# Hands-On Geometry

## *Using Origami & Compass Constructions for Advanced Explorations*

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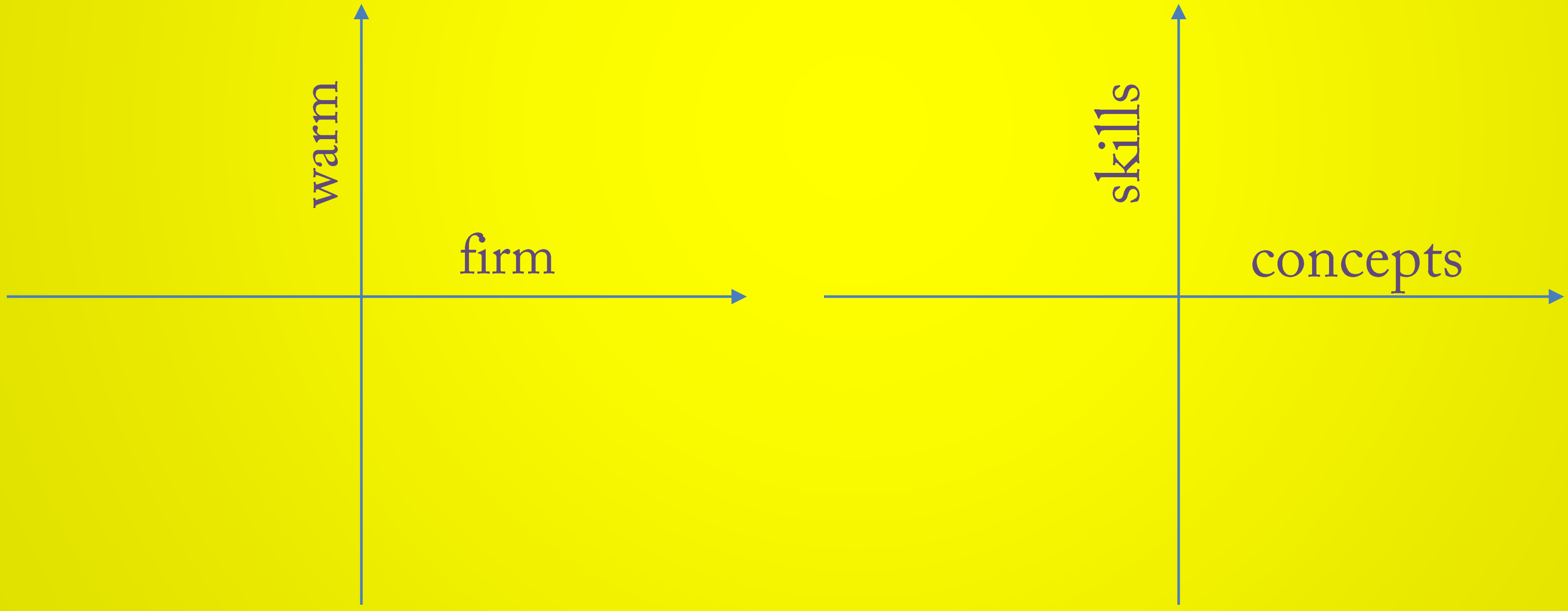
# Motivated, Resourceful, Self-Regulated Learners are…

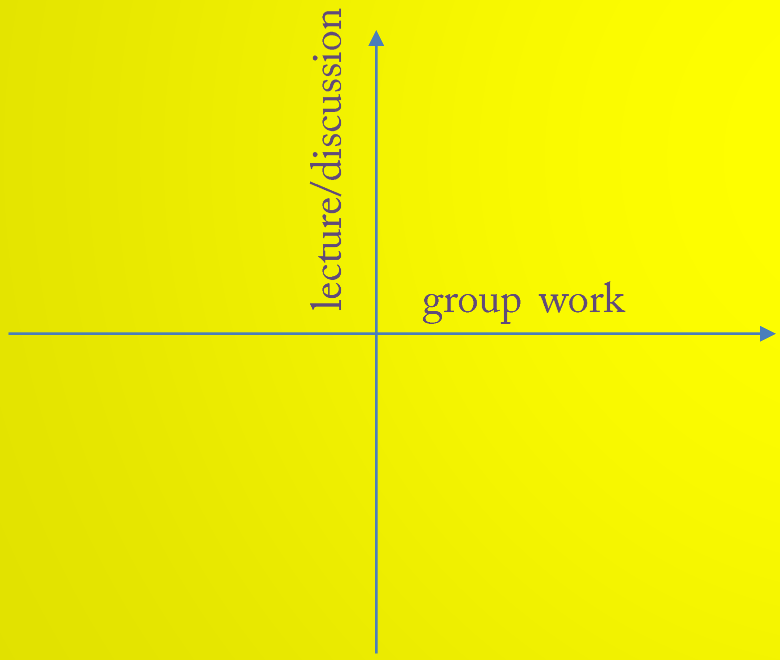
- self-appraising (who am I?)
- self-directed (who do I want to become?)
- self-assessing (how am I doing?)
- self-advocating (what is my voice?)

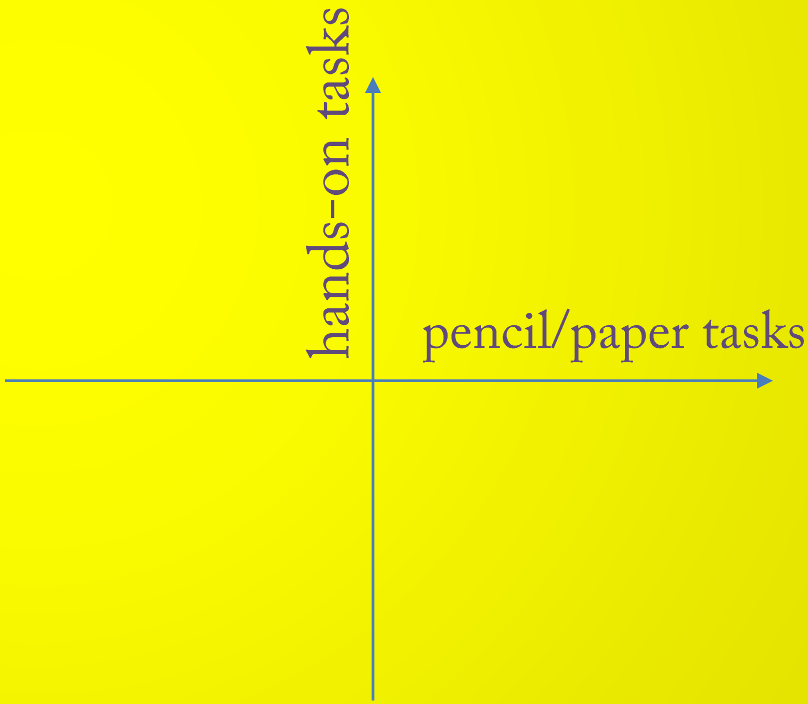
*There is no better way to empower your students and enliven their learning journey than to nudge them to be ACTIVE – in the classroom and out.*

# Are you a traditionalist?
# Are you a progressive teacher?
## …might you be both?



# A New Conversation:
# from either-or to both-and




Credit: Andy Crouch,
in *Strong and Weak* (2016)

…your students' learning will know no limits!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# The back-story to this talk…

- **A colleagueship, then a friendship & bday gift**
- **A whimsical course proposal…**
- **A lot of learning (quickly!) and a super-fun course with amazing kids**
- **A realization that these can be a-la-carte tasks**

Robert Lang, an inspiration…
Thomas Hull, a generous genius!

An ongoing consideration of the best mix of content, method, enjoyment…

# Don't let Standards Get in the Way

- The Common Core Standards include CCSS.MATH.PRACTICE.MP1 through 8
- Students frequently have the wrong idea about what math class is, and may find *Project Origami* confounding
- Address the needs of both failing and thriving math students: Do your students need to witness how math can be engaging, or how math can be authentic?

# Quick Participant Poll

- Geometry/Precalculus/Other…?
- Origami/Compass/(other?)

*We can tailor this a bit; also know that the entire set of slide images will be posted on the NCTM website or we could send you the actual PPT if you email us.*

# Our 'short list' of constructive origami tasks

- **Equilateral Triangle**: addtl challenge: max size?
- **Double-Angle Formulas**, Helix
- **Trisection**, Multi-section, **Fujimoto's method**
- **Modular Star Ring, Molly's Hex, Butterfly Bomb**
- **Parabola**, Solving Cubics (& Angle Trisection!)
- **TUPs**, 4-corners-to-a-point
- **Haga's Thm**, Mother & Baby Lines
- **Unit Polyhedra**: sonobe units & PHIZZ units

# And some recommended compass tasks

- **The Basics:**
  - Copying segments & angles, bisections
  - Perpendiculars & parallels
- Regular Polygons
  - **Tri, Quad, Hex, Dodec,** Penta*, Deca*
- **Number Line project** (Rosenthal 2013 online, and in appendix)
- Mosaic tilings, tesselations, etc.

# Other recommended practices:

- Math histories: students write 2-4 pages about their relationship with math over the years…a trove of information (and a message to them that you ask)!
- **Formative feedback, decoupled from delivery of [summative] grades.**
- **Required revisitation of all incorrect work on assessments**, and modeling/practice in class: homework revision and tutoring help.
- Keen attention in the first weeks to the form of students' written homework, and commentary on how and why to use the notation effectively.
- Strong exhortation to use study groups and TALK mathematics… again, this is modeled and practiced in class as a way to equip students.
- Use of $\tau$ (tau), the better circle constant, instead of $\pi$ (pi). (See The Tau Manifesto for a compelling argument in favor of this choice!)
- Book and movie recommendations for my students.
- Music, videos, and bad jokes distributed liberally.

# Resources/authors for further study and reflection:

- **Hull, Thomas, *Project Origami***
- **Fold-School,** a site with ready designs for three projects
- Brown, Roedinger, and McDaniel, *Make It Stick: The Science of Successful Learning*
- Daniel Willingham, *Why Don't Students Like School? A Cognitive Scientist Answers Questions About How the Mind Works and What It Means*
- Carol Dweck, *Mindset: The New Psychology of Success*
- Dan Pink, *Drive: The Surprising Truth About What Motivates Us*
- Angela Duckworth, *Grit*

Thank you for your interest and
for participating in this NCTM 2018 session.

Please send us a note by email to give feedback that you think would be helpful if we were to give this workshop again.

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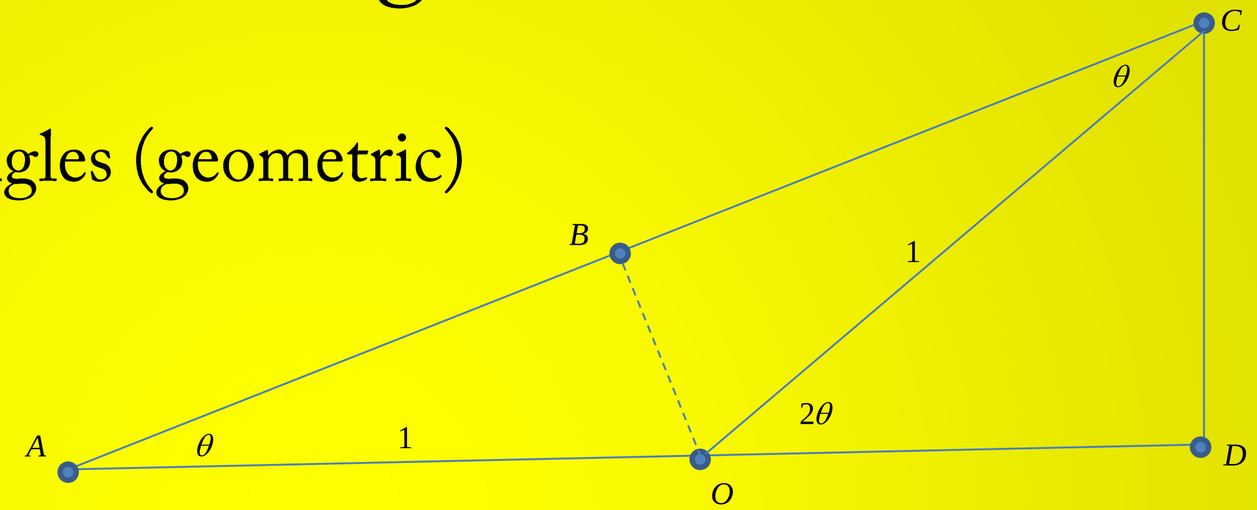
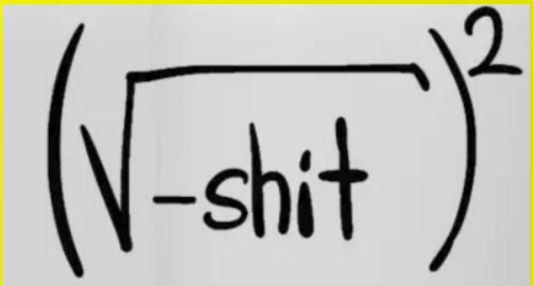
# Helping Double-Angle Formulas Stick

- Set up the triangles (geometric)



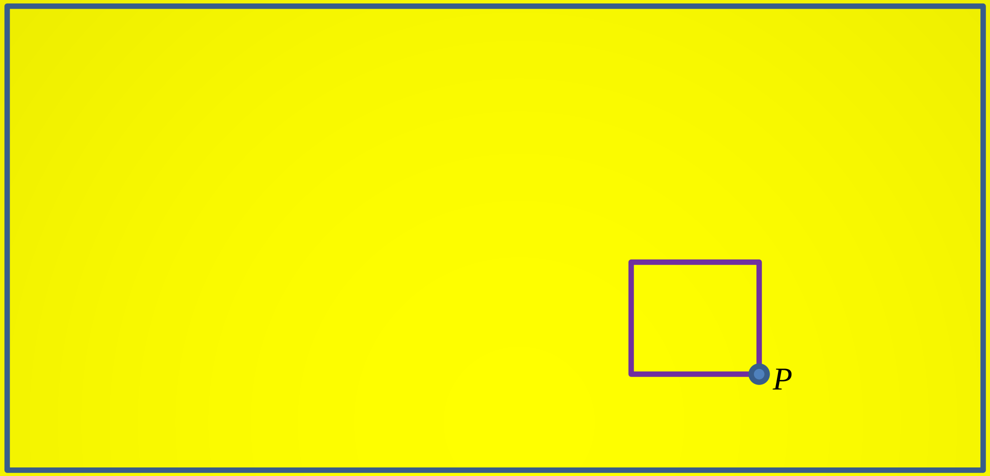

- Use the definitions (verbal & symbolic)
  - In ΔABO, write cos(θ) and sin(θ)…and consider the fold
  - In ΔCOD, write cos(2θ) and sin(2θ)…and consider CO
  - In ΔACD, write cos(θ) and sin(θ)

- Algebra moves (symbolic & procedural)



# Turned-Up Parts (TUPs)

- Fix a location for the square; mark the corners.




- Turn up point *P* to another position within the square. What do you see? What shape is the TUP? We might ask, "How can we tell how many sides a TUP will have?"
- Now turn up *P* to a position outside the square. Now what are the possibilities? (Can you justify why the regions are shaped this way?)
- Explore/Experiment…Conjecture…Prove

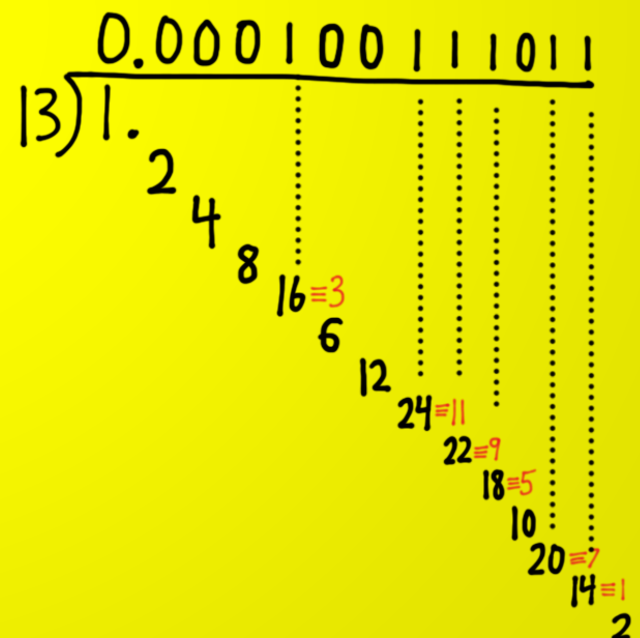
# Fujimoto Approximation

- Can students figure out how this trick works?
- Connect with counting in binary (see, for example, https://youtu.be/aAPrkb3hpDg)
- Connect with modular arithmetic

4:02

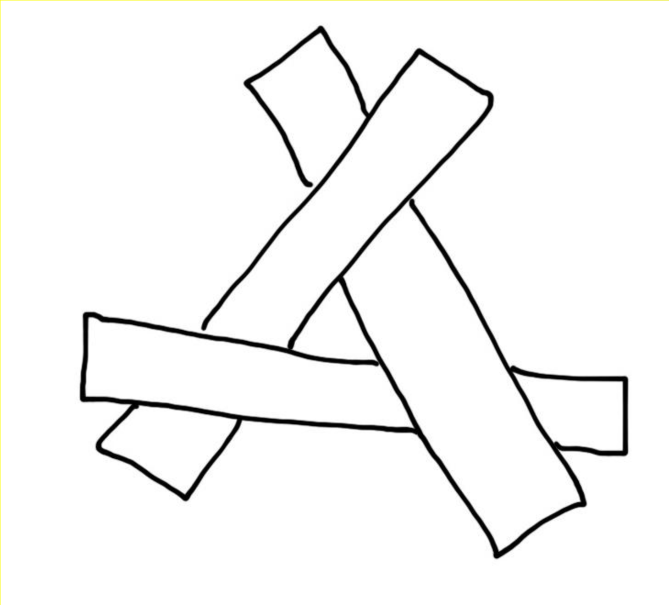
# Fujimoto Approximation

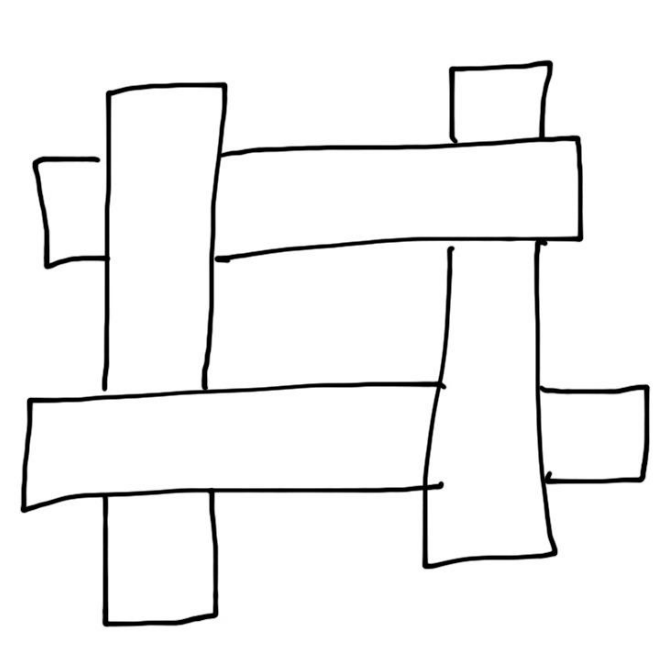
| | | |
|---|---|---|
| 12 | 1 | 1 |
| 6 | 7 | 1 |
| 3 | 10 | 0 |
| 8 | 5 | 1 |
| 4 | 9 | 1 |
| 2 | 11 | 1 |
| 1 | 12 | 0 |
| 7 | 6 | 0 |
| 10 | 3 | 1 |
| 5 | 8 | 0 |
| 9 | 4 | 0 |
| 11 | 2 | 0 |
| 12 | 1 | |



# Butterfly Bomb

- Sparse Directions
- The weave
- Guided by symmetry





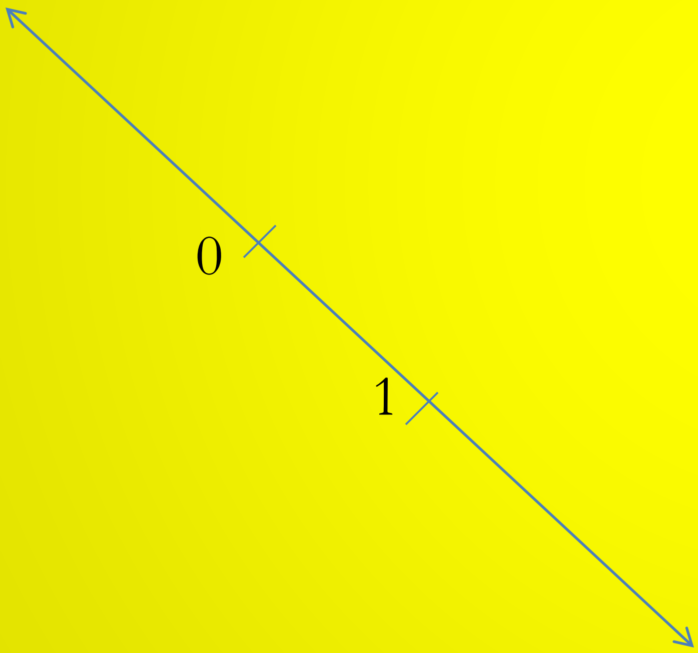
5 min optional excursus

# Appendix: Number Line Project

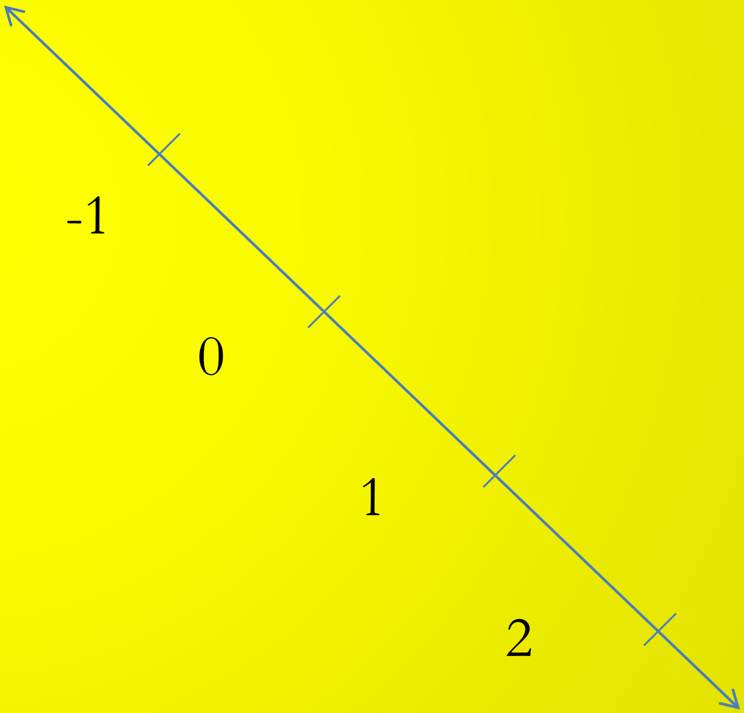
- Introduction/revisitation of number types
- Integer construction
  - Can we count 'all of' ($\aleph_o$)them?
- Rational number construction
  - Further extension: can we count 'all of' **these** (are there $\aleph_o$ as well)?
- Irrational number construction
  - How about these we're constructing? How about *all* irrationals?

# First, the **integers**

- Ruler Postulate



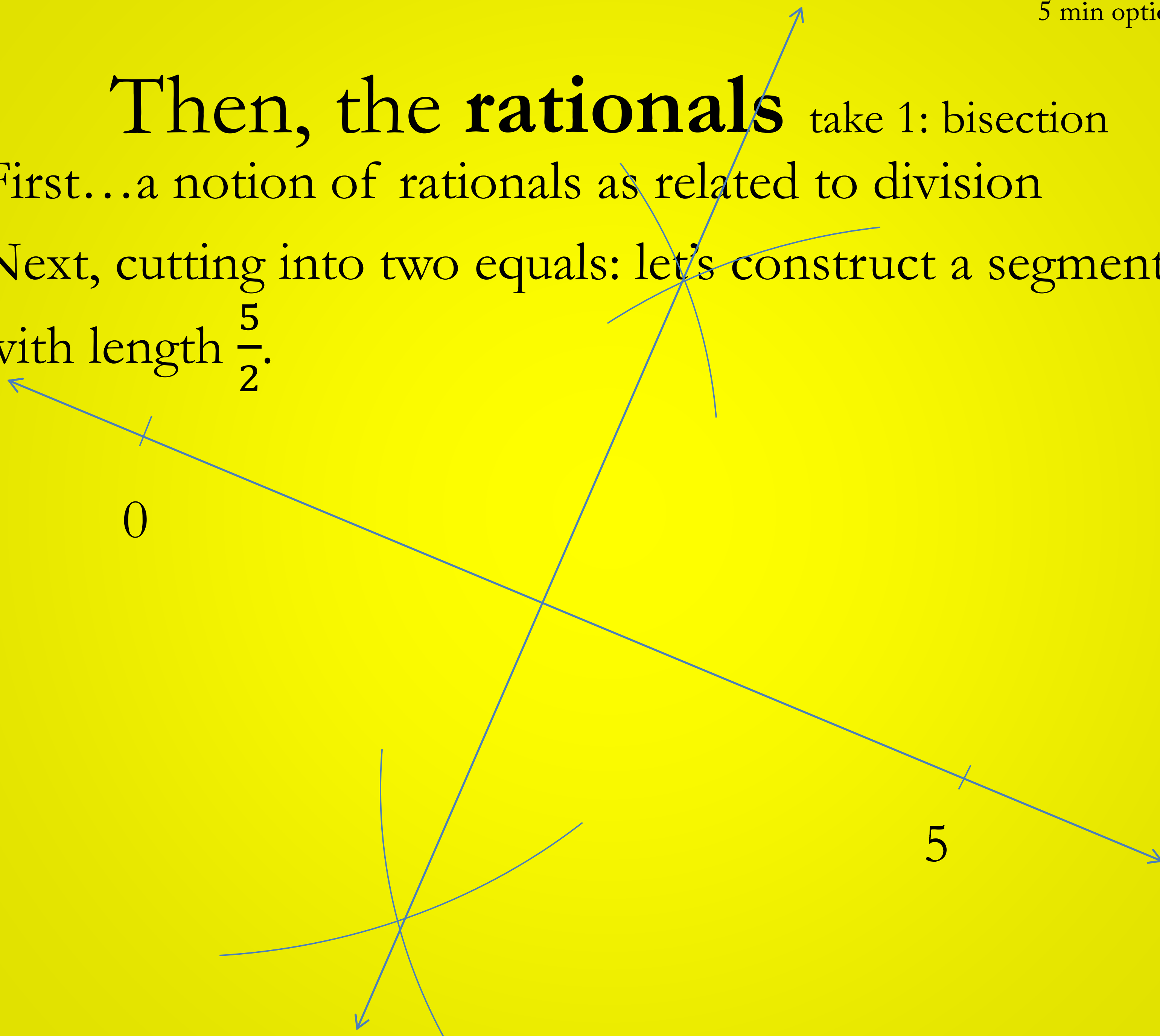

- Segment Addition



5 min optional excursus

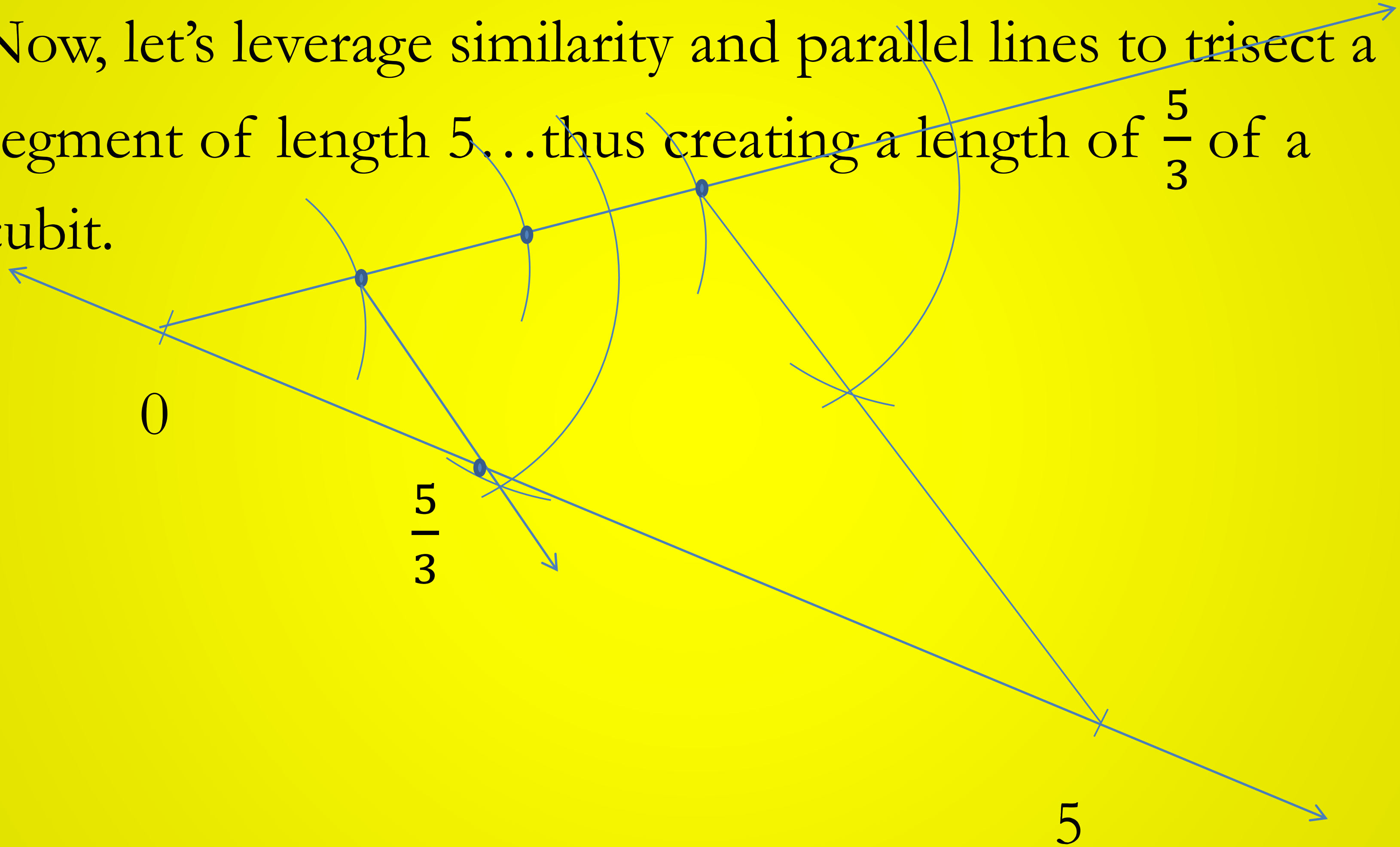
# Then, the **rationals** take 1: bisection

- First…a notion of rationals as related to division
- Next, cutting into two equals: let's construct a segment with length $\frac{5}{2}$.



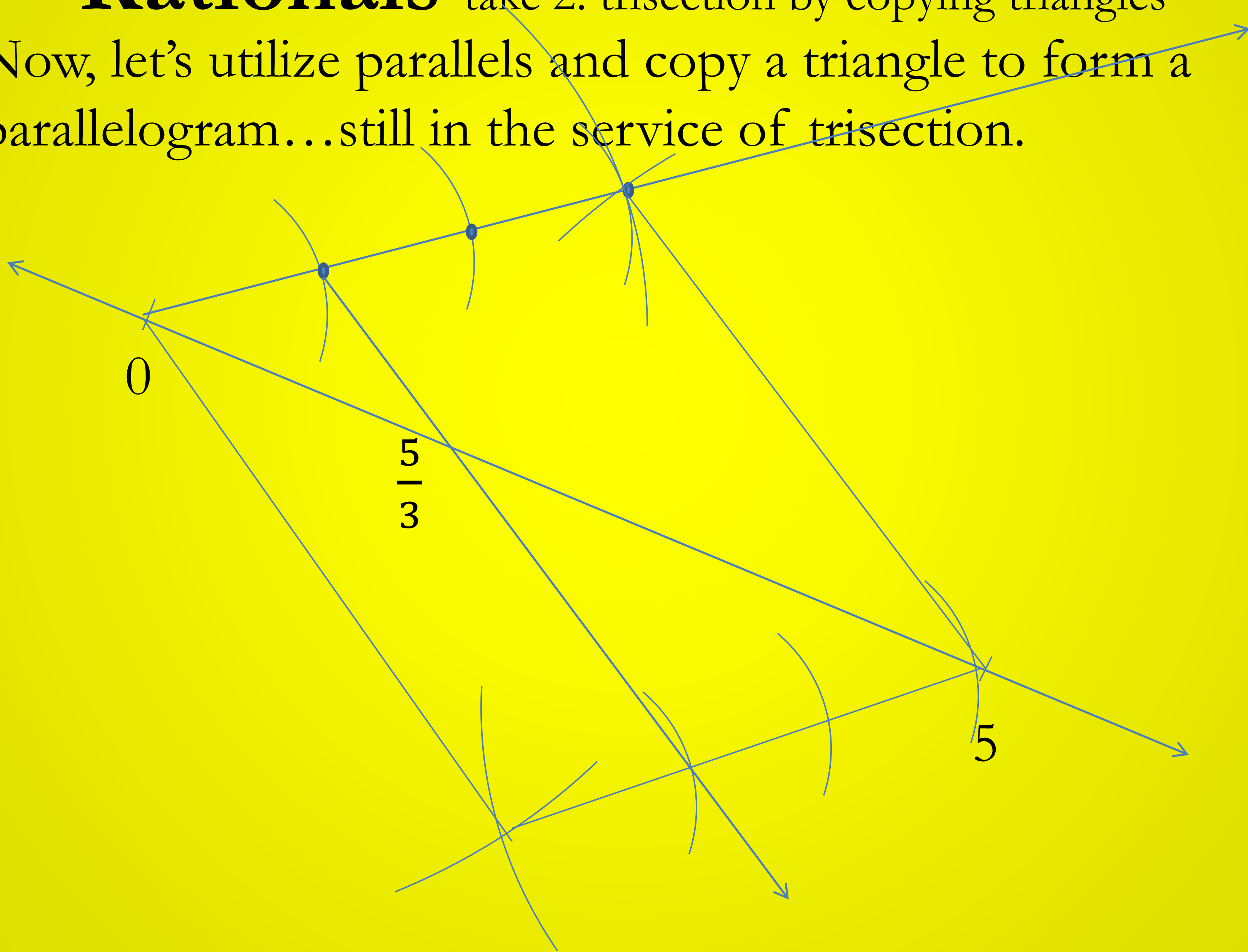
# **Rationals** take 2: trisection (and multisection)

- Now, let's leverage similarity and parallel lines to trisect a segment of length 5…thus creating a length of $\frac{5}{3}$ of a cubit.



# Rationals take 2: trisection by copying triangles

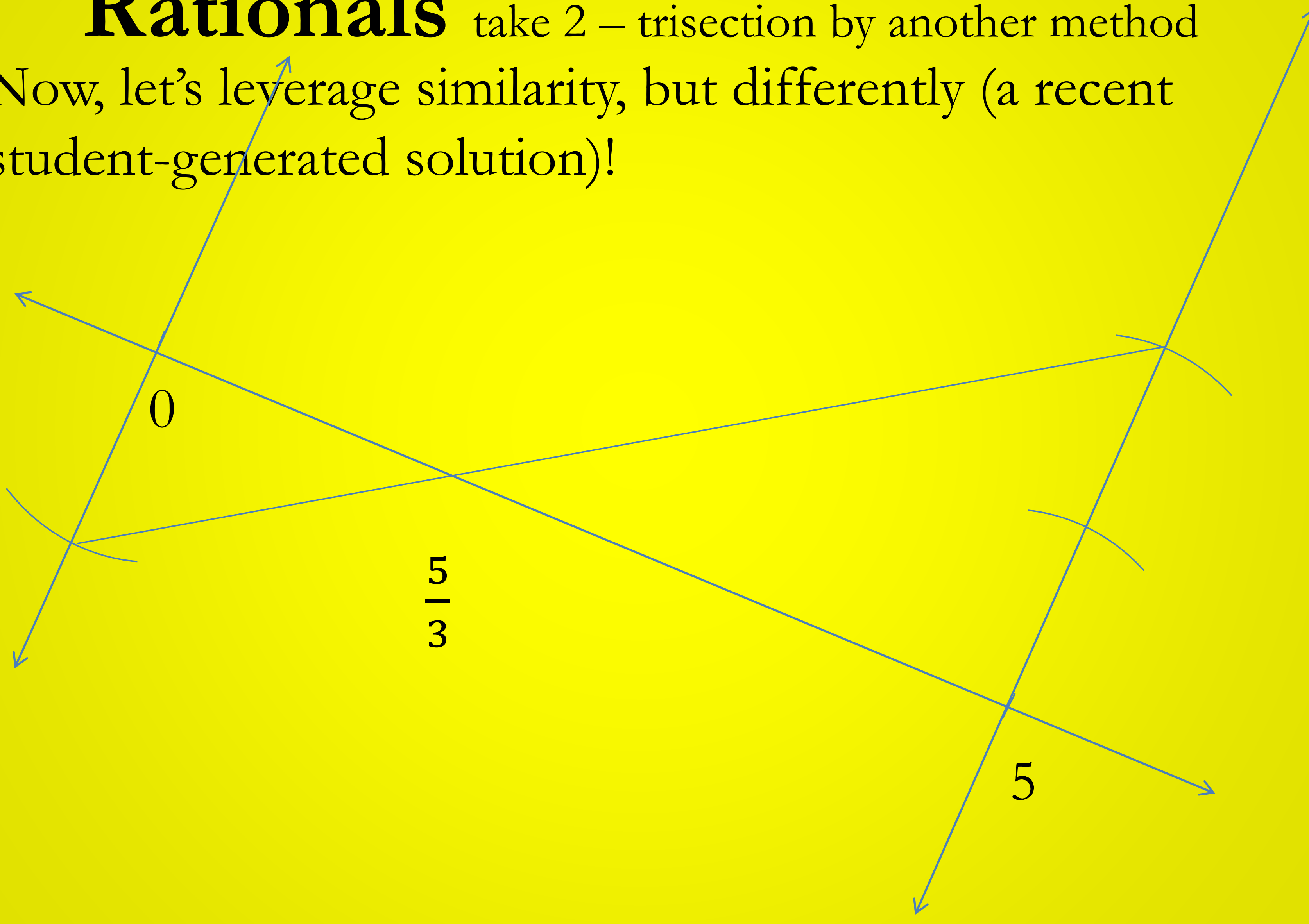
- Now, let's utilize parallels and copy a triangle to form a parallelogram…still in the service of trisection.



5 min optional excursus

# Rationals take 2 – trisection by another method

- Now, let's leverage similarity, but differently (a recent student-generated solution)!



# On to the irrationals

Let's try to construct segments of length $\sqrt{2}, \sqrt{13}, \sqrt{7}$, and $\sqrt{41}$.

Since considering square roots are tricky, let's consider squares.
W.W.P.D.? (What would Pythagoras do?)

$$l_1{}^2 + l_2{}^2 = h^2$$

1+1=2… so $\sqrt{1}^2 + \sqrt{1}^2 = \sqrt{2}^2$, or $1^2 + 1^2 = \sqrt{2}^2$.

In other words, if we have legs of length 1, our hypotenuse would have length $\sqrt{2}$.

Are there leg lengths that would help you construct $\sqrt{13}$?

4+9=13

What number sentence would help you construct $\sqrt{41}$?

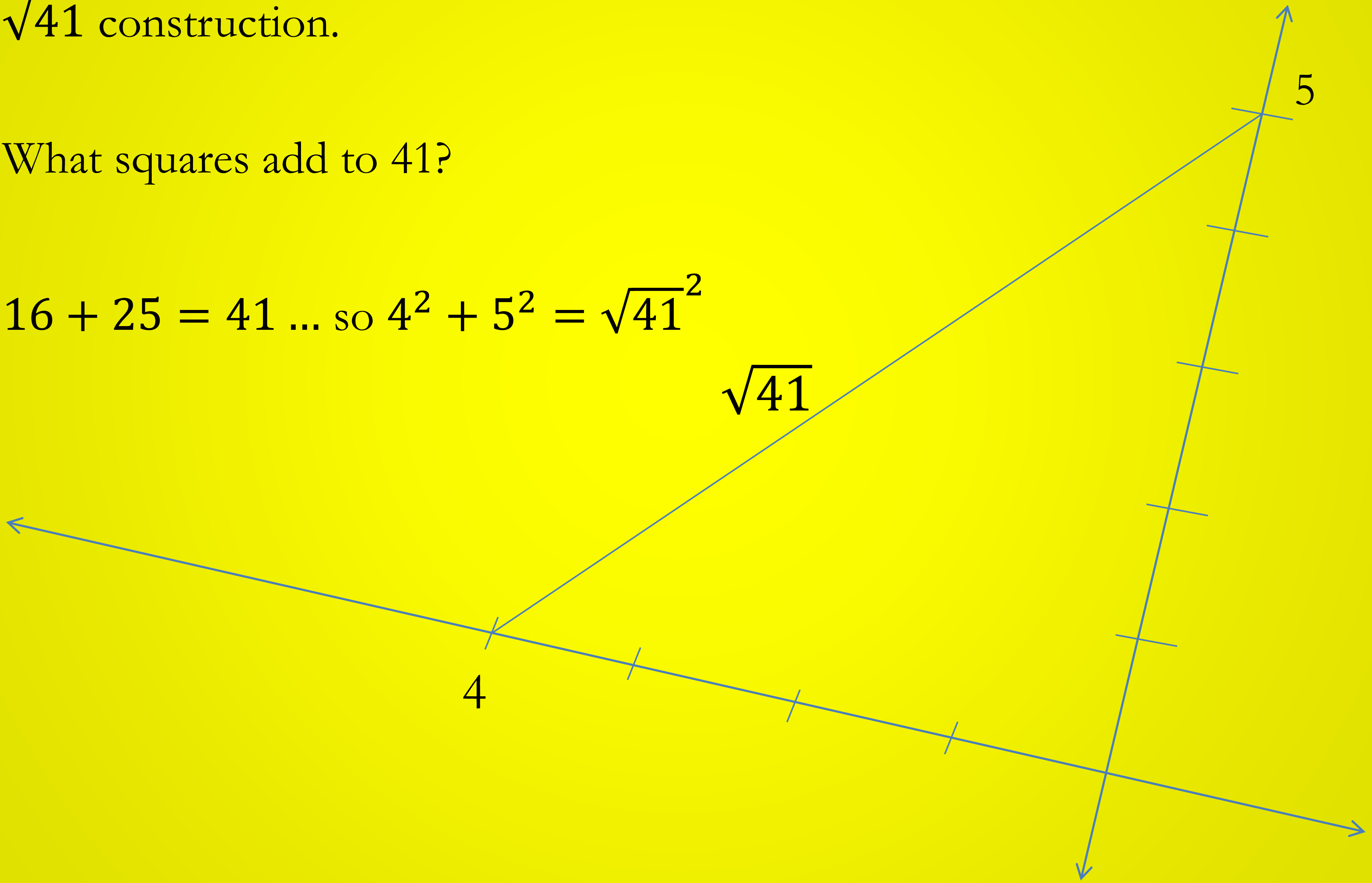
16+25=41

Hmmm…how about $\sqrt{7}$?

16-9=7………so 9+7=16

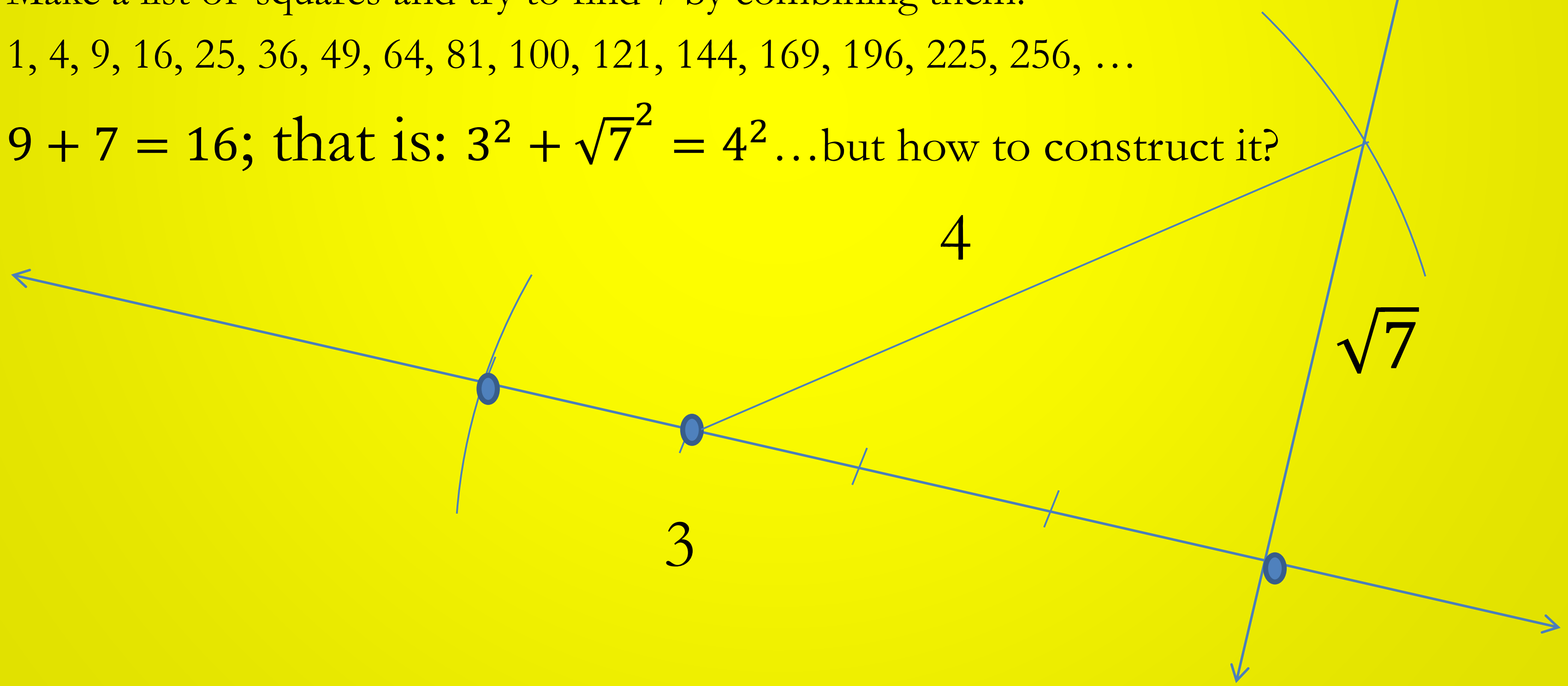
# Constructing irrationals

- $\sqrt{41}$ construction.
- What squares add to 41?
- $16 + 25 = 41$ … so $4^2 + 5^2 = \sqrt{41}^2$



# Constructing irrationals

- $\sqrt{7}$ construction.
- What squares add to 7?
- Hmmm…what to do?
- Make a list of squares and try to find 7 by combining them!
- 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, …
- $9 + 7 = 16$; that is: $3^2 + \sqrt{7}^2 = 4^2$…but how to construct it?



# Further resources/authors for study:

- On developing mathematical understanding (a classic):

  Courant and Robbins: *What is Mathematics?*

- Four books that Brent thinks every math teacher should consider having on the shelf:

  Richard Philips: *Numbers – Facts, Figures, and Fiction*

  Steven Strogatz: *The Joy of X*

  David Wells: *The Penguin Dictionary of Curious and Interesting Numbers*

  David Wells: *The Penguin Dictionary of Curious and Interesting Geometry*

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