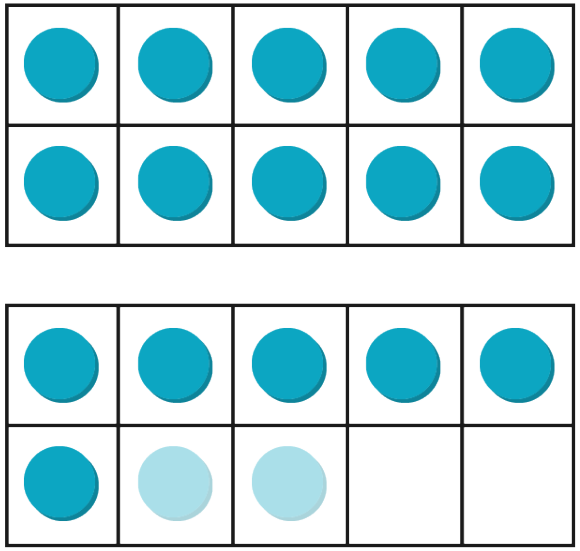
18 + 1 = ?

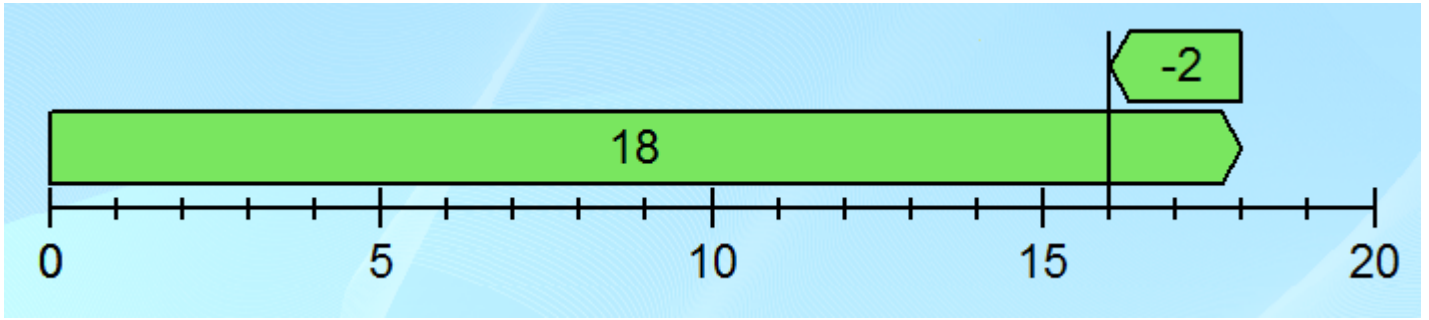






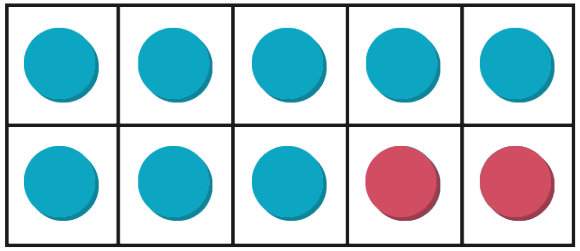
18 - 2 = ?

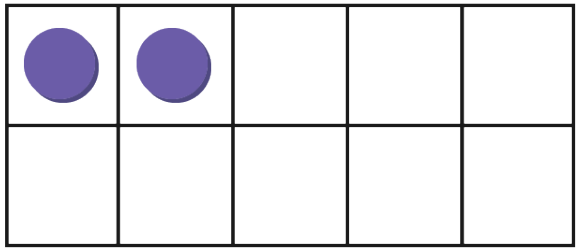


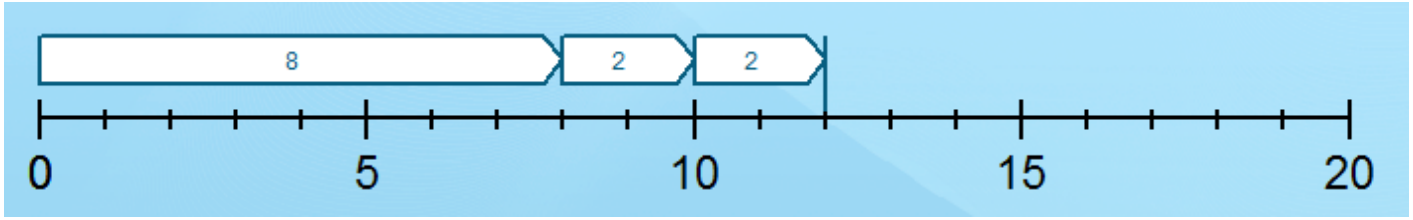


8 + 4 =

8 + 2 + 2 = ?

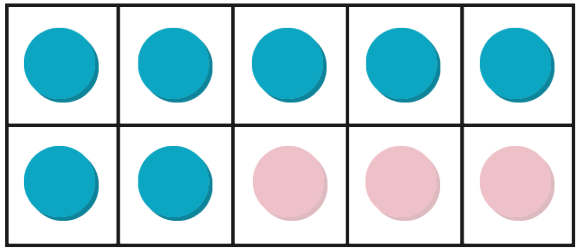


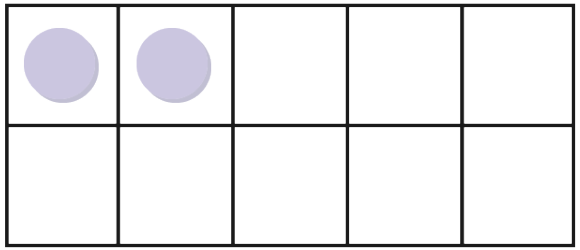


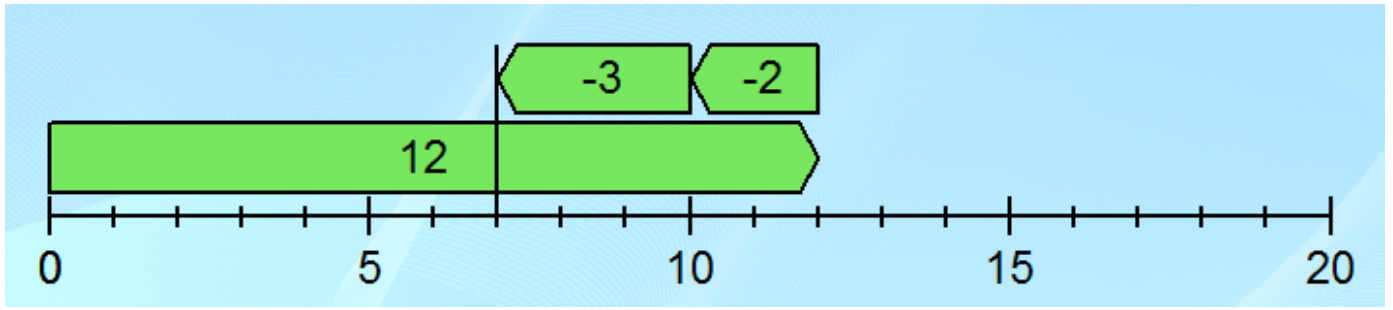


12 – 5 =

12 - 2 – 3 = ?







Grade 2 – Patterns in addition and subtraction to 100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Grade 2 – Patterns in addition and subtraction to 100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Repeated Addition and Repeated Subtraction

Multiplication and Division

	Unknown Product	Group Size Unknown ("How many in each group?" Division)	Number of Groups Unknown ("How many groups?" Division)
	$3 \times 6 = ?$	$3 \times ? = 18$, and $18 \div 3 = ?$	$? \times 6 = 18$, and $18 \div 6 = ?$
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example*. You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example*. You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example*. You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays,[4] Area[5]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example*. What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example*. A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example*. A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs \$6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example*. A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs \$18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example*. A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs \$18 and a blue hat costs \$6. How many times as much does the red hat cost as the blue hat? *Measurement example*. A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$, and $p \div a = ?$	$? \times b = p$, and $p \div b = ?$

Multiplication

Equal Groups

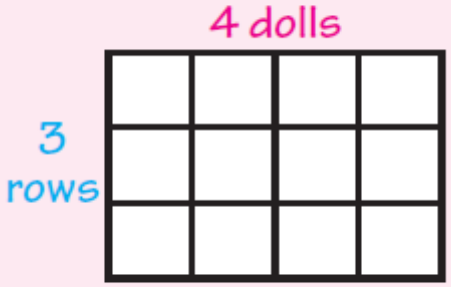
Josie has 3 toy cars. Each car has 4 wheels. How many wheels are there in all?

3 groups of 4

3 × 4 = 12

Arrays of Objects

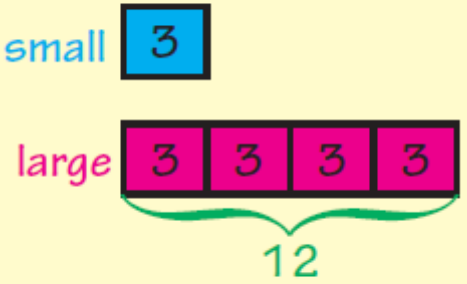
At the toy store, Kaitlin notices 3 rows of dolls with 4 dolls in each row. How many dolls are at the toy store?



3 × 4 = 12

Compare

A small balloon costs $3.00. The large balloon costs 4 times as much. How much does the large balloon cost?

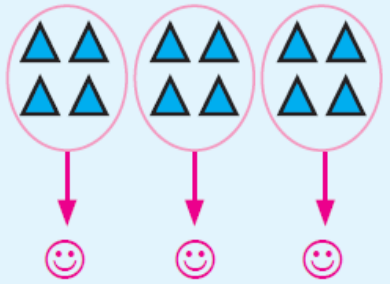


$3 × 4 = $12

Division

Equal Groups

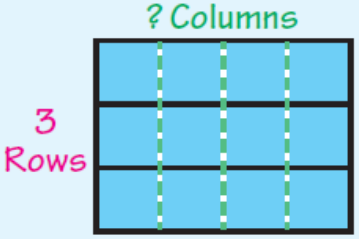
If 12 toys are divided equally among 3 brothers, how many toys will each brother receive?



$12 \div 3 = 4$

Arrays of Objects

If 12 apples are arranged into an array with 3 rows, how many apples will be in each row?

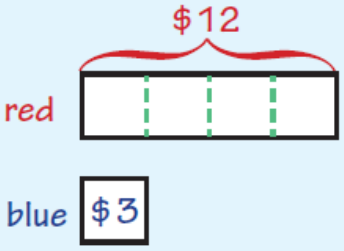


$12 \div 3 = \square$

$12 \div 3 = 4$

Compare

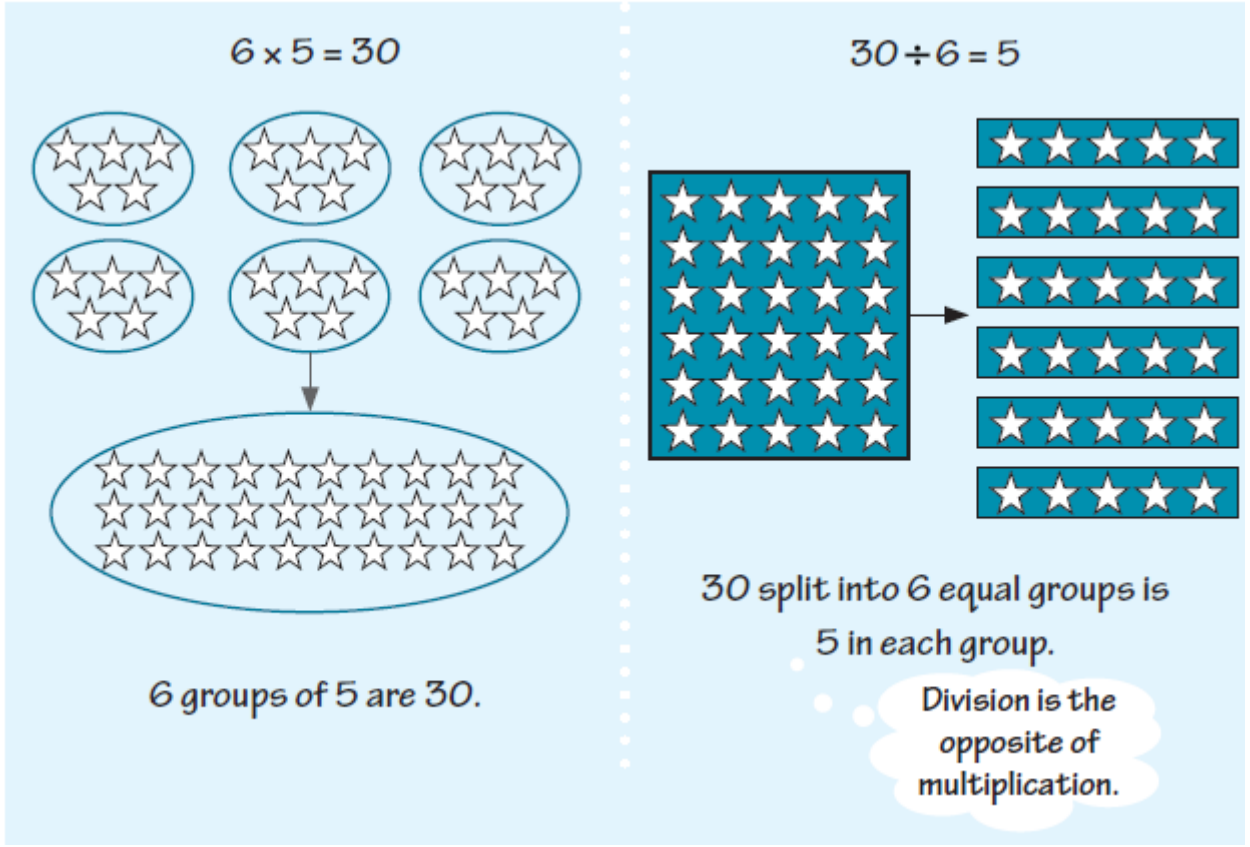
A red hat costs $12 and a blue hat costs $3. How many times more expensive is the red hat than the blue hat?



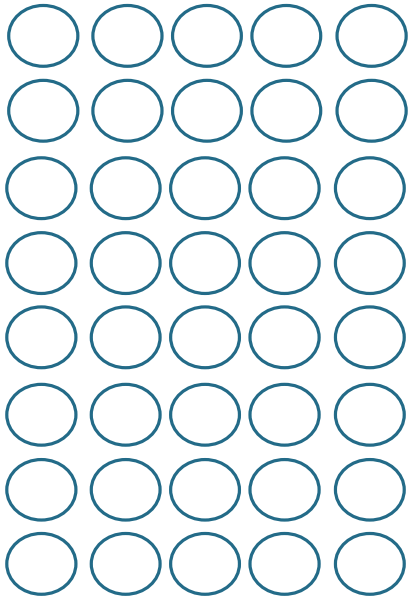
$12 \div \square = 3$

$12 \div 4 = 3$

Grade 3 – Multiplication and Division

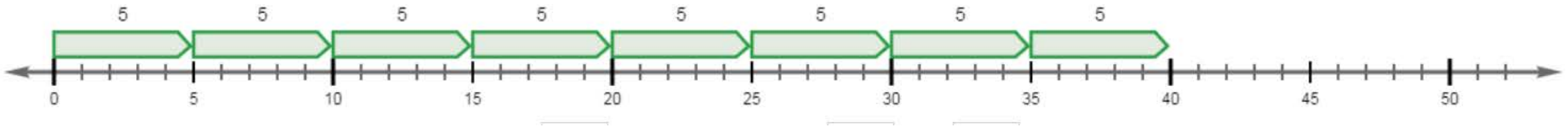


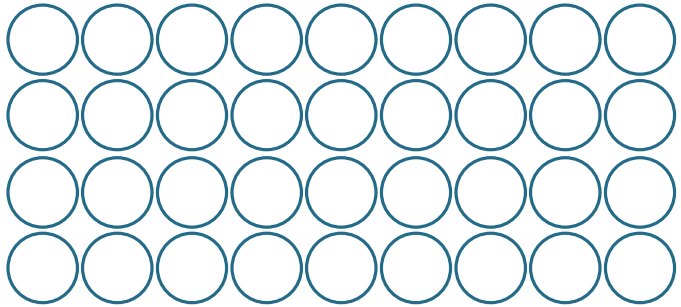
	Unknown Product	Group Size Unknown ("How many in each group?" Division)	Number of Groups Unknown ("How many groups?" Division)
	$3 \times 6 = ?$	$3 \times ? = 18$, and $18 \div 3 = ?$	$? \times 6 = 18$, and $18 \div 6 = ?$
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example*. You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example*. You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example*. You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays,[4] Area[5]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example*. What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example*. A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example*. A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs \$6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example*. A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs \$18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example*. A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs \$18 and a blue hat costs \$6. How many times as much does the red hat cost as the blue hat? *Measurement example*. A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$, and $p \div a = ?$	$? \times b = p$, and $p \div b = ?$



40
35
30
25
20
15
10
5

8 x 5

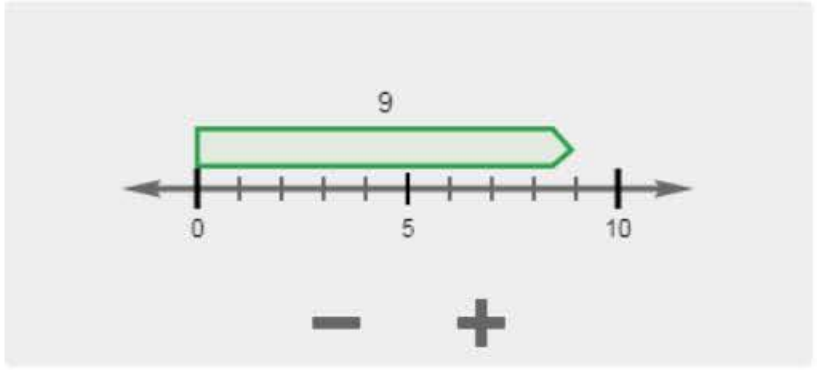


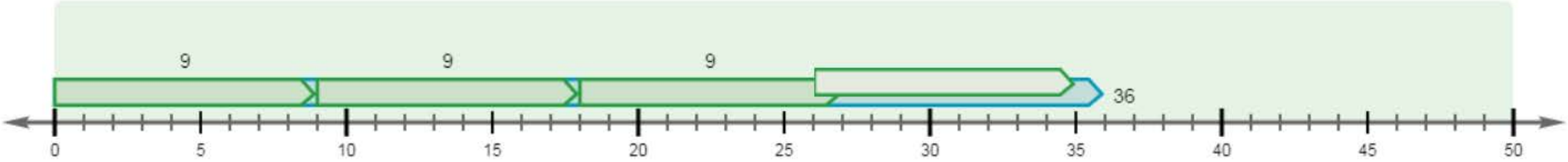


4

9

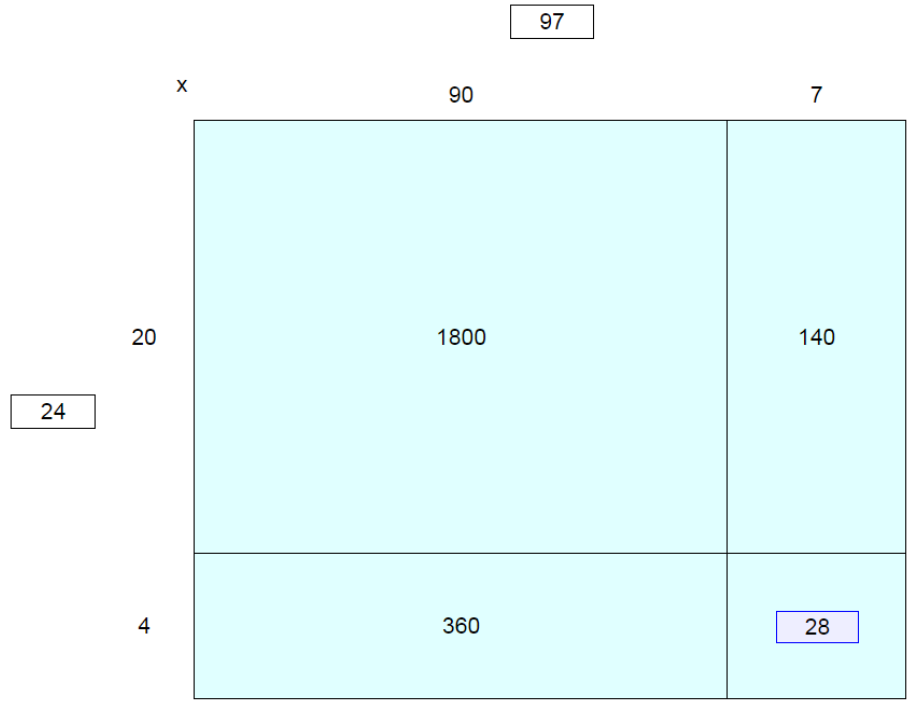
36 ÷ 9





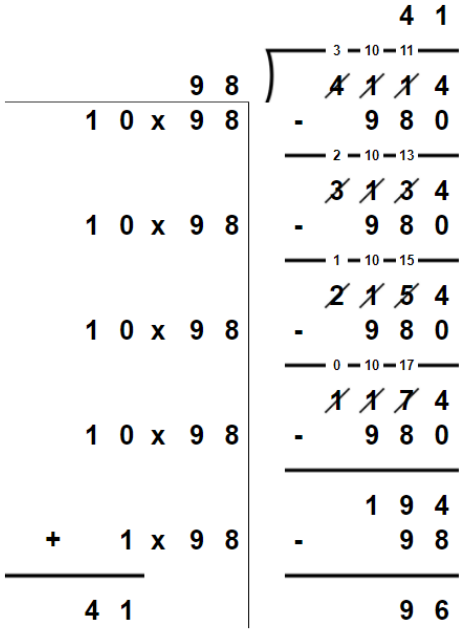
Grade 4 – Multiplication, Larger Numbers

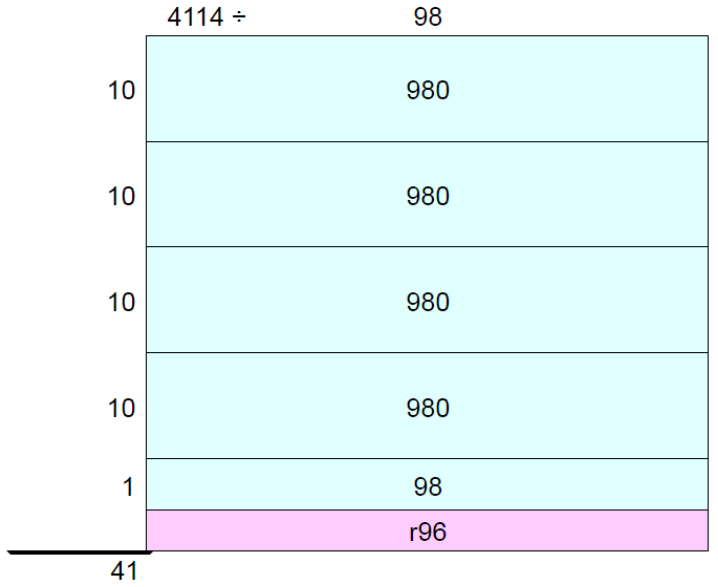
$$
\begin{array}{rrrr}
 & & 9 & 7 \\
 & \times & 2 & 4 \\
\hline
 & & 2 & 8 \\
 & 3 & 6 & 0 \\
 & 1 & 4 & 0 \\
+ & 1 & 8 & 0 & 0 \\
\hline
\end{array}
$$



97 × 24 = Product

Grade 4 – Division, Larger Numbers

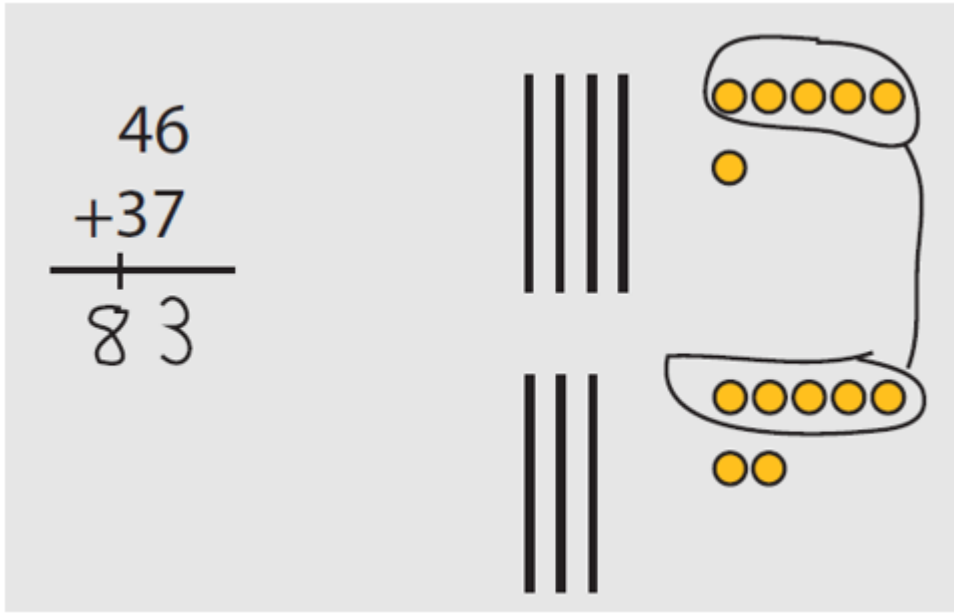




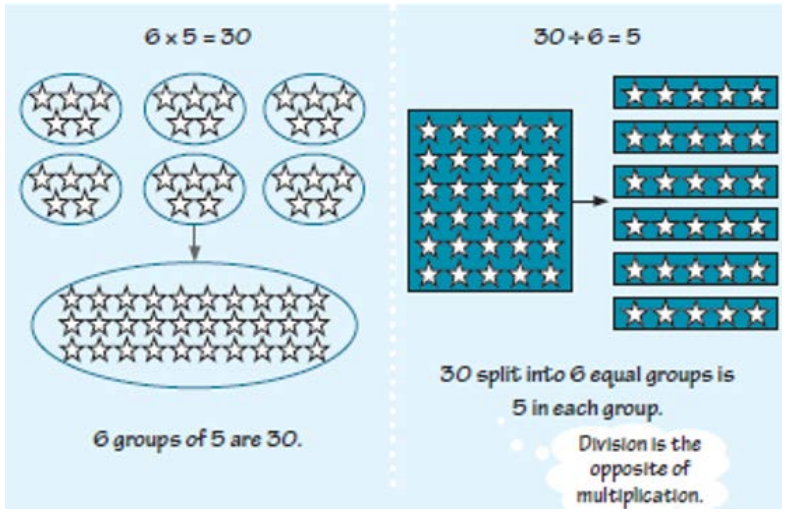
4114 ÷ 98 = 41 r 96

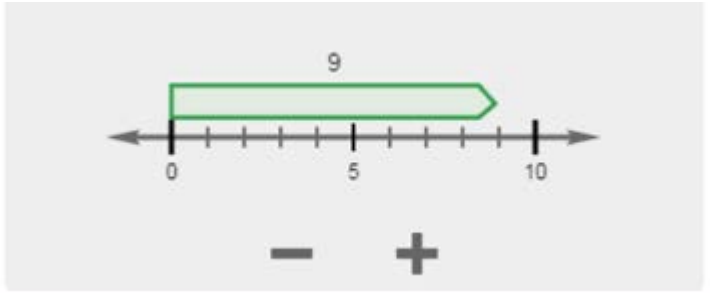
A Difficult 5th Grade Problem

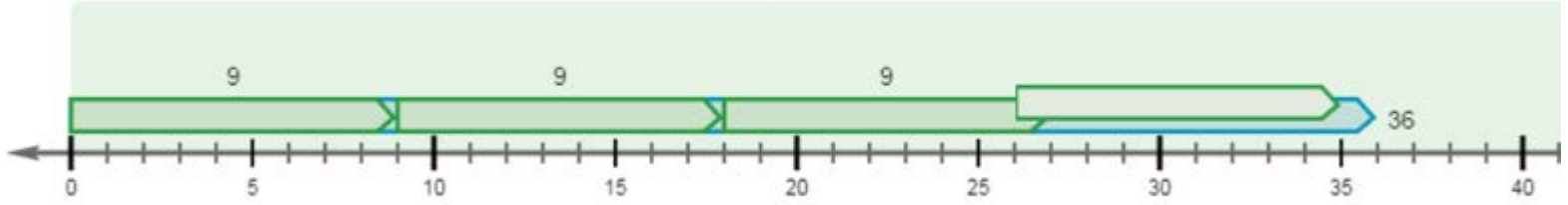
$8.4 \div 1.4 = ?$



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

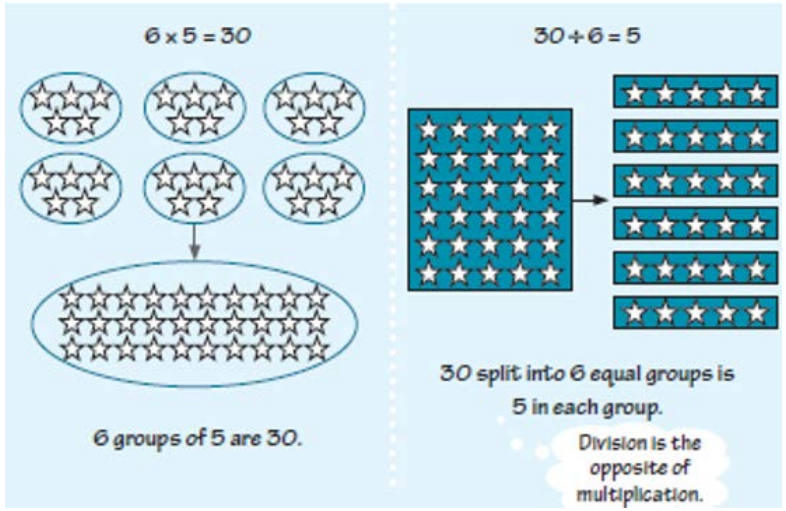


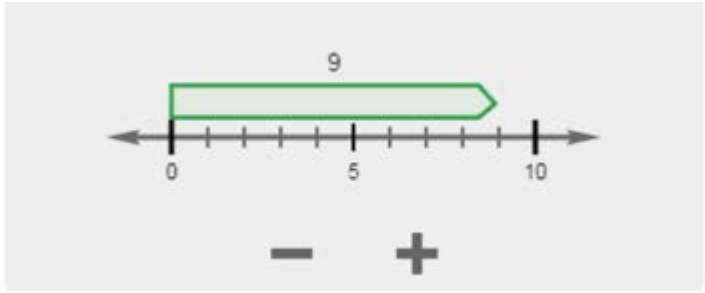


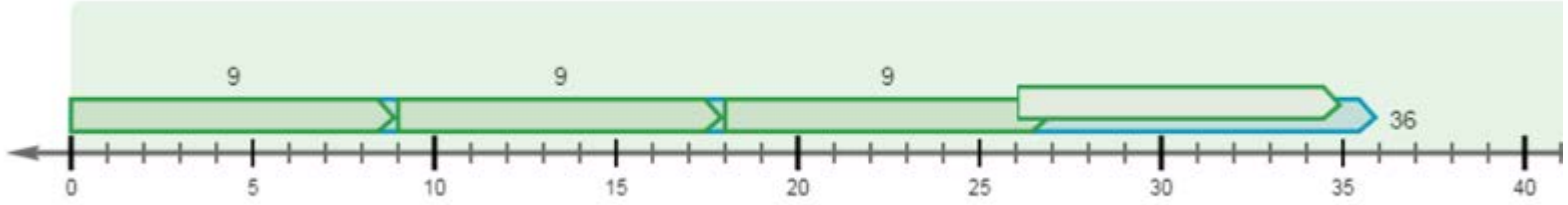


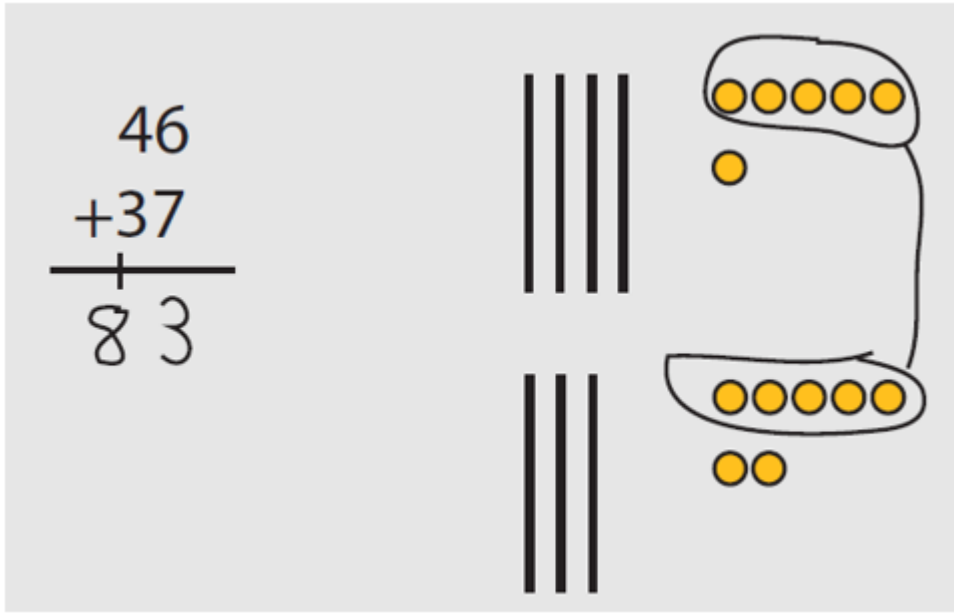
Addition is all about base ten

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

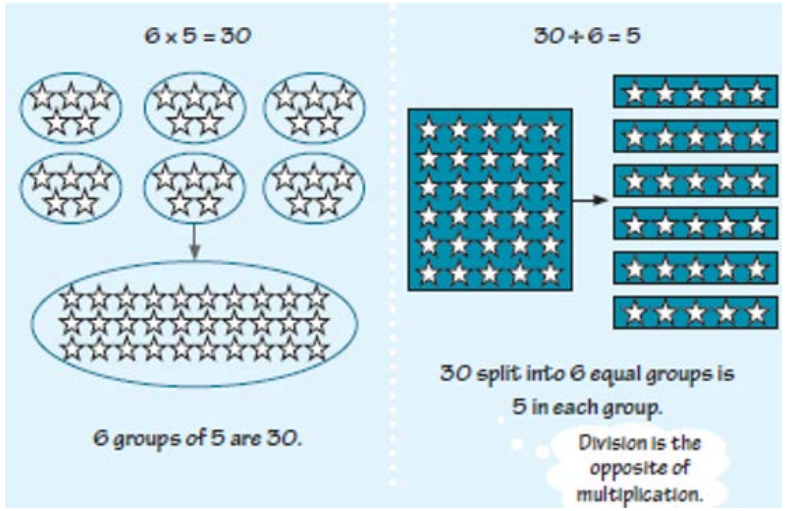


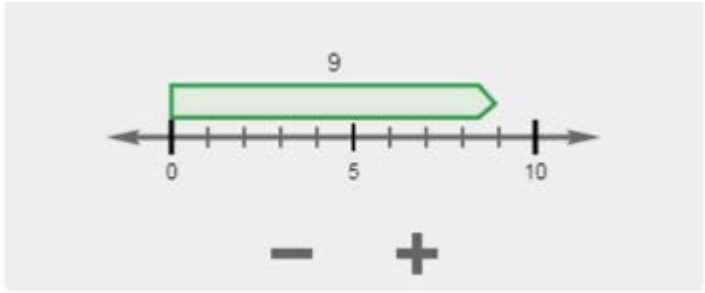


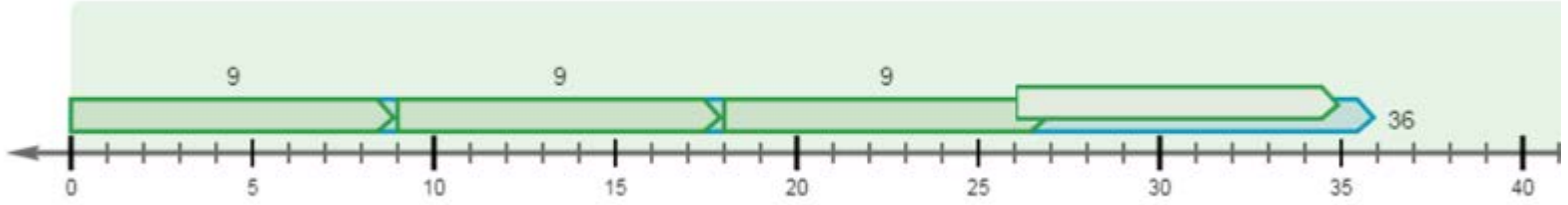


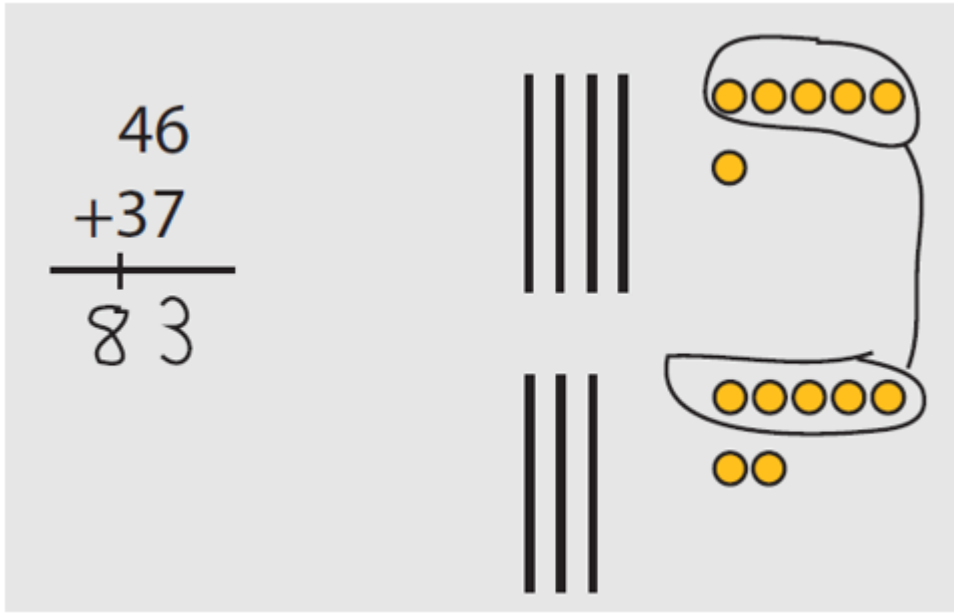


Multiplication is repeated addition.



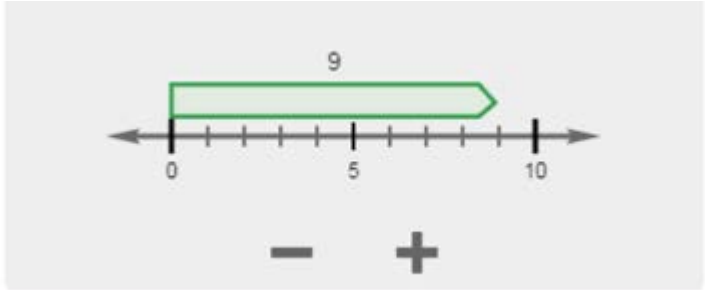


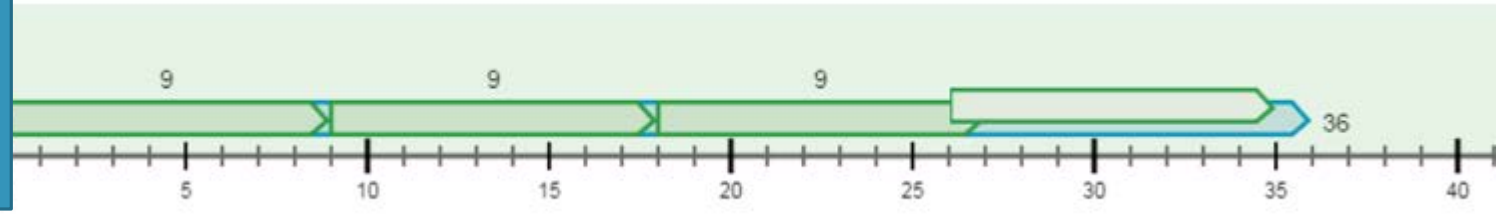


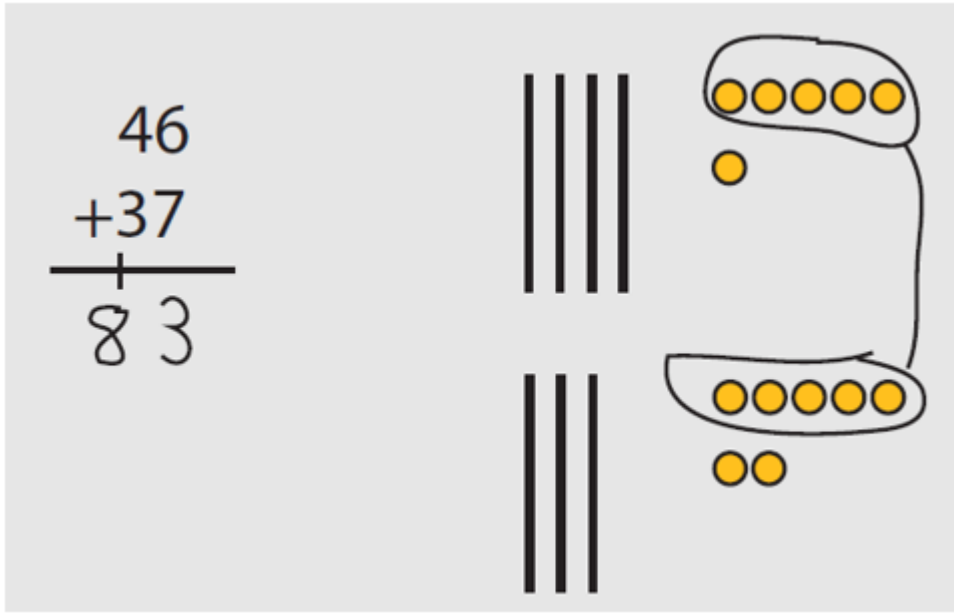


1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

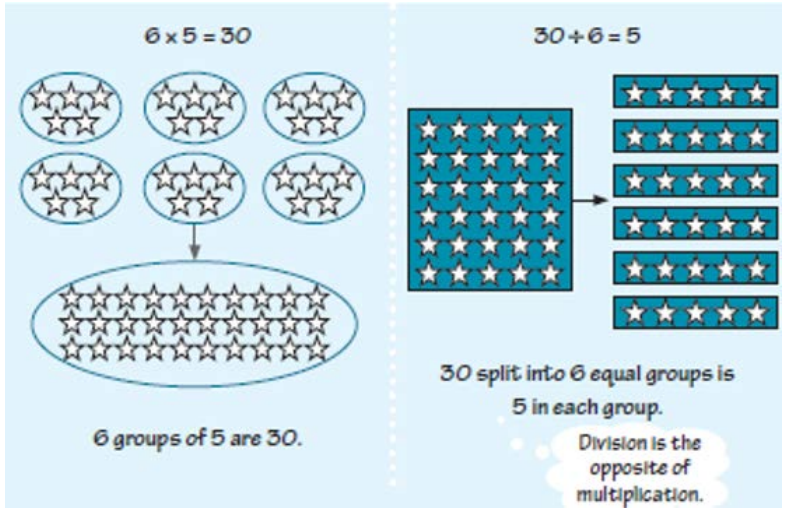
Multiplication and division are the same.



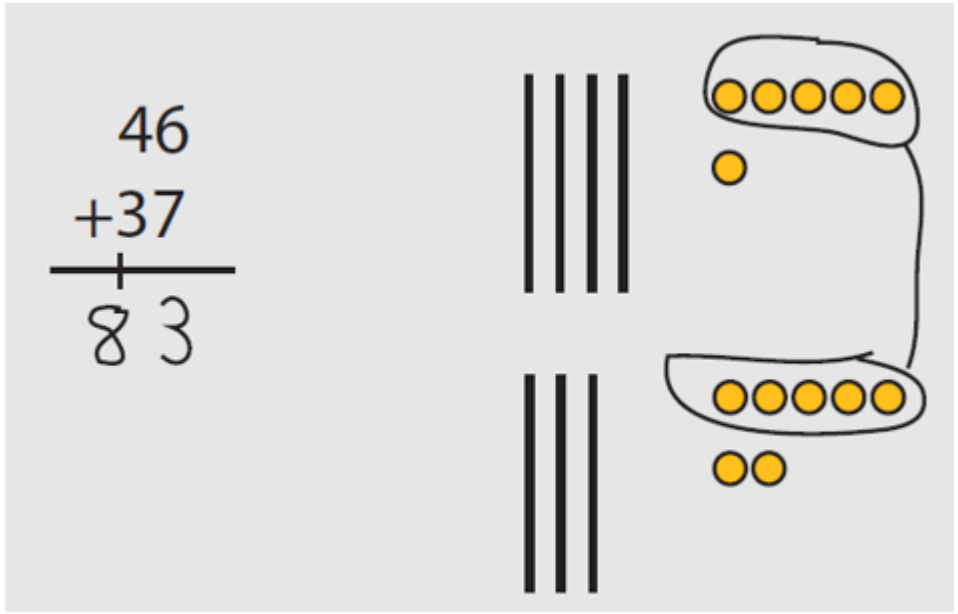




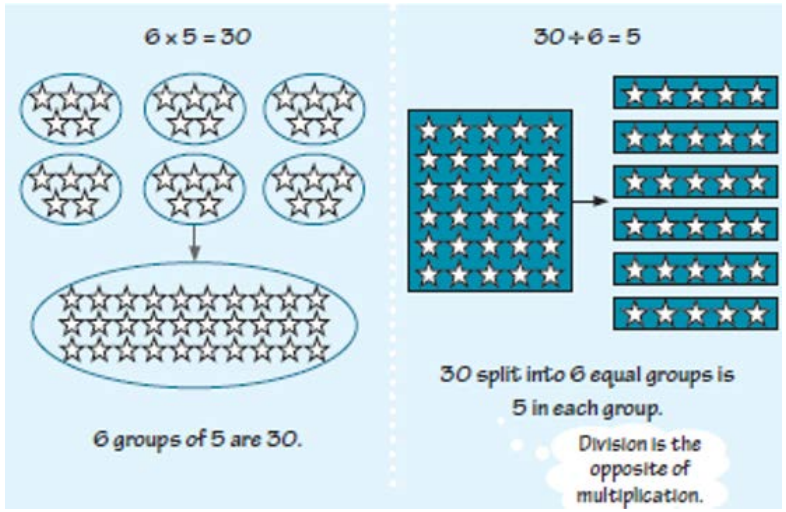
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

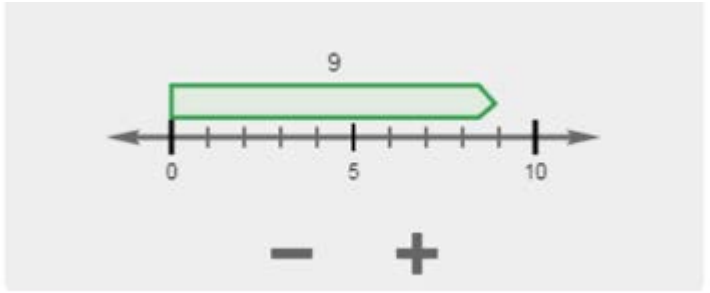


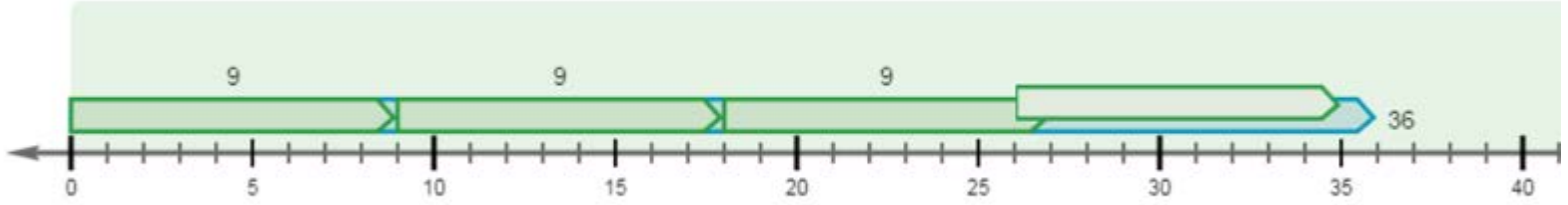
Division is repeated subtraction.



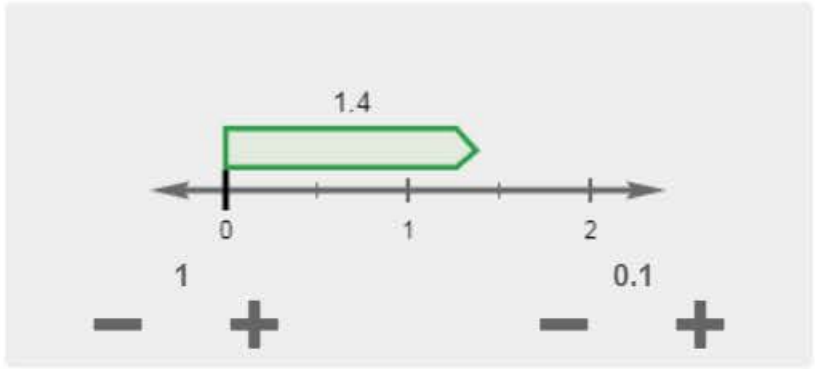
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

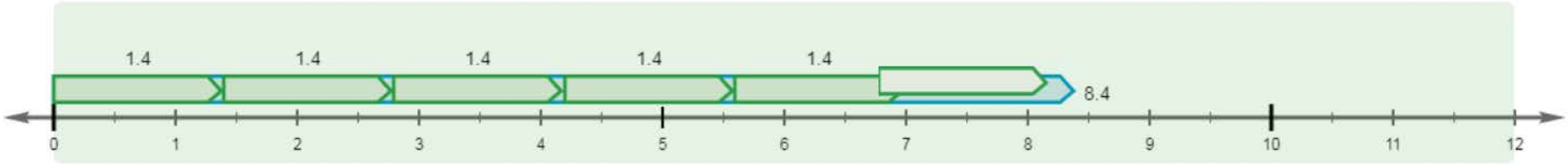






8.4 ÷ 1.4 = ?





? groups of 1.4 are in 8.4

$$8.4 \div 1.4 = ?$$

Let’s call this 1.4.

$$8.4 \div 1.4 = ?$$

Now we have 2.8.

$$8.4 \div 1.4 = ?$$

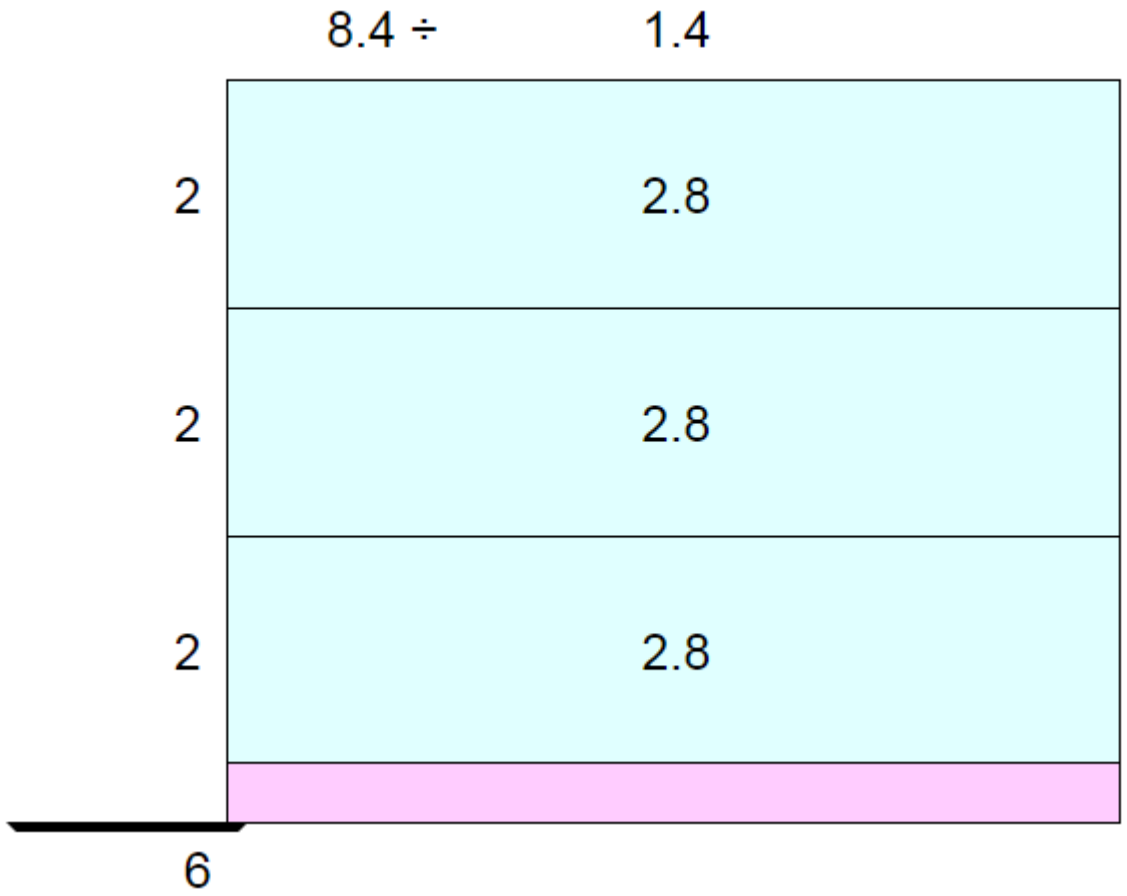
4.2!

8.4 ÷ 1.4 = ?

Voila!

8.4 ÷ 1.4 = ?

Voila!

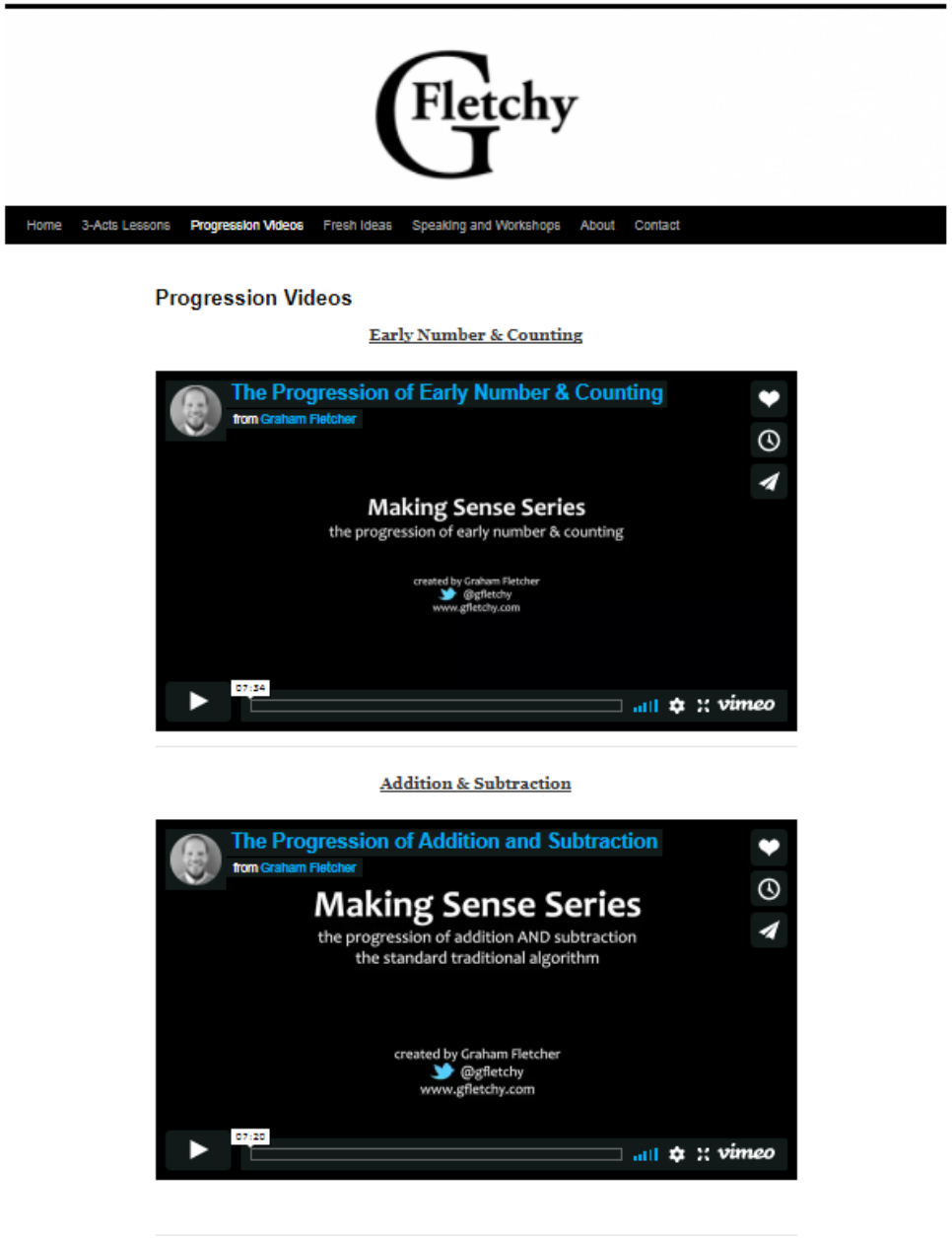


		6
	1.4	8.4
2 x	1.4	- 2.8
		5.6
2 x	1.4	- 2.8
		2.8
+ 2 x	1.4	- 2.8
		0.0
	6	

8.4 ÷ 1.4 = 6

Videos by Graham Fletcher

https://gfletchy.com/progression-videos/



1 to 1 Million: Number Sense Progressions for K-5

NCTM 2018

Arjan Khalsa

akhalsa@activatelearning.com

@arjankhalsa

www.arjankhalsa.com

www.conceptuamath.com

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning