A Meaningful Approach to the Quadratic Formula

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- **Ken was tutoring his granddaughter in high school algebra.**
- **Her textbook had a section on the vertex of a parabola that connected it to the axis of symmetry.**
- **She didn't see any meaning in the formula, as stated**
- $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ **is much more complex than** $x = \frac{-b}{2a}$
- **We could not find the formula stated in a different form in any textbook.**

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

know that squares thing we did the other day? Can we solve quadratic equations that way or do we have to use the formula?

Students learn this and also learn to apply it to solving quadratic equations, but …

You can solve them that way, but why do you ask?

Does it have any *meaning* for them?

Do they know where it comes from?

Last year when we had to use the formula, I never understood it, but doing it this way makes sense to me.

It’s worth deriving the quadratic formula, if for no other reason than doing so prevents it from just falling out of the sky one day.

The method of completing the square can be taught in a progression until a coefficient of $a \neq 1$, or fractions in the solution, won’t bother them. Then they’ll be ready to derive of the formula.

$$x^2 + 6x - 7 = 0$$
$$x^2 + 4x - 7 = 0$$
$$2x^2 + 4x - 5 = 0$$
$$2x^2 + 3x - 7 = 0$$
$$ax^2 + bx + c = 0$$

So, is there any meaning in the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}?$$

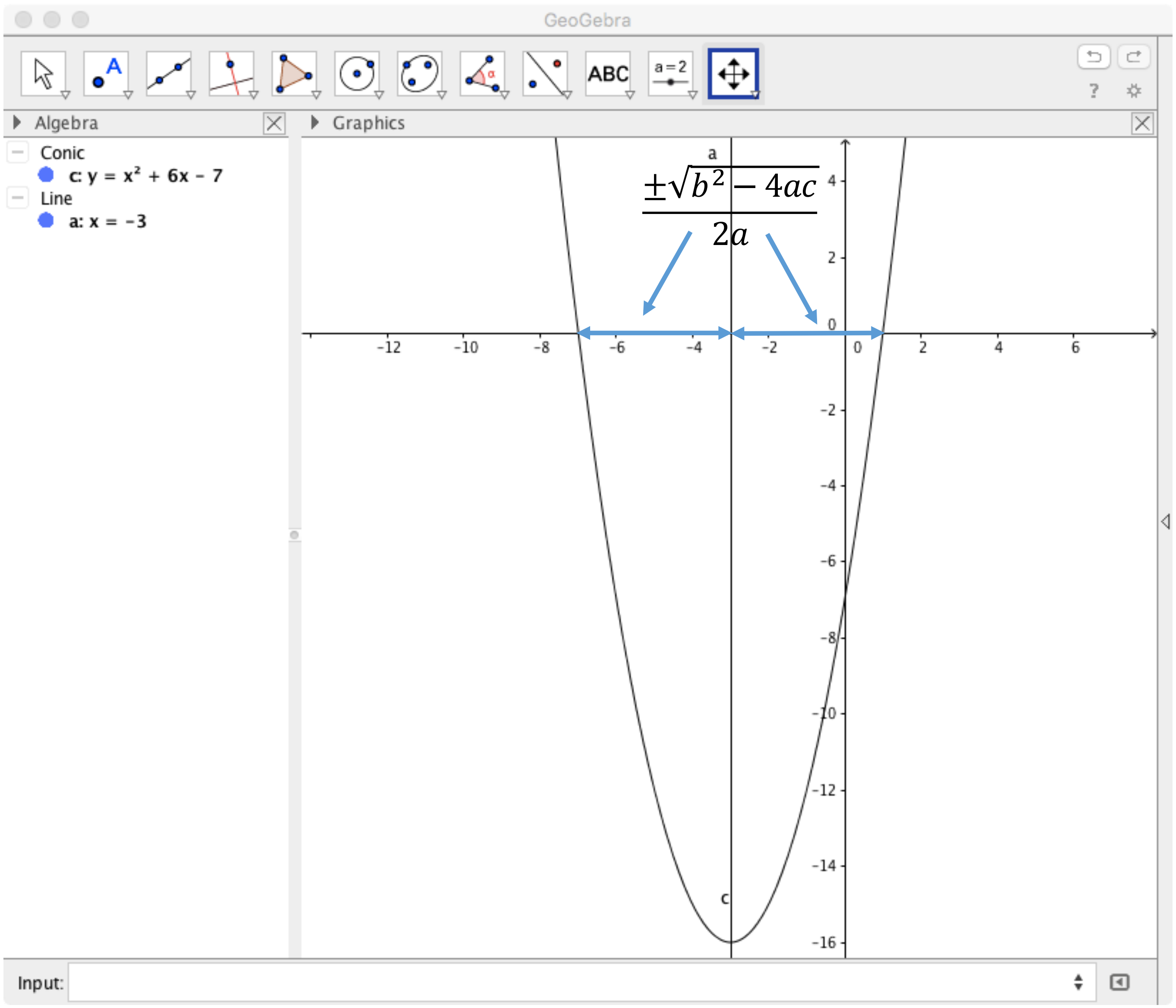
What if we separate the expression into two separate terms?

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

How might this format help with meaning?

What does the first term mean in terms of the graph of $y = ax^2 + bx + c$?

What does the second term mean?

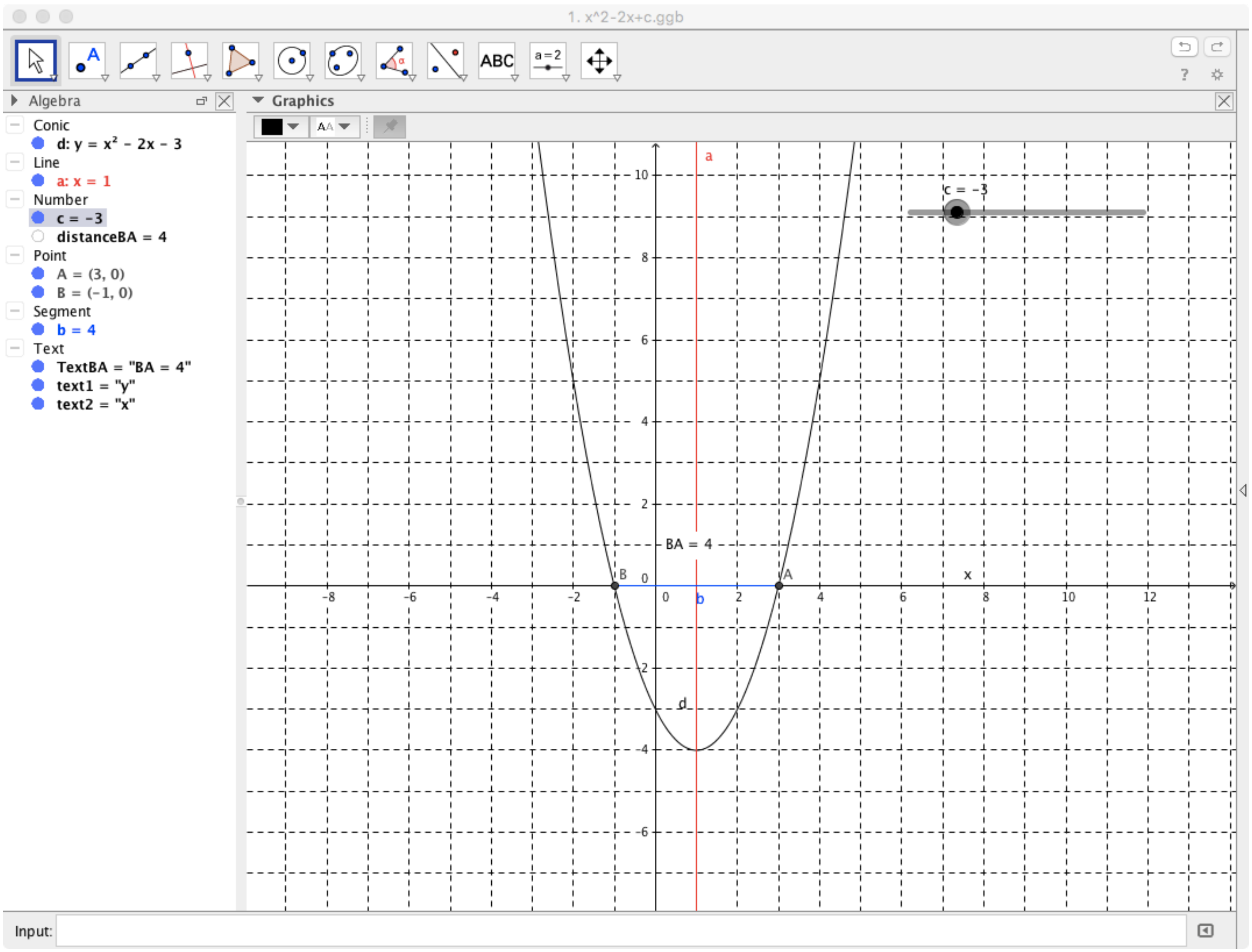


GeoGebra
Algebra
Graphics
Conic
c: y = x² + 6x − 7
Line
a: x = −3
$\frac{\pm\sqrt{b^2 - 4ac}}{2a}$
a
c
Input:

GeoGebra Activity #1

In this first GeoGebra activity, we want students to:

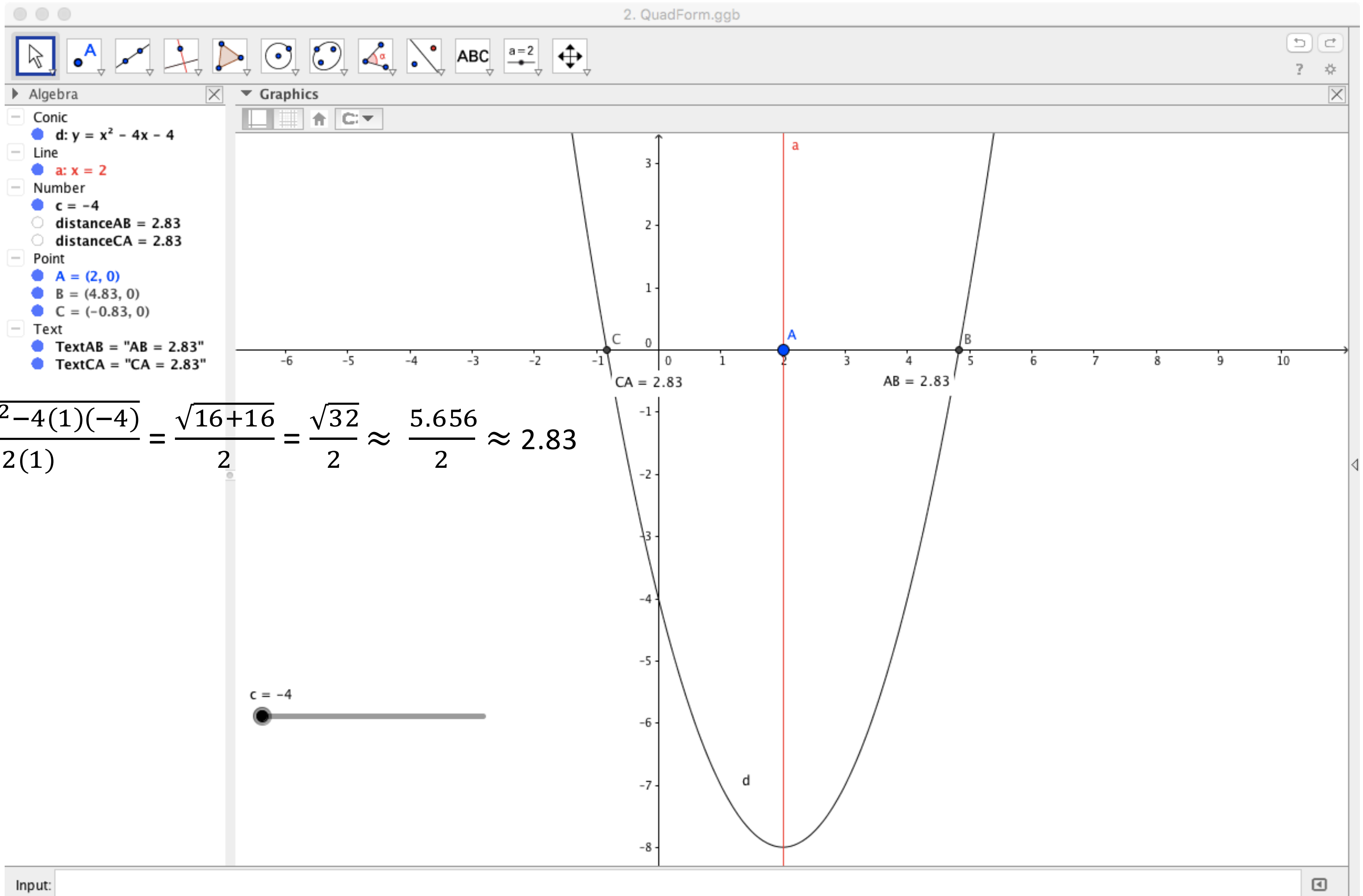
- **observe the effect on the graph of a quadratic function if *c* changes while *a* and *b* remain constant.**
- **observe that the graph of the parabola moves up or down along the axis of symmetry as *c* changes while *a* and *b* remain constant.**
- **observe that changing *c* while *a* and *b* remain constant affects the distance between the *x*-intercepts.**
- **verify that the *x*-intercepts are the roots of the equation $ax^2 + bx + c = 0$**


1. x^2-2x+c.ggb
Algebra
Graphics
Conic
d: $y = x^2 - 2x - 3$
Line
a: x = 1
Number
c = −3
distanceBA = 4
Point
A = (3, 0)
B = (−1, 0)
Segment
b = 4
Text
TextBA = "BA = 4"
text1 = "y"
text2 = "x"
c = −3
BA = 4
Input:

GeoGebra Activity #2

In the second GeoGebra activity, we want students to:

- **observe that as the parabola moves up or down along the axis of symmetry, the roots remain equidistant from the axis of symmetry.**
- **verify that the equal distances that ggb reports for CA and AB are given by $\frac{\pm\sqrt{b^2-4ac}}{2a}$.**

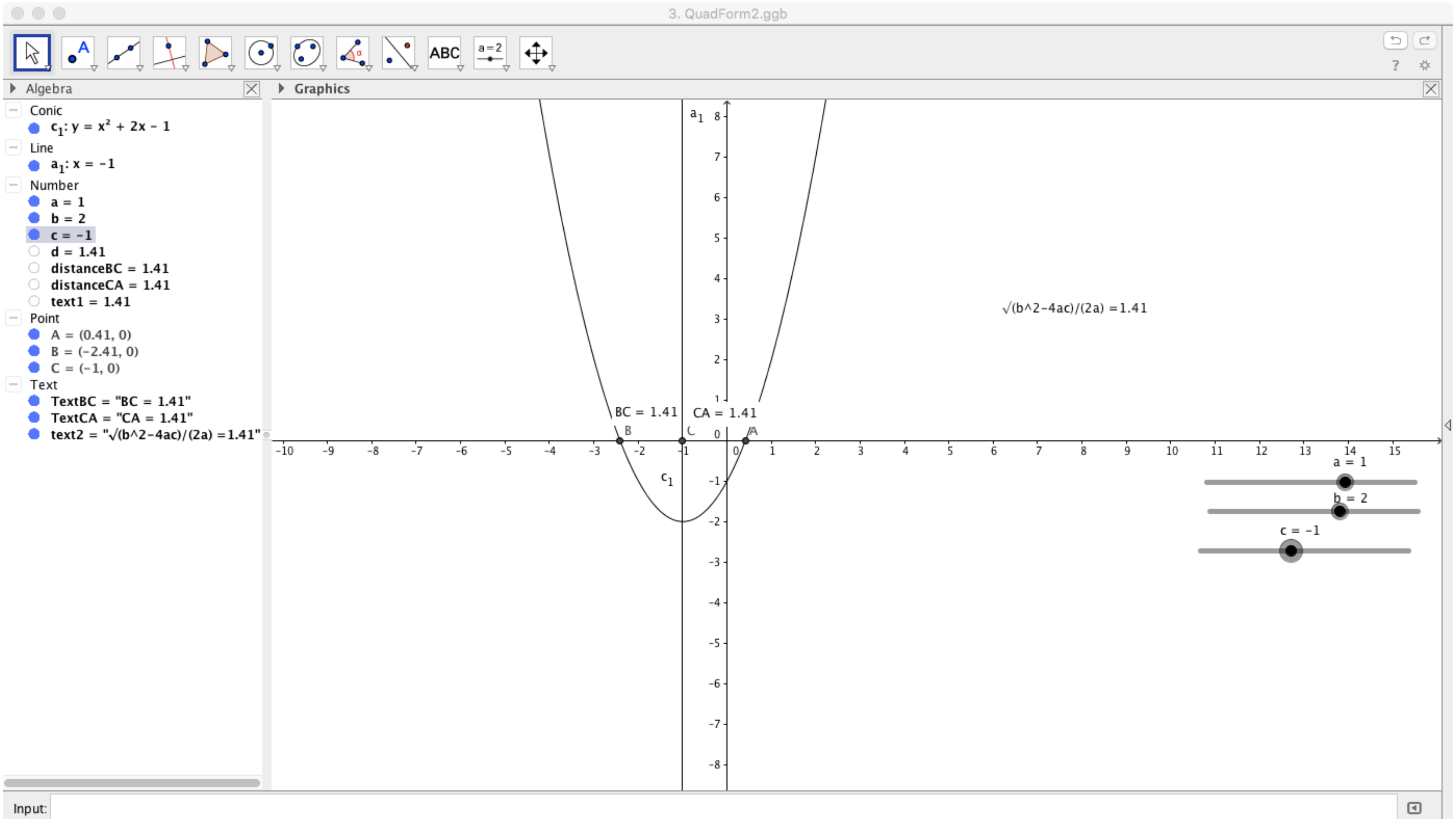


$$\frac{\sqrt{(-4)^2-4(1)(-4)}}{2(1)} = \frac{\sqrt{16+16}}{2} = \frac{\sqrt{32}}{2} \approx \frac{5.656}{2} \approx 2.83$$

GeoGebra Activity #3

In our final GeoGebra activity for today, we want students to:

- **observe what happens as *a*, *b*, and *c* are changed, one at a time.**
- **Observe the relationship between $\frac{\pm\sqrt{b^2-4ac}}{2a}$ and the distances BC and CA.**
- **reflect on what happens when *a* = 0.**
- **reflect on why ggb sometimes returns "$\frac{\pm\sqrt{b^2-4ac}}{2a}$ =?".**
- **connect a graph that has no *x*-intercepts with complex roots.**


3. QuadForm2.ggb
Algebra
Conic
c1: y = x² + 2x − 1
Line
a1: x = −1
Number
a = 1
b = 2
c = −1
d = 1.41
distanceBC = 1.41
distanceCA = 1.41
text1 = 1.41
Point
A = (0.41, 0)
B = (−2.41, 0)
C = (−1, 0)
Text
TextBC = "BC = 1.41"
TextCA = "CA = 1.41"
text2 = "√(b^2−4ac)/(2a) =1.41"
Graphics
√(b^2−4ac)/(2a) =1.41
BC = 1.41
CA = 1.41
a = 1
b = 2
c = −1
Input:

So, we think that one small change in the way the formula is presented can make a big difference in the way students understand it.

All well and good if the equation has real roots, but what if it does not? Suppose we have $x^2 - 2x + 5 = 0$

What is the graphical consequence?

$y = x^2 - 2x + 5$ does not intersect the x-axis.

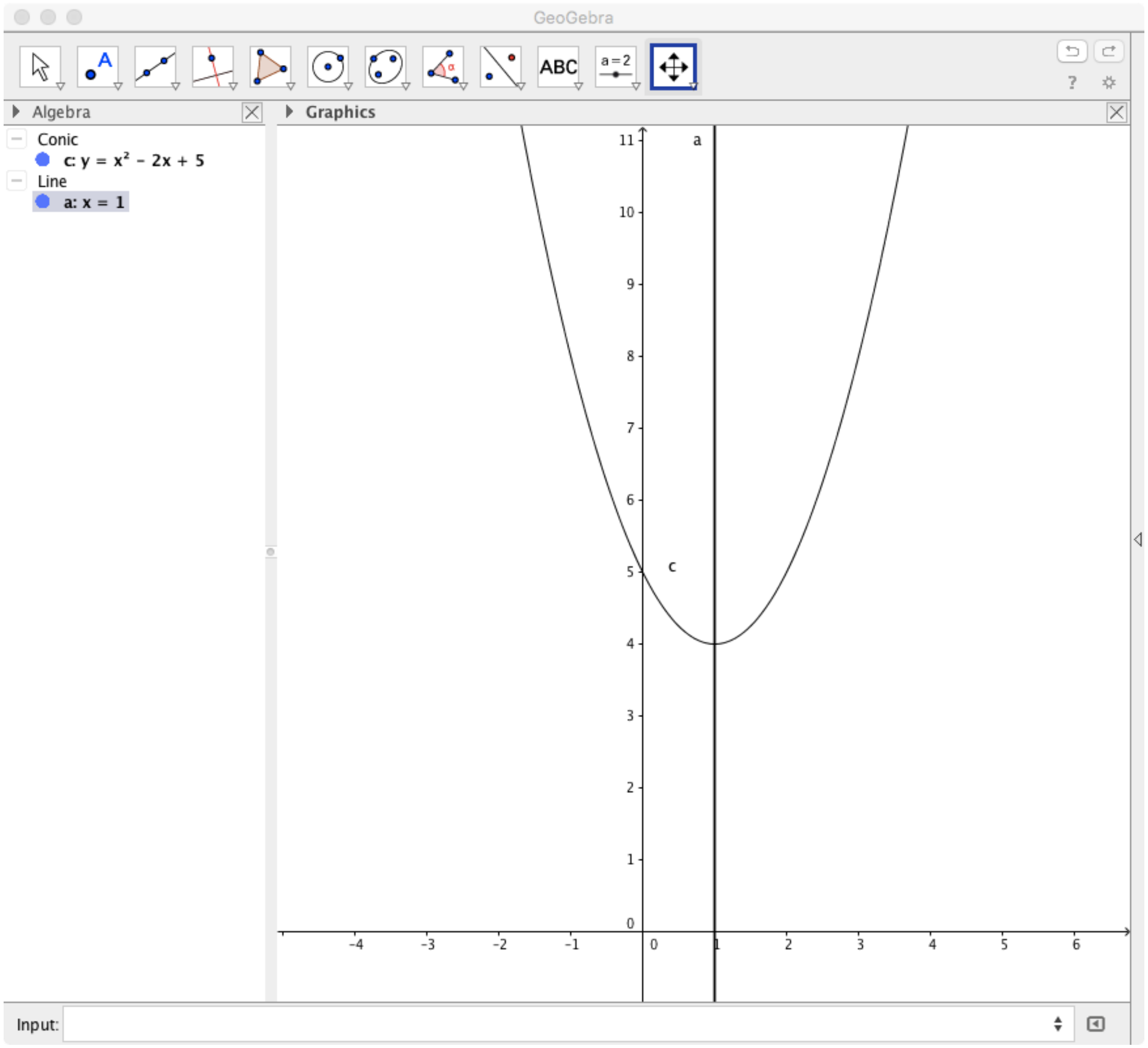
What is the algebraic consequence?

$$x = \frac{-(-2)}{2(1)} \pm \frac{\sqrt{(-2)^2 - 4 \cdot 1 \cdot 5}}{2(1)} = \frac{2}{2} \pm \frac{\sqrt{-16}}{2} = 1 \pm 2i$$

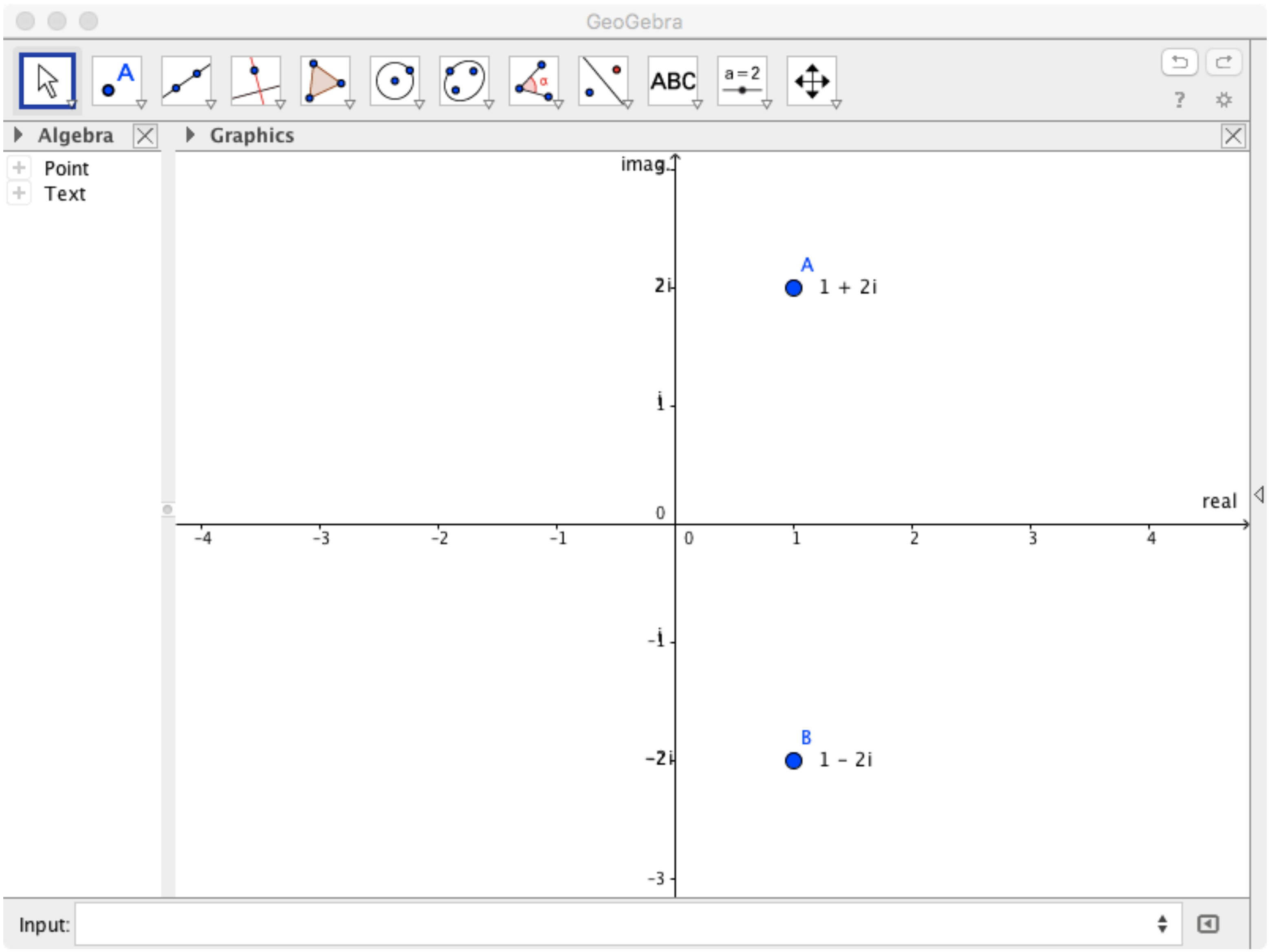
Can we graph those imaginary roots?

Yes, in the complex plane.

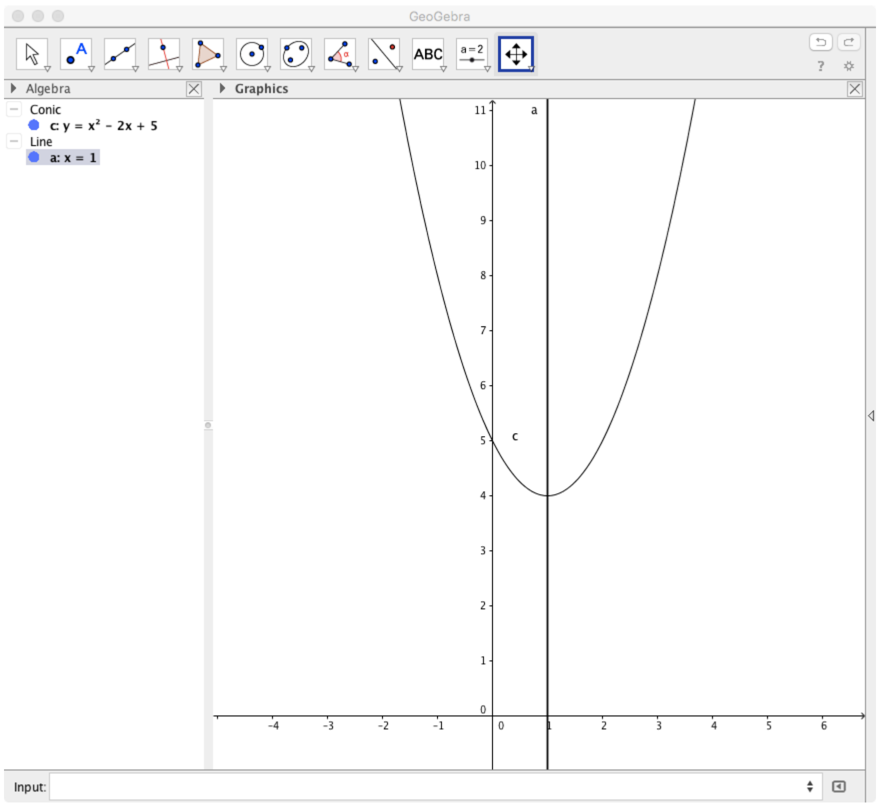
How do we know that without the graph?

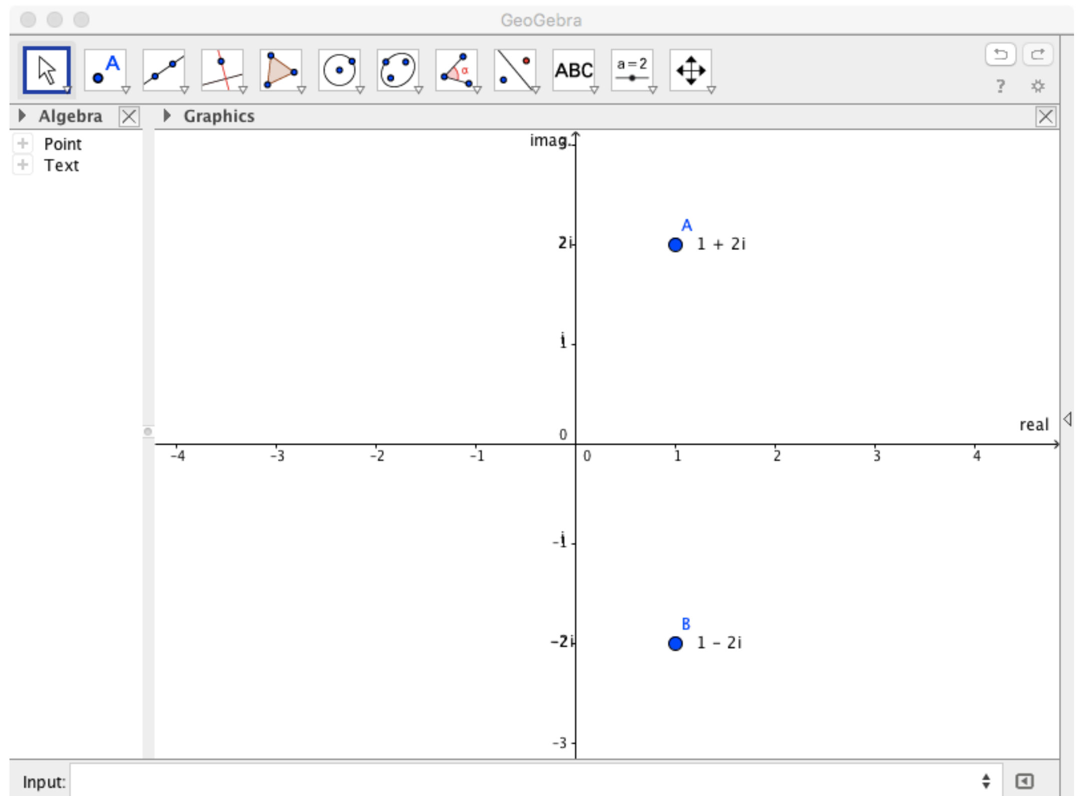


GeoGebra
Algebra
Conic
c: y = x² − 2x + 5
Line
a: x = 1
Graphics
a
c
Input:

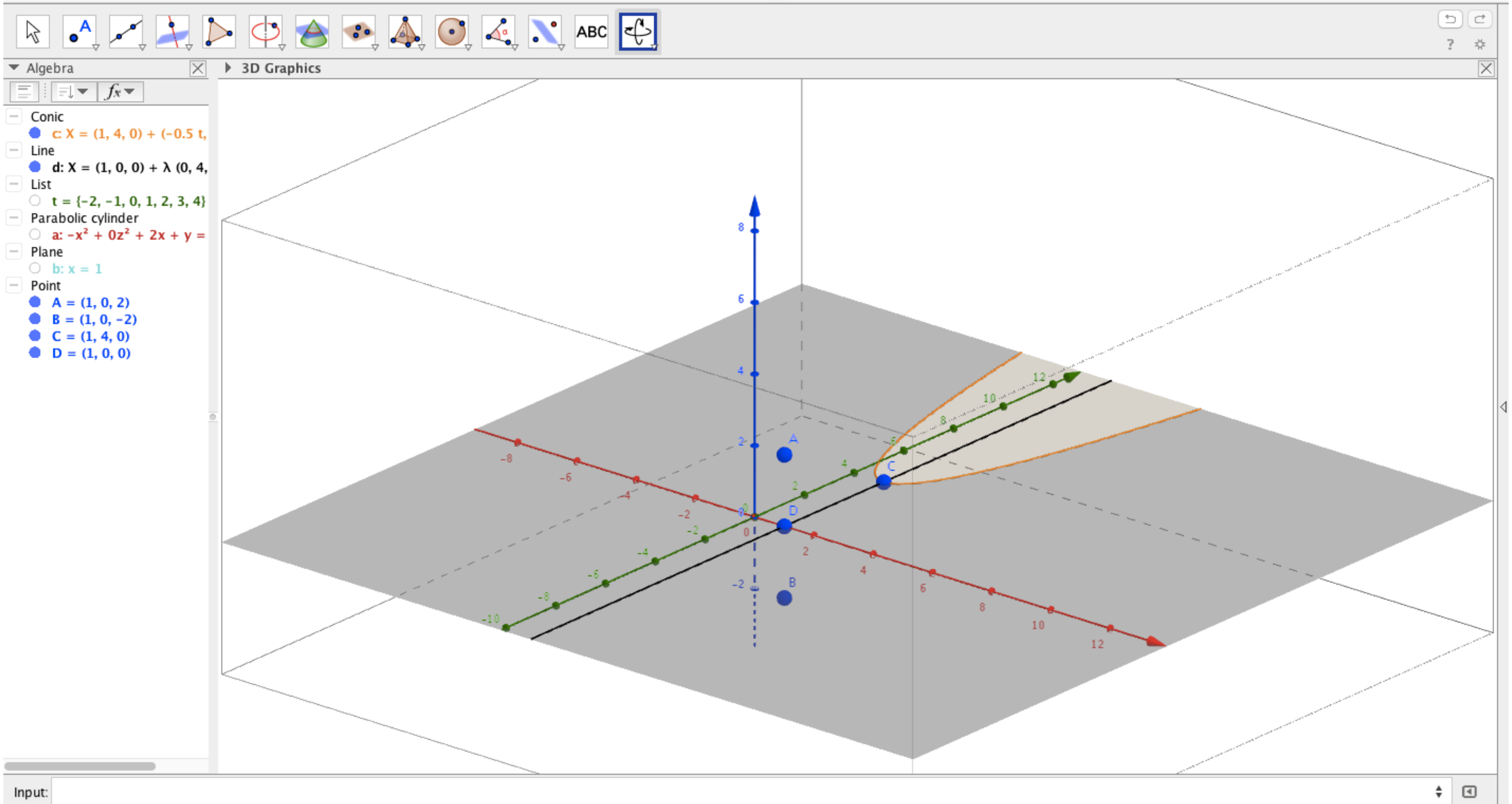

GeoGebra
Algebra
Graphics
Point
Text
imag.
real
A
1 + 2i
B
1 − 2i
2i
i
-i
-2i
-3
0
-4
-3
-2
-1
0
1
2
3
4
ABC
a = 2
Input:

Now, is there any relationship between these two graphs?



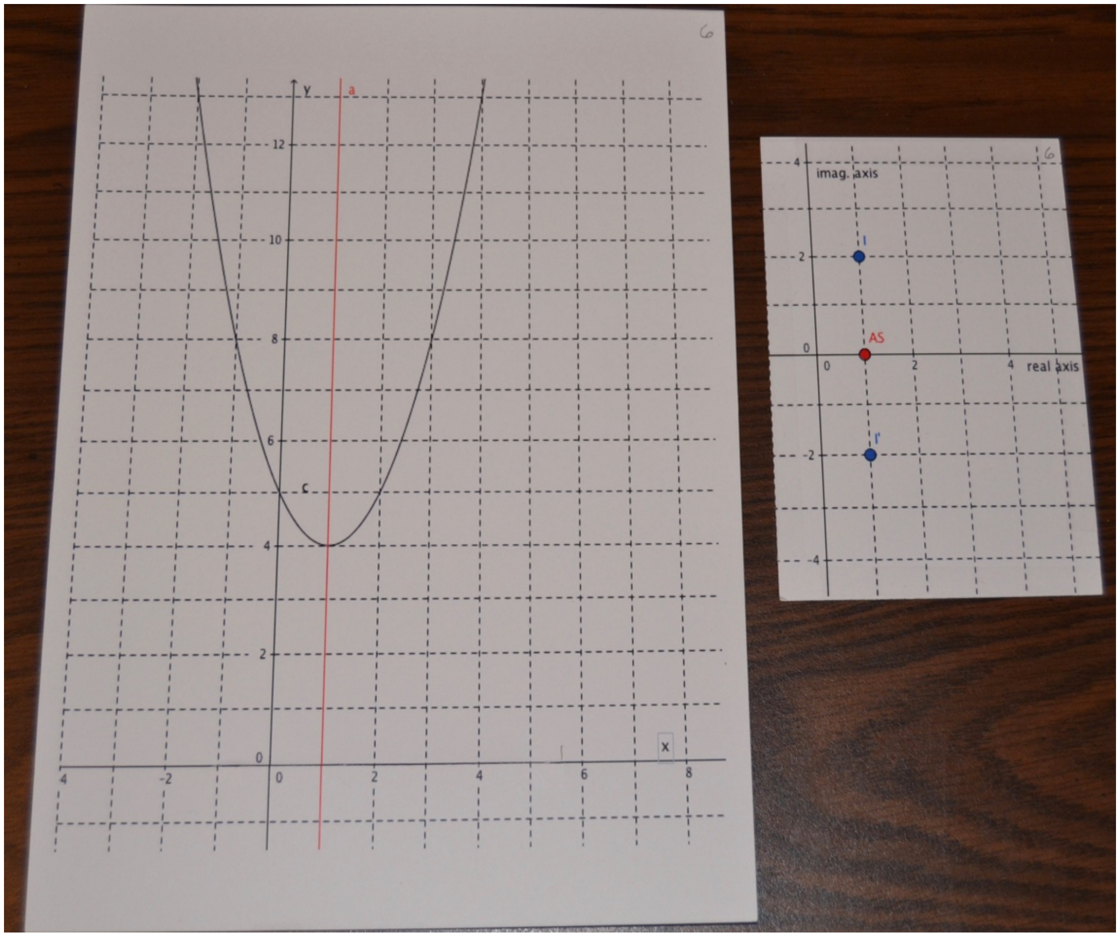


Do these two graphs have anything in common?

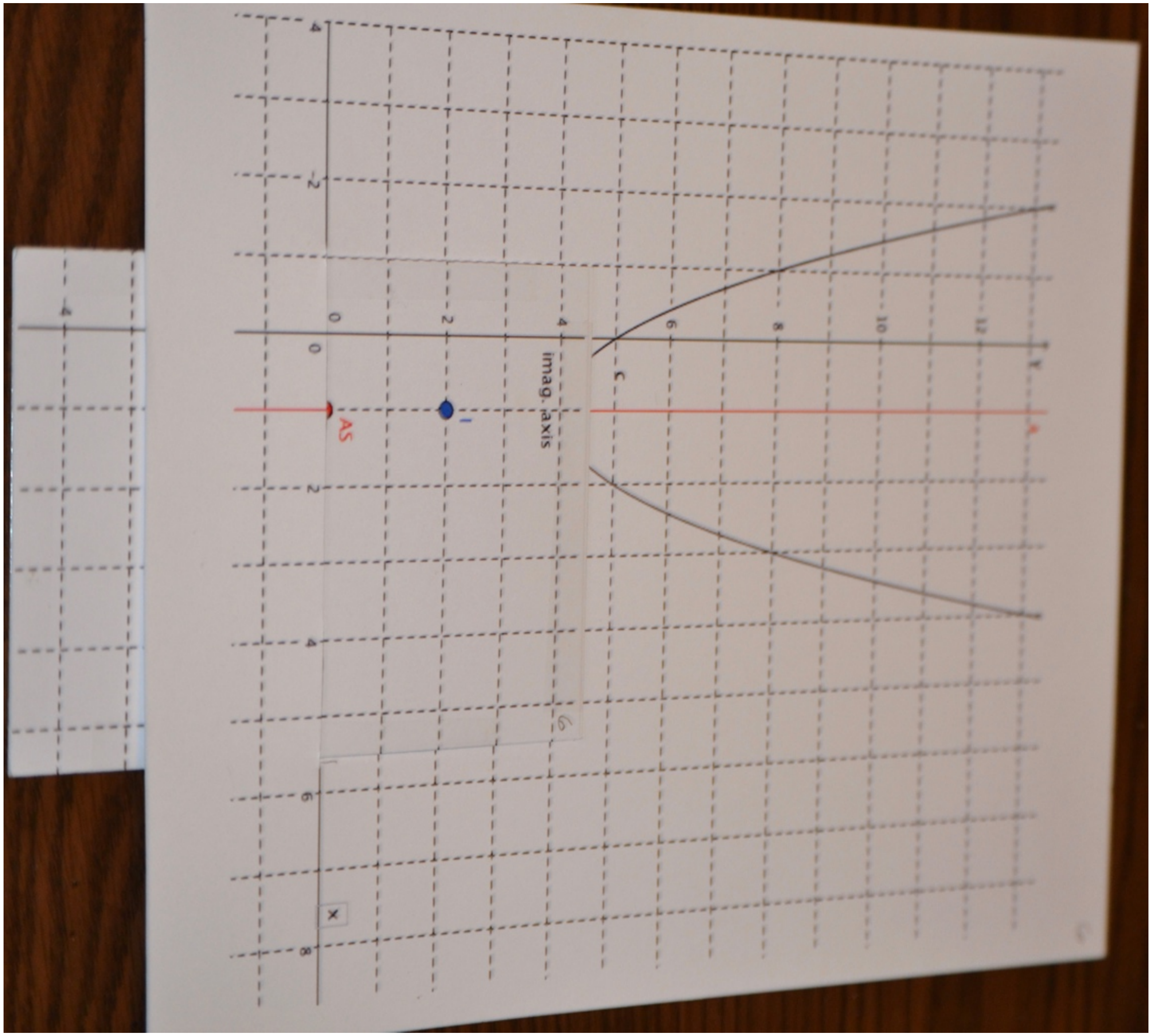


The trouble with this graph is that many students have trouble visualizing a two dimensional representation of three dimensions.

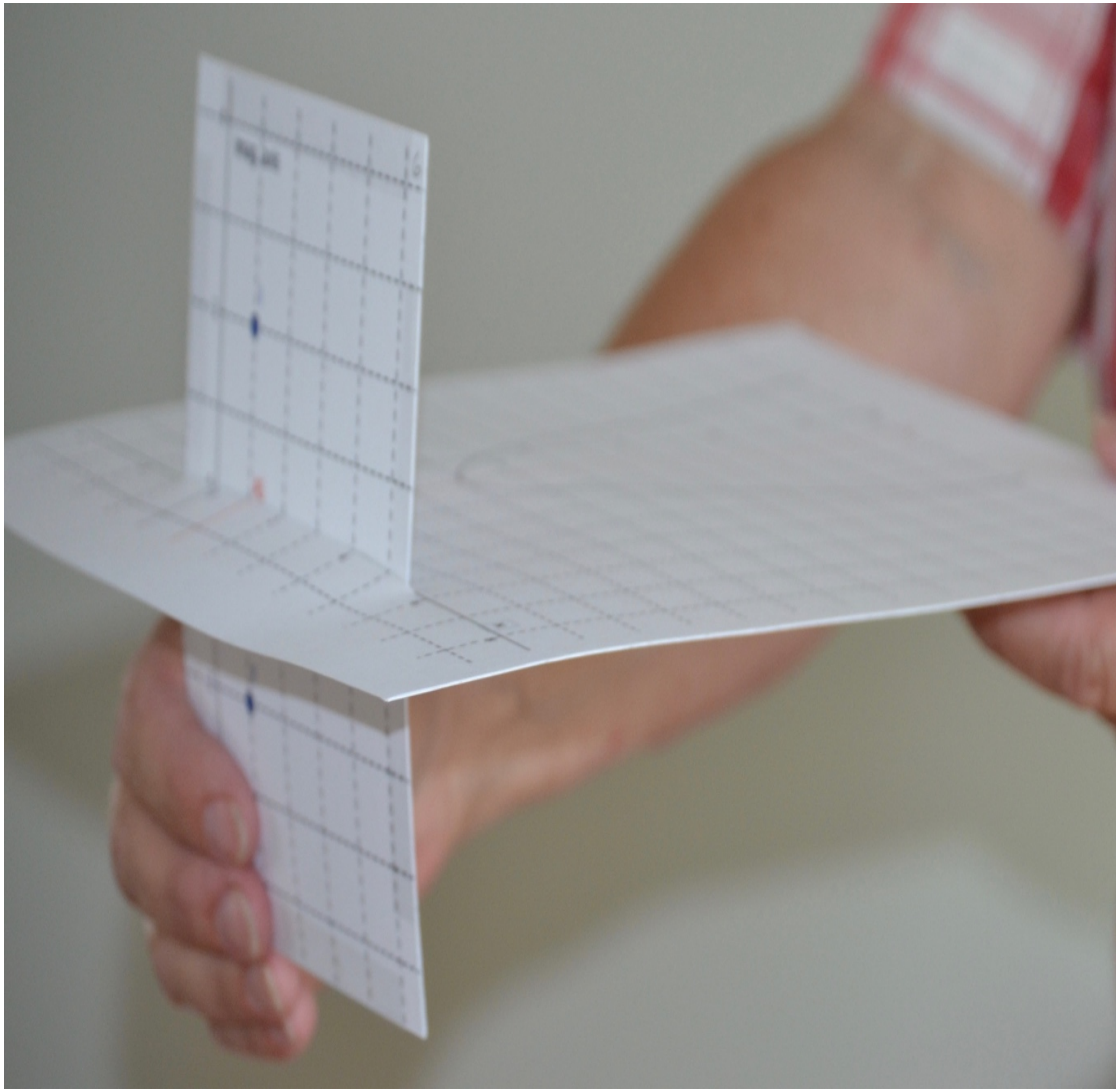
- I had an idea of how to help students to better visualize that graph.



- **Next, I slit the *x*-axis of the graph of the parabola just long enough to allow siding the complex graph in until the real axis of the complex graph coincided with the *x*-axis of the Cartesian graph of the parabola.**


imag. axis
AS
y
x

- **Finally, when the *x*-axis and the real axis coincide, the complex graph is rotated until perpendicular to the Cartesian graph.**
- **This shows the complex roots above and below the plane of the parabola.**



When I have used this activity in H.S. classes, many students say that without the model (which they saw first), the two dimensional representation of the three dimensional graph would not have made sense to them.

- **More details about graphing complex roots of a quadratic equation can be found at:**

Melliger, Carmen, "How to Graphically Interpret the Complex Roots of a Quadratic Equation." ***University of Nebraska-Lincoln MAT Exam Expository Papers,(35).***

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