STUDENT THINKING AND QUADRATIC FUNCTIONS: BUILDING PROCEDURAL FLUENCY

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Effective Teaching Practices

1. Select mathematical goals.
2. Implement challenging tasks.
3. Facilitate meaningful student discourse.
4. Pose purposeful questions.
5. Connect multiple representations.
6. Develop procedural fluency through conceptual understanding.
7. Promote productive struggle.
8. Elicit and Use Evidence of Student Thinking

Principles to Actions, NCTM, 2014

Fluency

Students are flexible when choosing the method or strategy to solve problems.

Principles to Actions: Ensuring Mathematical Success for All, National Council of Teachers of Mathematics (NCTM), Reston, Va.: NCTM 2014. 42

Fluency & Conceptual Understanding

Fluency is not restricted to computation. Fluency includes fraction operations, measurement formulas and algebraic procedures.

Principles to Actions: Ensuring Mathematical Success for All, National Council of Teachers of Mathematics (NCTM), Reston, Va.: NCTM 2014. 42

Standard Algorithm

Fluency with an algorithm is based on understanding the process and mathematical approach, not the way the steps are recorded.

Fuson, Karen C. and Sybilla Beckmann,
"Standard Algorithms in the Common Core State Standards", *National Council of Supervisors of Mathematics (NCSM) Journal of Mathematics Education Leadership*, 14, no. 1 [2012/2013], 14-30

Procedural Fluency

Students can explain the procedure they used.

Students can explain the mathematical basis for their work.

Students pick a strategy that seems to work best for a situation.

Principles to Actions: Ensuring Mathematical Success for All, National Council of Teachers of Mathematics (NCTM), Reston, Va.: NCTM 2014. *47*

What does Procedural Fluency look like?

(2)

$$\frac{3}{4} \div \frac{1}{2} = \frac{3 \div 1}{4 \div 2} = \frac{3}{2} = 1\frac{1}{2}$$

Why do we teach completing the square?

1) To derive the quadratic formula.
2) To change a standard form equation of a parabola into "graphing" form.
3) To have another method to solve quadratic equations.
4) To make connections among symbols, graphs and tables of quadratic functions.
5) To find the center of a circle, an ellipse or a hyperbola from standard form and to find the vertex of a parabola in standard form. These forms assist us in graphing conic sections and in finding other features of conic sections.
6) And ….

After our goal is selected …

- In advance questions and moves are planned to promote student discourse.
- The tasks involve connecting multiple representations.
- The process of implementing the task is designed to promote productive struggle. The different approaches make the mathematical concepts accessible to all students. Peer support and teacher moves encourage student involvement.
- Our tasks involve connecting area concepts with symbolic and graphing concepts from Algebra to build a conceptual understanding of completing the square, as well as having a reason to use the process.

What is procedural fluency?

Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems

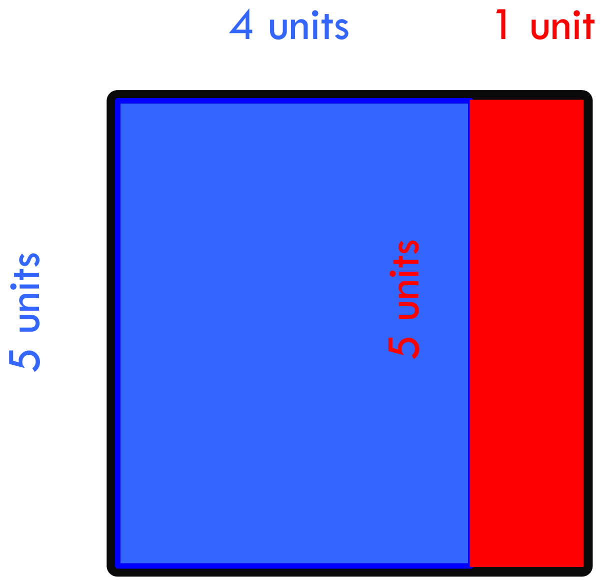
Principles to Actions: Ensuring Mathematical Success for All (NCTM 2014, p.42)

Procedural fluency & Flexibilty

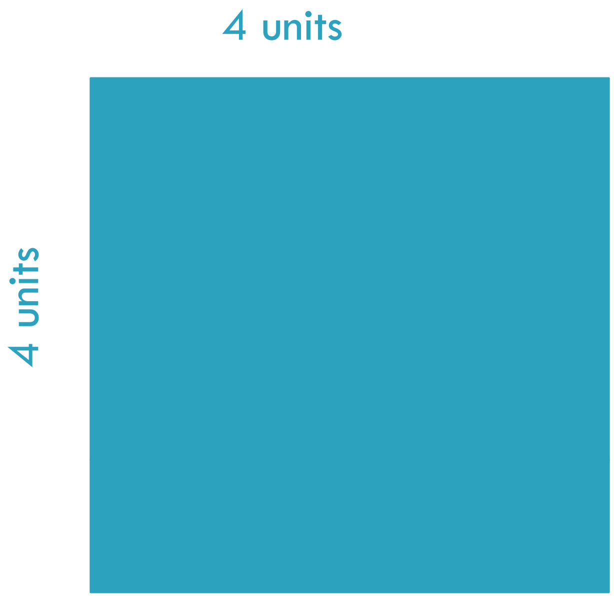
Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems

Principles to Actions: Ensuring Mathematical Success for All (NCTM 2014, p.42)

Connecting to Geometry

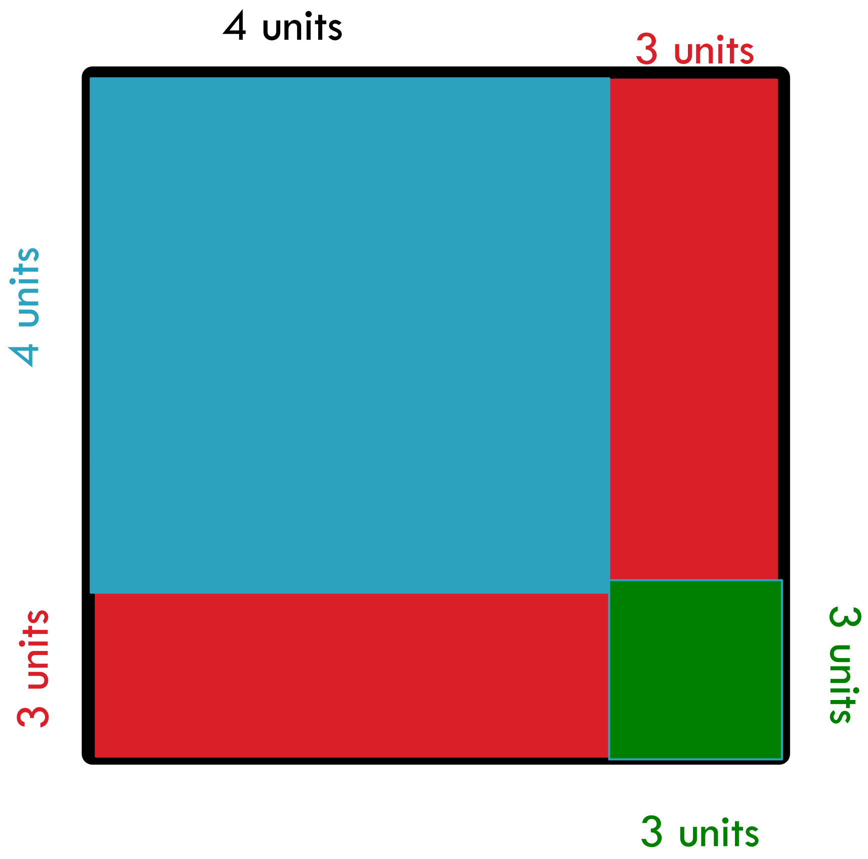


More Geometry – enlarging the square



This is a square – but we want to create a larger square by adding rectangles. How can you make a square with 7 units per side?

More Geometry – enlarging the square



What do you notice about the areas that are added?

What is the area of the entire shape?

What's next?

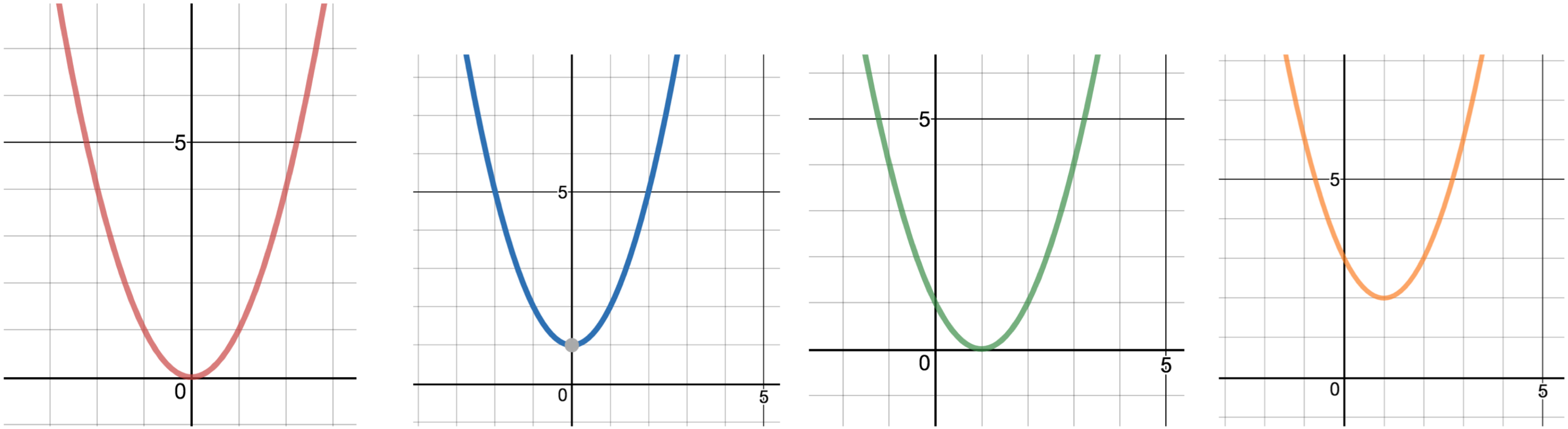
Let students try a few enlarging squares problems as warm ups to get the idea while transitioning to considering quadratic functions.

Our classroom process seems to be shifting to another focus, so be sure to let students know these ideas will connect.

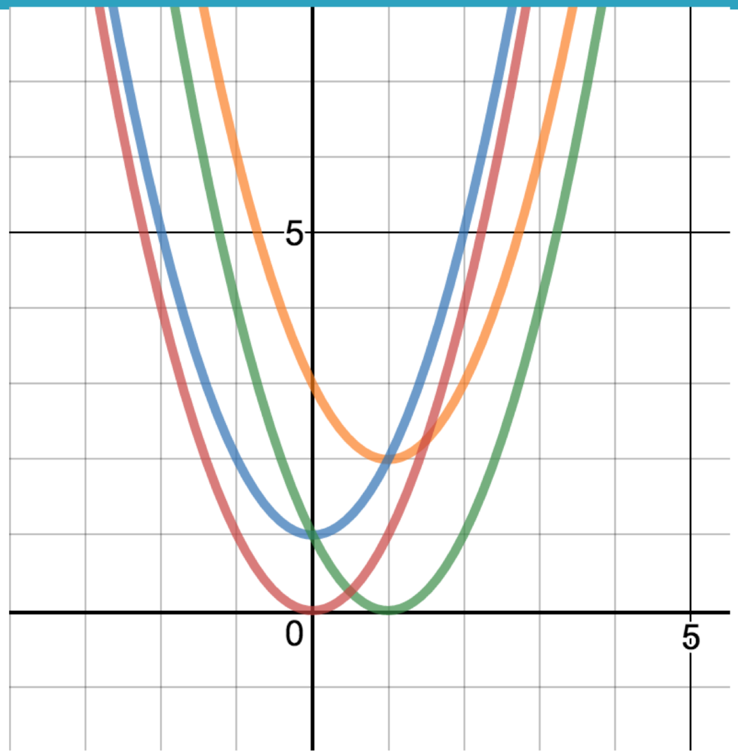
Relating to Symbolic forms and Graphs

Let's look at the graphs of $y = x^2$, $y = x^2 + 1$, $y = (x - 1)^2$ and $y = (x - 1)^2 + 2$

What do you notice?



Relating to Graphs and Symbolic forms



The vertex is at

(0, 0)

(0, 1)

(1, 0)

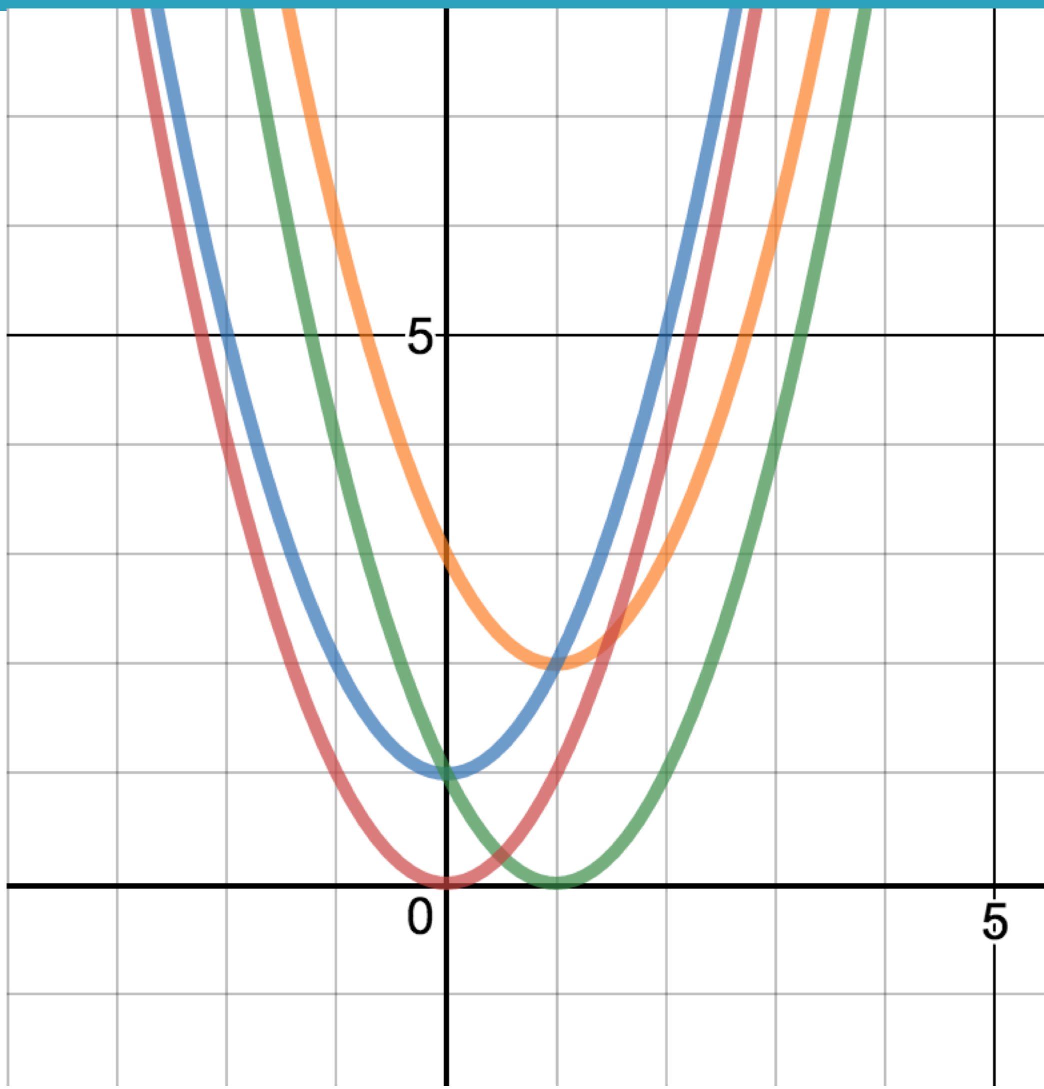
(1, 2)

Axis of symmetry?

Consider the form
$y = (x - h)^2 + k$

$y = x^2$, $y = x^2 + 1$,
$y = (x - 1)^2$ and $y = (x - 1)^2 + 2$

Relating to Graphs and Symbolic forms



$y = (x - h)^2 + k$

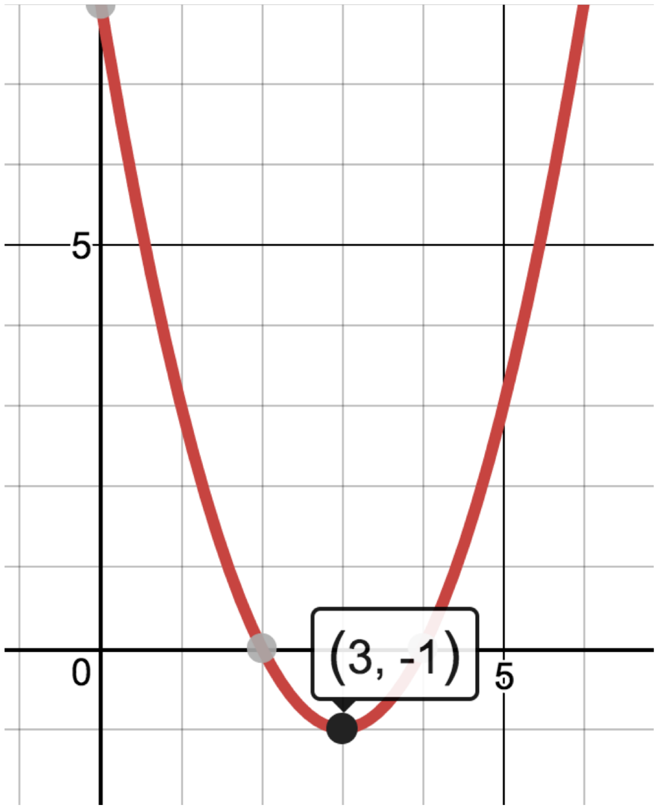
How does this help us graph the function?

Relating to Graphs and Symbolic Forms

What if the function is expressed as

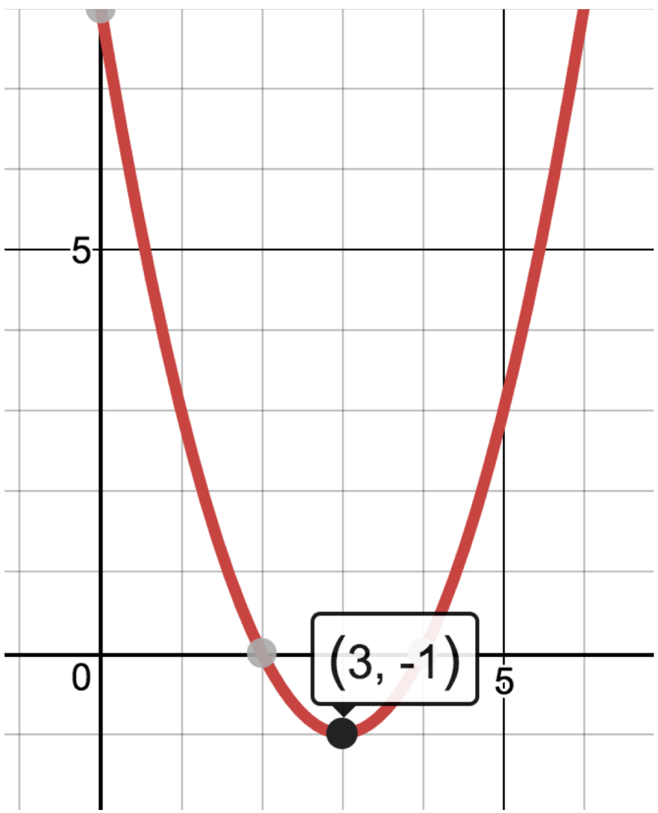
$y = x^2 - 6x + 8$?

Where is the vertex?



Relating to Graphs and Symbolic Forms

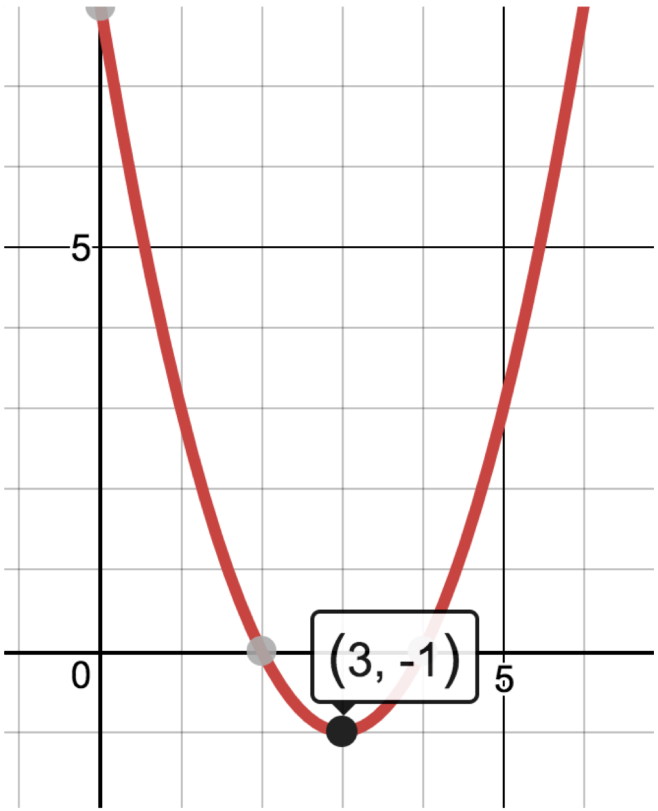
Since the graph is really the function $y = x^2$ shifted three units to the right and down one unit, let's consider just the graph shifted three units to the right.

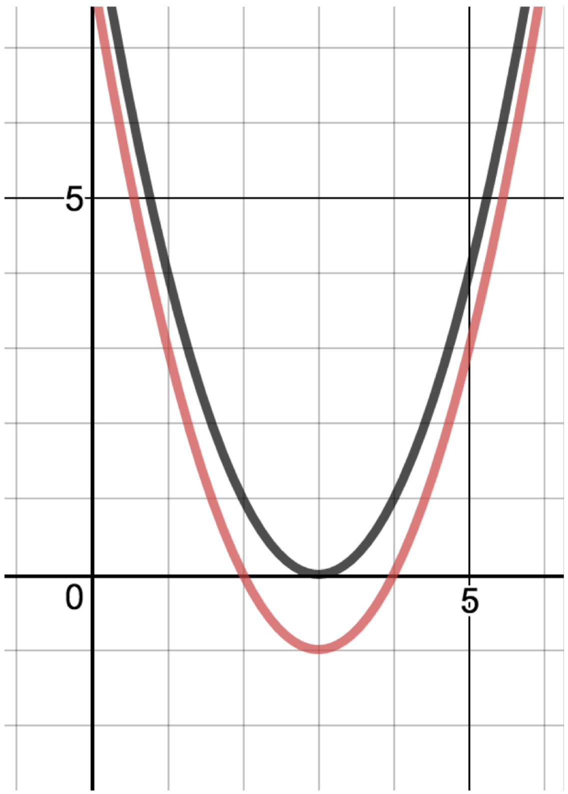


Relating to Graphs and Symbolic Forms

How does this relate to the graph of

$y = (x - 3)^2$?

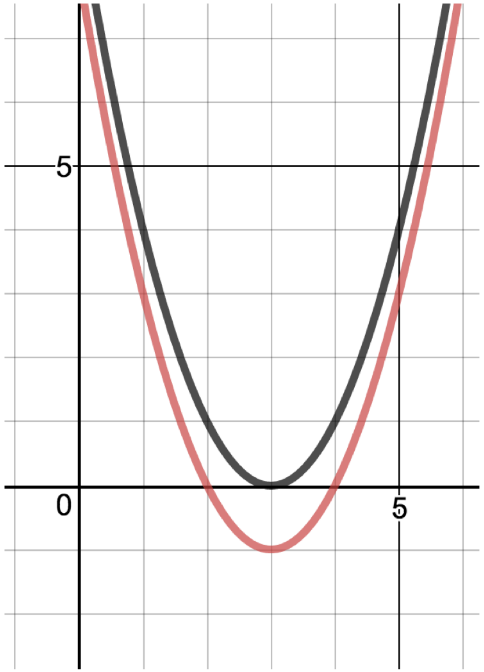




Relating to Graphs and Symbolic Forms

There is a vertical shift of $y = (x - 3)^2$ down one unit, that is $y = x^2 - 6x + 8$ is shifted down one unit from $y = (x - 3)^2$.

$y = (x - 3)^2 - 1$



Relating to Graphs and Symbolic Forms

Symbolically, how does $y = x^2 - 6x + 8$ relate to the "perfect square" $y = (x - 3)^2$?

$y = (x - 3)^2 = x^2 - 6x + 9 = x^2 - 6x + 8 + 1$

So, $x^2 - 6x + 8 = x^2 - 6x + 8 + 1 - 1$

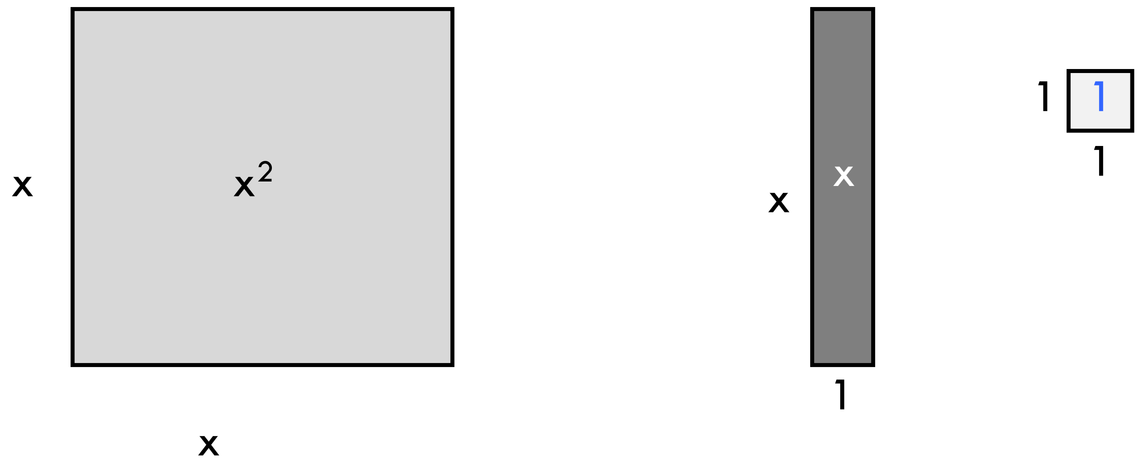
$= (x - 3)^2 - 1$

Relating to Graphs and Symbolic Forms

We can always look at the graph and reverse engineer to find the graphing or vertex form of the quadratic function – but what about when we want to graph the function directly from the standard form of the function?

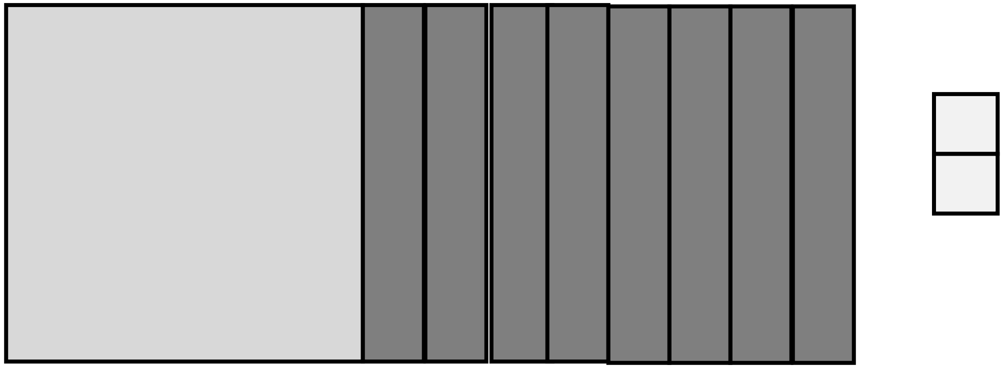
Let's look at $y = x^2 + 8x + 2$.

Back to Geometry



We learned how to make a square with Geometry and in Algebra I using algebra tiles.

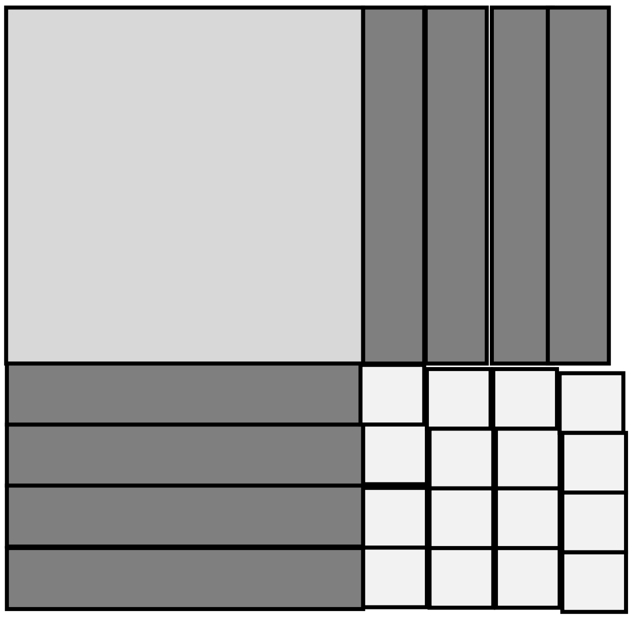
Back to Geometry



$x^2 + 8x + 2$

Back to Geometry

How do we use 8x? What happens with the 2?

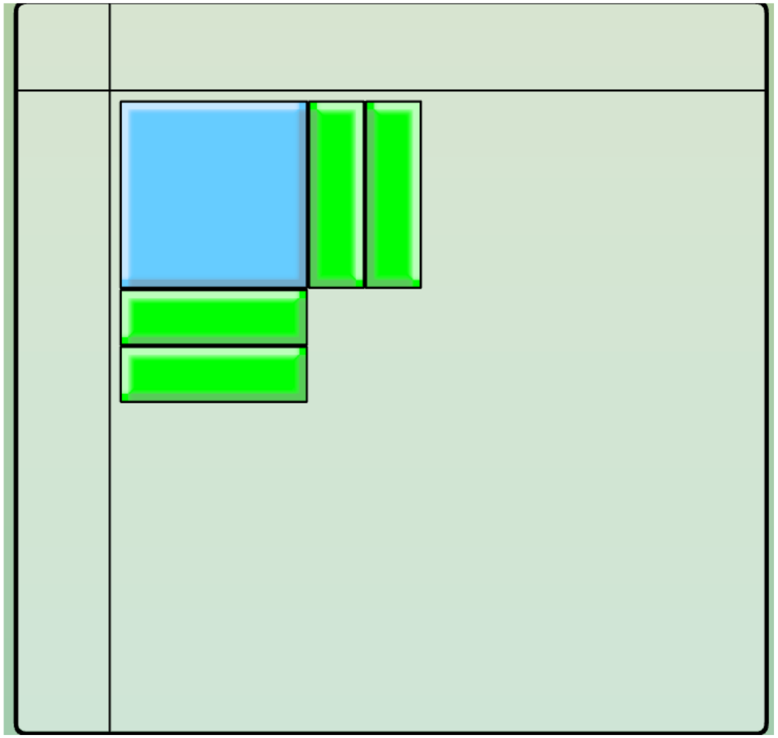


$$x^2 + 8x + 2 = x^2 + 8x + 2 + 14 - 14$$

$$= x^2 + 8x + 16 - 14 = (x + 4)^2 - 14$$

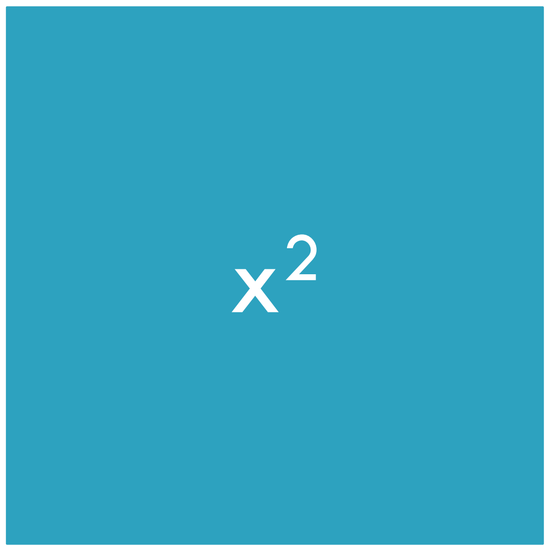
Algebra tiles part 2

http://illuminations.nctm.org/Activity.aspx?id=3482



More Geometry

$y = x^2 + 16x - 8$



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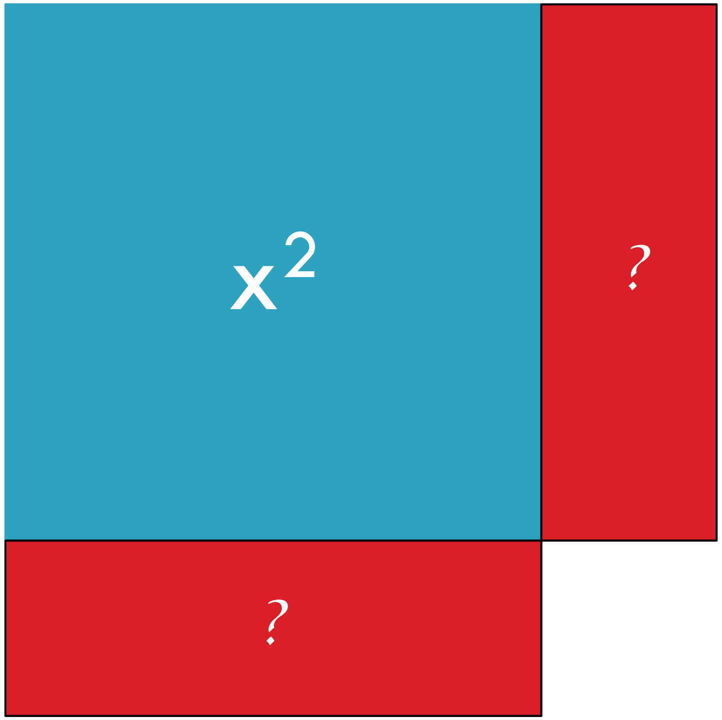
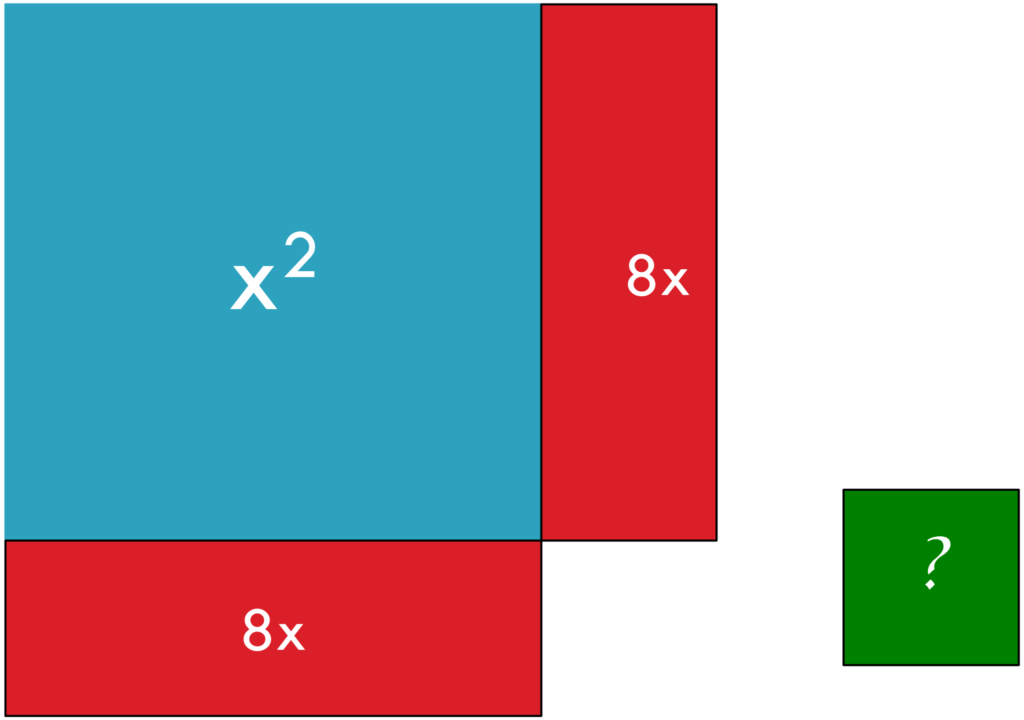


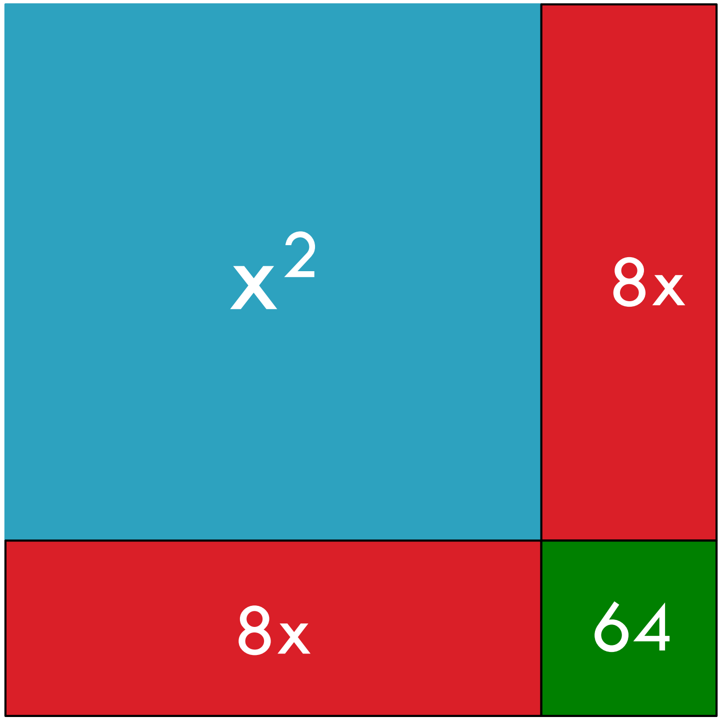
More Geometry

$y = x^2 + 16x - 8$



More Geometry

$y = x^2 + 16x - 8 = x^2 + 16x - 8 + 64 - 64$



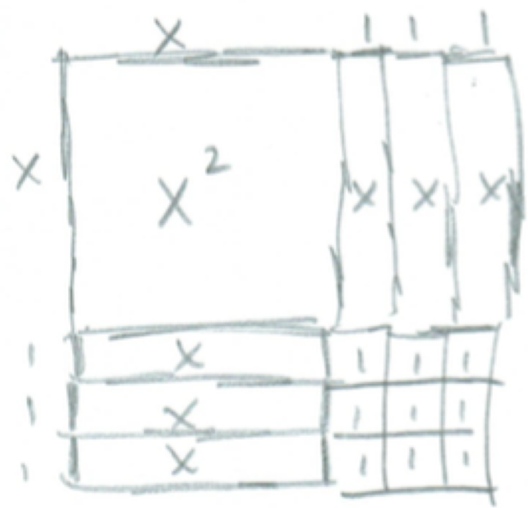
$y = (x + 8)^2 - 72$

Student work

$$y = x^2 + 6x + 7$$

$$y = (x^2 + 6x + 9) + 7 - 9$$

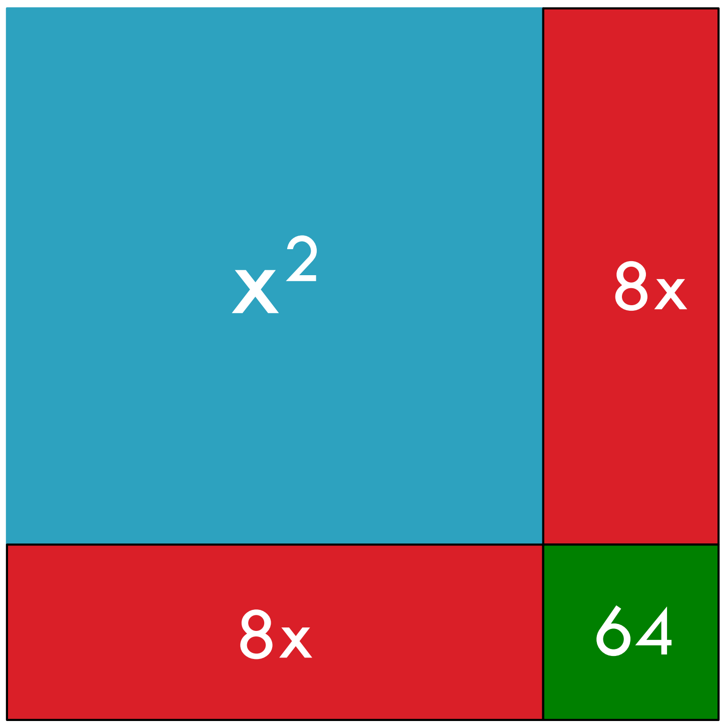
$$y = (x+3)^2 - 2$$



The 6x is split into 2 equal parts to make a square

More Geometry

$y = x^2 + 16x - 8 = x^2 + 16x - 8 + 64 - 64$



What is the vertex?

Are there x-intercepts? How do you know?

$y = (x + 8)^2 - 72$

Finding the x-intercepts

Let's find the x-intercepts of $y = x^2 + 16x - 8$

$x^2 + 16x + 64 - 64 - 8 = 0$

$(x + 8)^2 = 72$

$x + 8 = \pm\sqrt{72}$

$x = -8 \pm\sqrt{72}$

Estimate -8 ± 8.4

There's a coefficient in front of x^2!

$y = {}^{-}x^2 - 2x - 11$

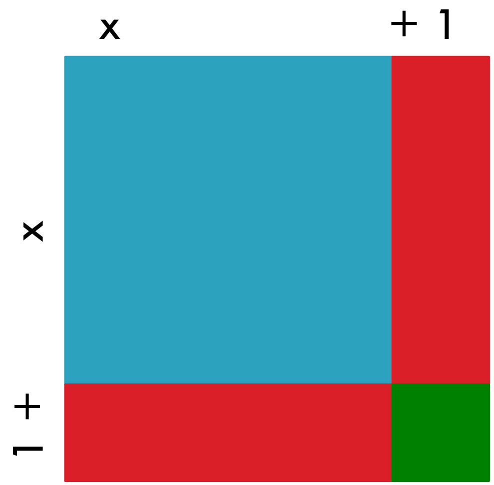
One way is

$-y = x^2 + 2x + 11$

$-y = x^2 + 2x + 1 - 1 + 11$

$-y = (x + 1)^2 + 10$

$y = -(x + 1)^2 - 10$

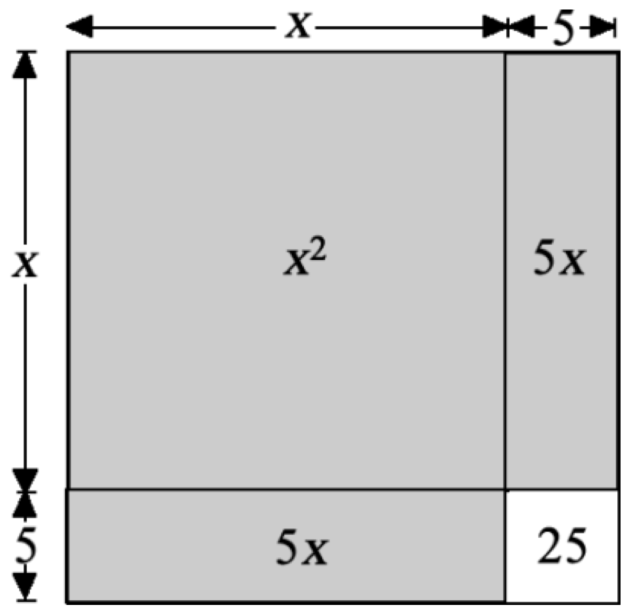


Solving an equation

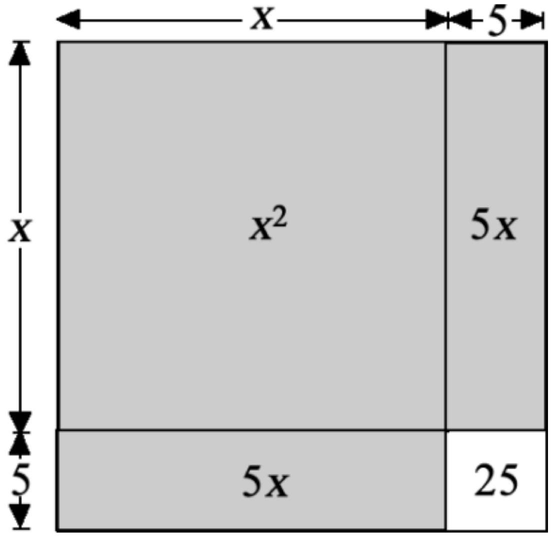
How can we use this area model to solve the equation: $x^2 + 10x = 144$?

Squaring it Away

As we saw before, we can rearrange the figure so that it is close to being a square, and then find the area of the missing piece.



Squaring Away



$$(x+5)^2 = 144 + 25$$

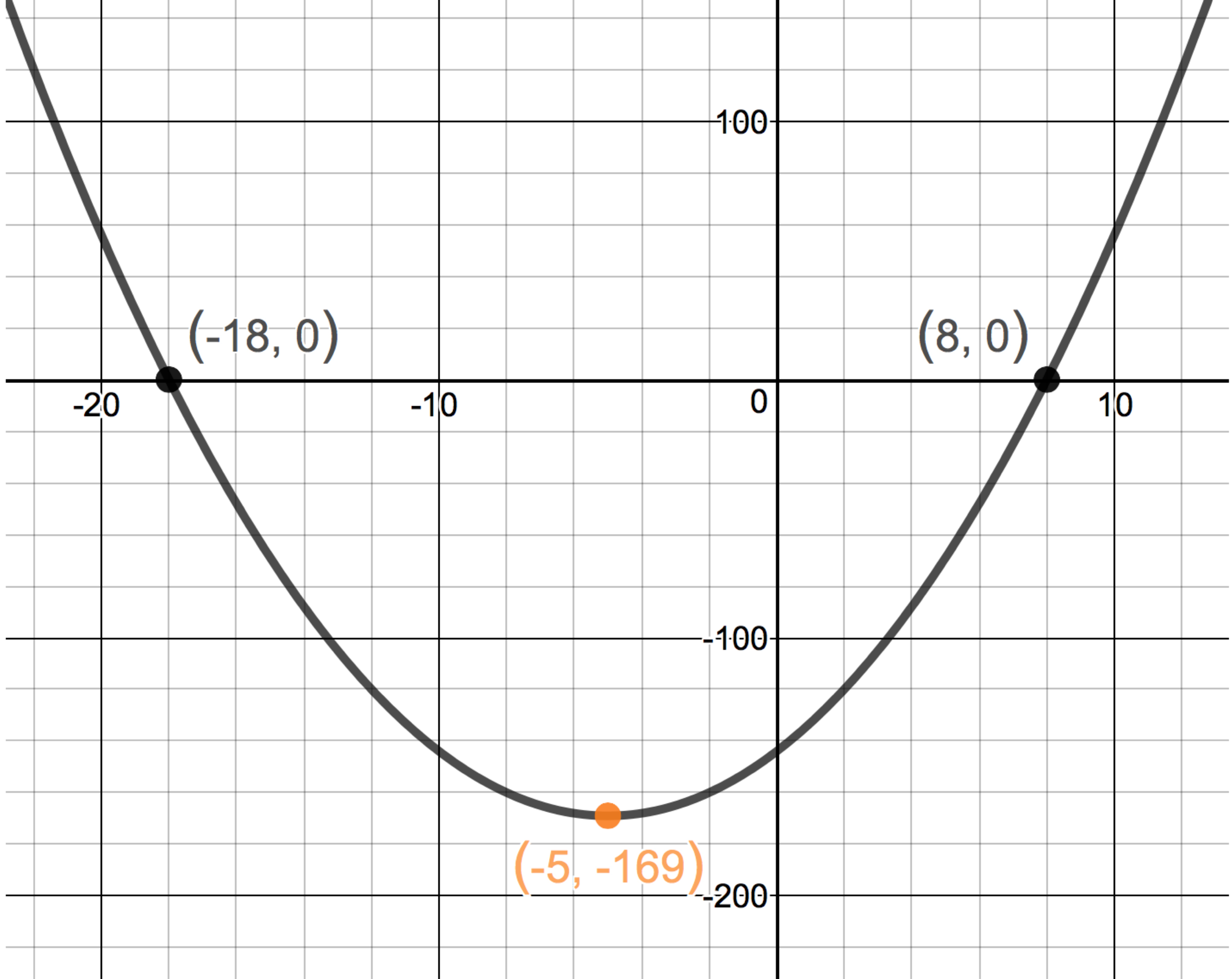
$$(x+5)^2 = 169$$

$$\sqrt{(x+5)^2} = \sqrt{169}$$

$$x + 5 = \pm 13$$

$$x + 5 = 13 \text{ or } x + 5 = -13$$

$$x = 8 \quad \text{or} \quad x = -18$$



(-18, 0)
(8, 0)
(-5, -169)
100
-100
-200
-20
-10
0
10

Questions for Students

- Can we use this procedure, easily, for all quadratic equations?
- Can you develop a general rule for completing the square in the following expressions: $x^2 + kx$ or $x^2 - kx$?
- How can we use this procedure to develop a formula for finding the solutions of any quadratic equation: $ax^2 + bx + c = 0$?

Do I have to draw those pictures?

- We can also look at our problem a different way.
- We want to rewrite the function so that

  $y = x^2 + 10x - 144$, is expressed with no linear term. That is, it looks like $(x - h)^2 + k$ for some numbers h and k.

  $x^2 + 10x - 144 = (x - h)^2 + k$

Mindful Manipulation

$x^2 + 10x - 144 = (x - h)^2 + k$

$x^2 + 10x - 144 = x^2 - 2hx + h^2 + k$

$-2h = 10$

$h = -5$

$h^2 + k = -144$ and

$(-5)^2 + k = -144$

$k = -169$

$x^2 + 10x - 144 = (x + 5)^2 - 169$

Graham, K., Cuoco, A., & Zmmermann, G. (2010). *Focus in High School Mathematics: Reasoning and Sense Making in Algebra*. Reston, VA: NCTM

Mindful Manipulation

You try: $y = x^2 - 20x + 104$

$$x^2 - 20x + 104 = (x - h)^2 + k$$
$$x^2 - 20x + 104 = x^2 - 2hx + h^2 + k$$

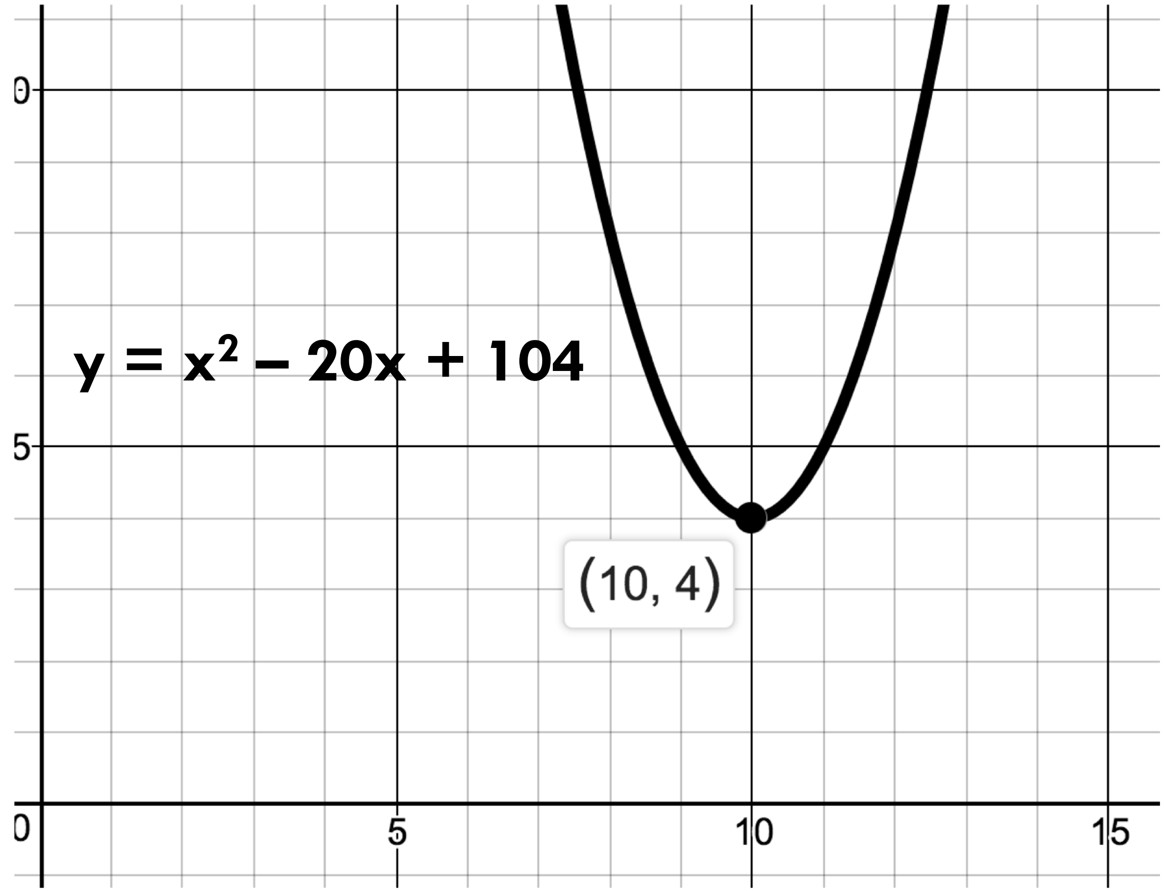
$$-2h = -20$$
$$h = 10$$

$$h^2 + k = 104 \text{ and}$$
$$(10)^2 + k = 104$$
$$k = 4$$

$$x^2 - 20x - 104 = (x - 10)^2 + 4$$

The connections

Connect back to the graph to check your solutions and solidify the concept of the roots with the quadratic solutions.



Work Sample

$$x^2+7x=32$$

$$x^2+7x-32=(x-m)^2+p$$

$$x^2+7x-32=x^2-2mx+m^2+p$$

So $7x=-2mx$

$$m=-7/2$$

Then $m^2+p=\left(-\frac{7}{2}\right)^2+p=-32$

$$p=\frac{-49}{4}+-32$$

$$=\frac{-49}{4}+\frac{-128}{4}$$

$$=\frac{-177}{4}$$

So $x^2+7x-32=(x-7/2)^2-\frac{177}{4}$

Then when $=0$

$$(x-7/2)^2=+177/4$$

$$x=\frac{7}{2}\pm\frac{\sqrt{177}}{2}$$

DONE

The coefficient of x^2 is not one.

$$y = 2x^2 + 5x + 6$$

$$2x^2 + 5x + 6 = 2(x - h)^2 + k$$

$$2x^2 + 5x + 6 = 2x^2 - 4xh + 2h^2 + k$$

$$5x = -4xh$$

$$h = \frac{-5}{4}$$

$$6 = 2h^2 + k$$

$$6 = \frac{25}{16} + k$$

$$k = \frac{71}{16}$$

Generalizing

$$y = ax^2 + bx + 6$$

$$ax^2 + bx + 6 = a(x - h)^2 + k$$

$$ax^2 + bx + 6 = ax^2 - 2axh + 2h^2 + k$$

$$bx = -2axh$$

$$h = \frac{b}{-2a} = -\frac{b}{2a}$$

What was that?

What did you notice?

How can that help us to create an algorithm?

How can our Geometry/area models help us create an algorithm?

An Algorithm

What algorithms would you want your students to create? How does this algorithm relate to conceptual understanding?

What about the quadratic formula?

Find vertex, axis of symmetry + zeros

$$y = ax^2 + bx + c$$

$$y = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$y = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right)$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$$V\left(\frac{-b}{2a},\ c - \frac{b^2}{4a}\right)$$

axis of symmetry: $x = \frac{-b}{2a}$

$$a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c = \frac{b^2 - 4ac}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Zeros: $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Conic Sections

Graph $x^2 + 4y^2 + 4x – 24y + 36 = 0$

$x^2 + 4x + 4(y^2 – 6y) + = 0$

$x^2 + 4x + 4 – 4 + 4(y^2 – 6y + 9) – 36 + 36 = 0$

$(x + 2)^2 + 4(y – 3)^2 – 4 = 0$

$(x + 2)^2 + 4(y – 3)^2 = 4$

Ellipse

Center at (-2, 3) Minor axis is vertical and length 1

Major axis is horizontal and length 4

Procedural Fluency

How will you develop procedural fluency?

Begin with tasks that *develop conceptual understanding by building on students' informal knowledge*

Move to tasks that support students to *develop informal strategies to solve problems*, where students use their own invented strategies and "short-cuts" and engage with a variety of strategies created by their peers

Refine informal strategies to develop fluency with standard methods and procedures (algorithms or formulas). Engage students in considering how to make invented strategies or short cuts more efficient, such as *explicitly comparing and contrasting different methods to solve the same problem*

Taking Action for High School Mathematics, NCTM, 2017, Boston, Dillon, Smith, Miller, And Briars, 2016

Thank You!

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