Creating Fractals with Complex-Valued Functions

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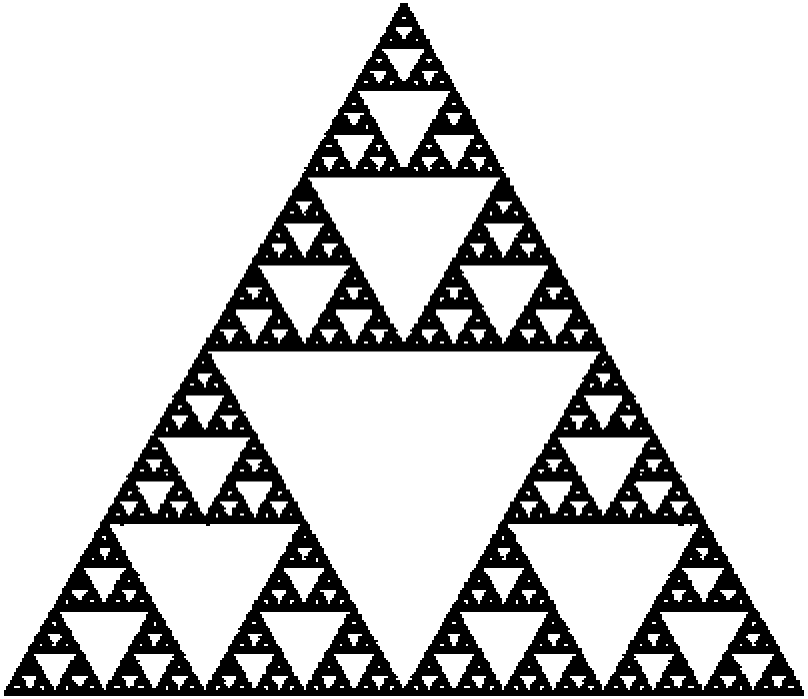
Johns Hopkins Center for Talented Youth

When we think of fractals…



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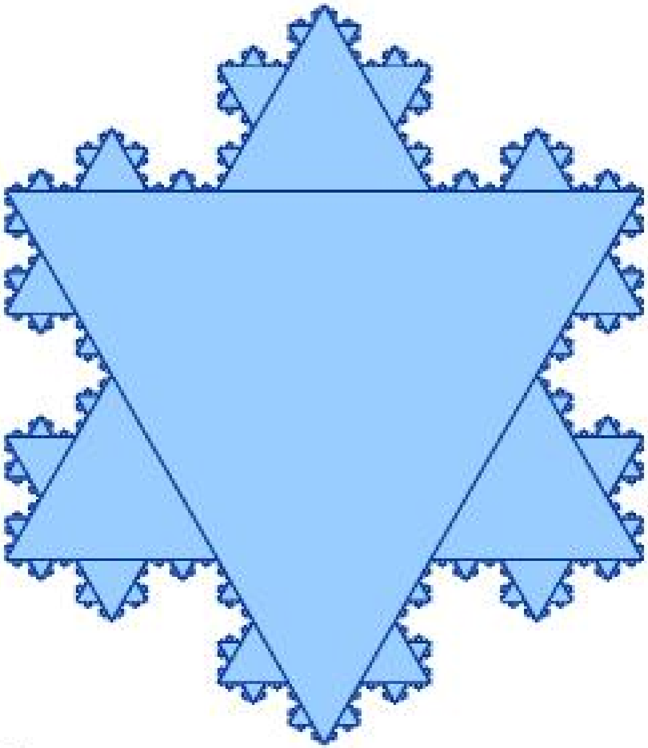

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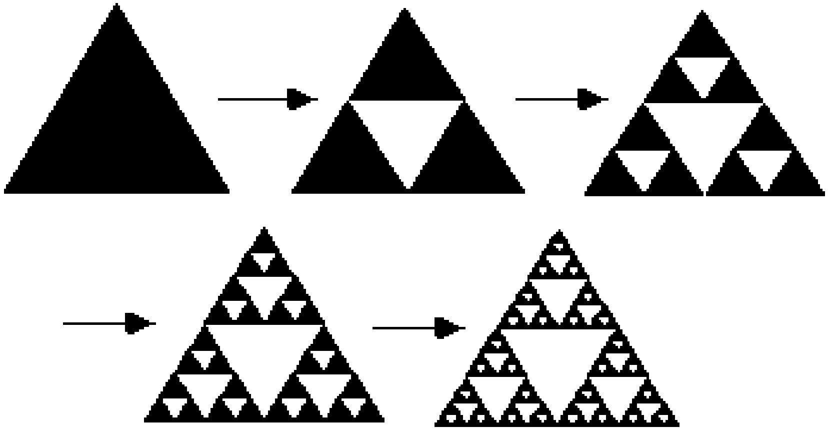
When we think of fractals…



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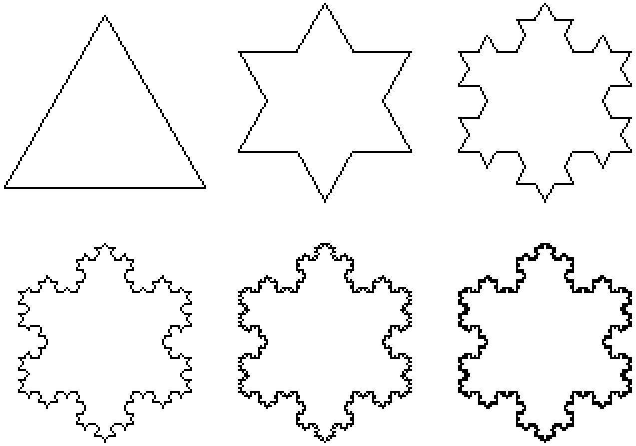

From http://www.oxfordmathcenter.com/drupal7/node/417

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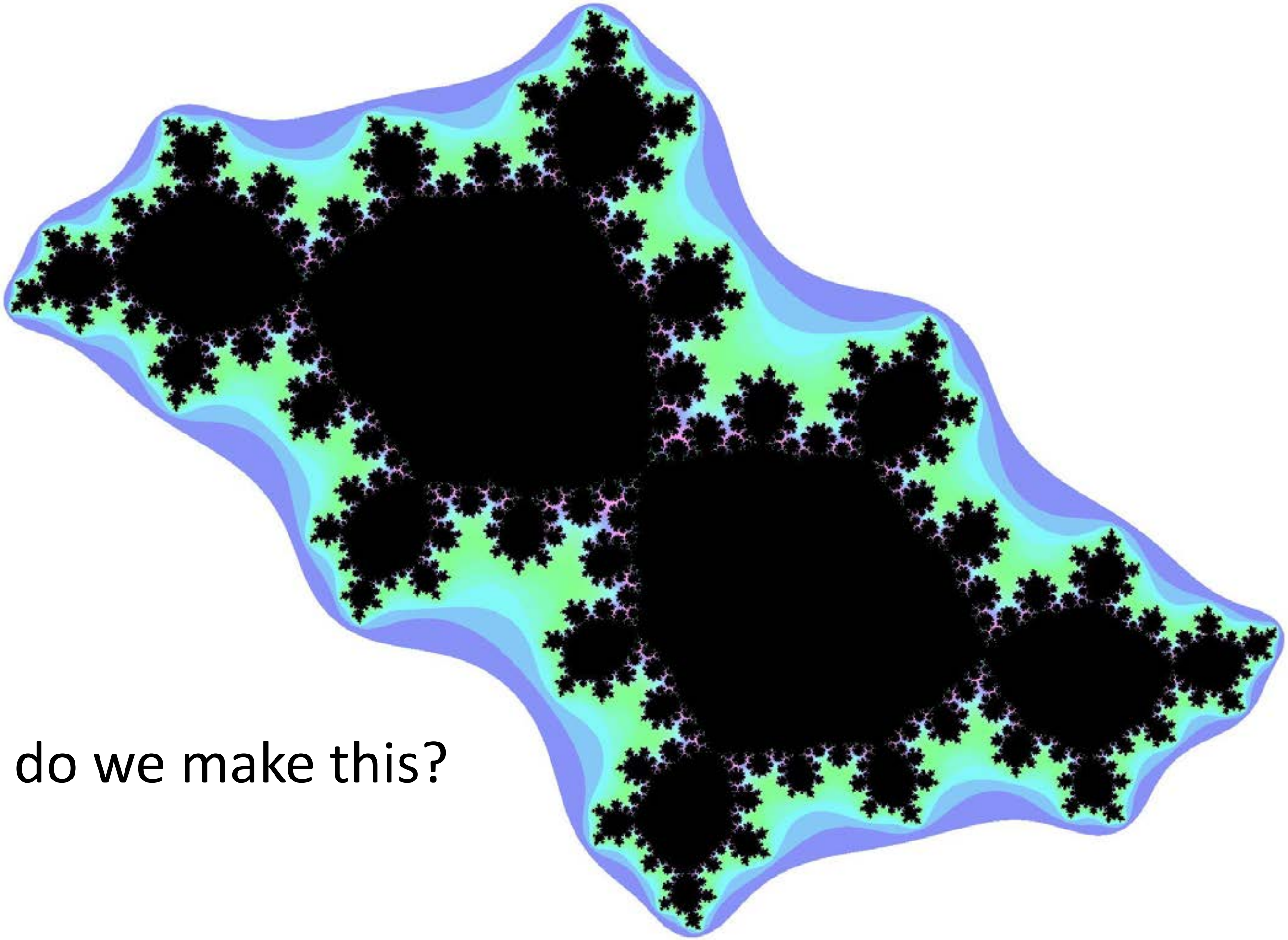
When we think of fractals…



How do we make this?

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Practice for Students

Students will:

- Multiply complex numbers and complex polynomials
- Compose functions
- Plot points on the complex plane
- Compute the modulus of a complex number, and understand the relation of the modulus to distance

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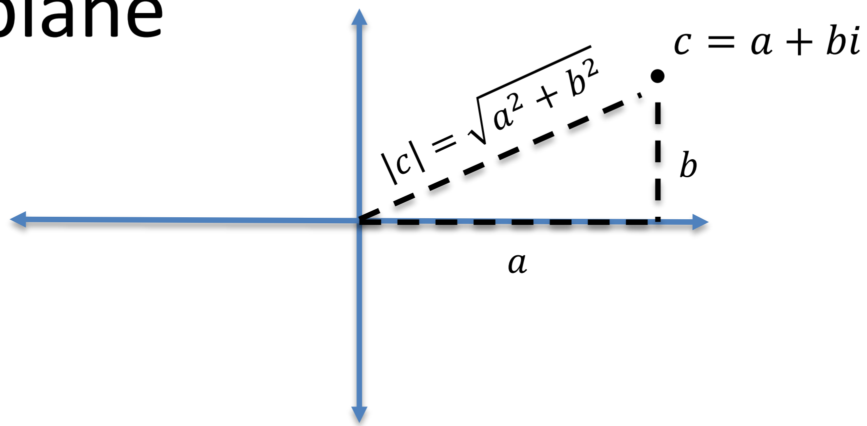
Definitions

Complex modulus $|c|$:

- If $c = a + bi$

$$|c| = \sqrt{a^2 + b^2}$$

- Is the distance between c and 0 on the complex plane



Definitions

Complex function:

- Domain and range are subsets of complex numbers
- Coefficients can be complex numbers
- $f(z)$ where z denotes a complex variable

Definitions

Iteration of a function:

- Repeated composition of a function with itself

$$f(z),$$
$$f\big(f(z)\big),$$
$$f\Big(f\big(f(z)\big)\Big),$$
$$\vdots$$

Notation:

$$f^n(z) = \underbrace{f\Big(f\big(f\cdots\big(f(z)\big)\dots\big)\Big)}_{n\text{ times}}$$

Student Task: Iteration

- Expand the first 5 iterates of each function:
$$f(z) = 2z^2 + z + i$$
$$g(z) = iz + 1.$$
- Expand the first 5 iterates of $h(z) = iz^2$. Can you find an explicit formula for $h^n(z)$?
- Find a complex function f such that
$$f^4(z) = z$$
but $f(z), f^2(z), f^3(z) \neq z$.

Definitions

Orbits:

If c is a complex number and f a complex function, the sequence of numbers

$$c, f(c), f^2(c), \dots, f^n(c), \dots$$

is called the orbit of c

Example: $f(z) = z^2; c = 2$
Orbit of 2:

$$2, 4, 16, 256, \dots$$

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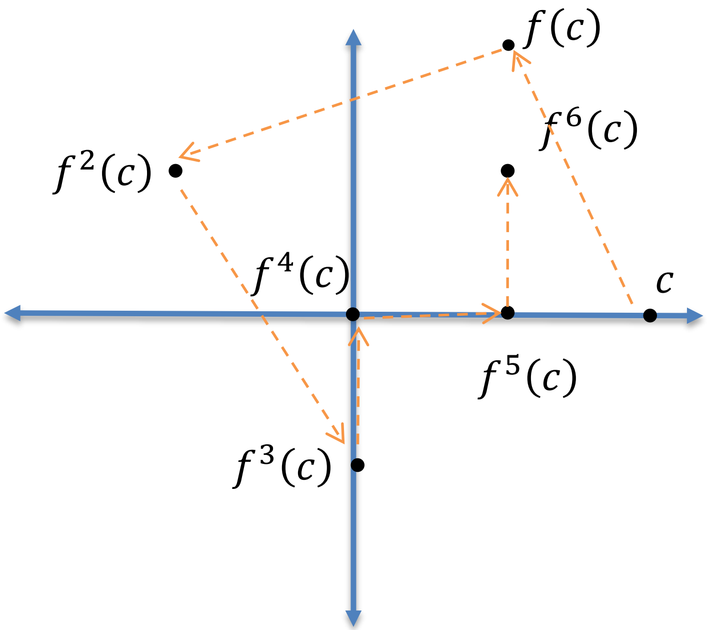
Student Tasks: Orbits

- For $f(z) = iz + 1$, compute the first 7 numbers in orbit of $c = 2$.
- Plot these values on the complex plane, drawing arrows between the points to show the behavior of the point c under iteration of f.

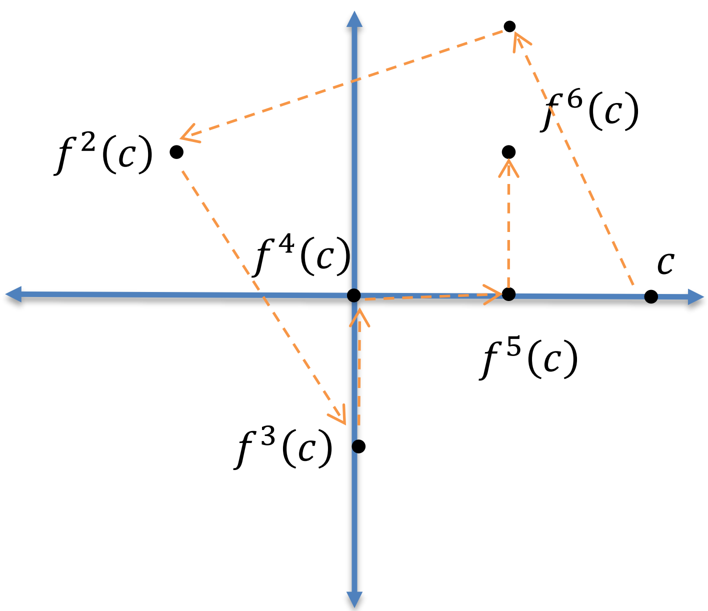
c	2
$f(c)$	$1 + 2i$
$f^2(c)$	$-1 + i$
$f^3(c)$	$-i$
$f^4(c)$	0
$f^5(c)$	1
$f^6(c)$	$1 + i$



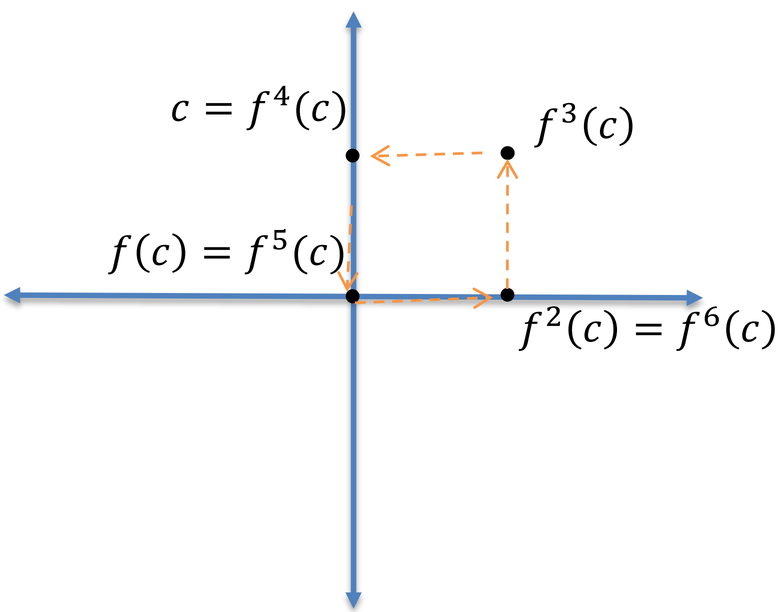
Student Tasks: Orbits

- For $f(z) = iz + 1$, compare the orbits of different points. How are they similar? How are they different?

$\boldsymbol{c = 2}$



$\boldsymbol{c = i}$



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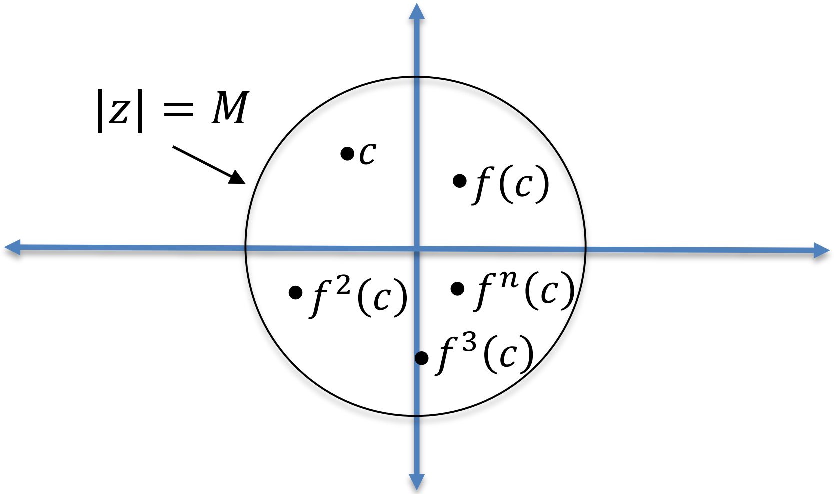
Questions about Orbits

For a complex function $f(z)$:

- Which points c in the complex plane have a bounded orbit $c, f(c), f^2(c), \ldots$?

c has a bounded orbit means we can find a number M so that

$$|f^n(c)| < M \text{ for all } n$$

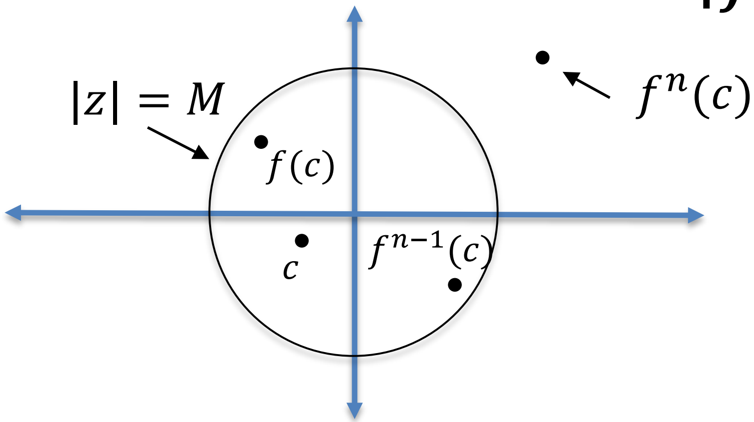


Questions about Orbits

For a complex function $f(z)$:

- Which points c have an unbounded orbit?
- An <u>unbounded orbit</u> means:

For any number M we pick, we can always find an n such that the modulus $|f^n(c)| > M$



Questions about Orbits

For a complex function $f(z)$:

- If the orbit of c is unbounded, what is the smallest n such that $|f^n(c)|$ is bigger than a given bound that we choose?

Example: $f(z) = 2z$;

Orbit of $c = 1$:

$$1, 2, 4, 8, 16, \dots, 2^k, \dots$$

What is the smallest n so that $|f^n(1)| > 2$?

Answer: $n = 2$

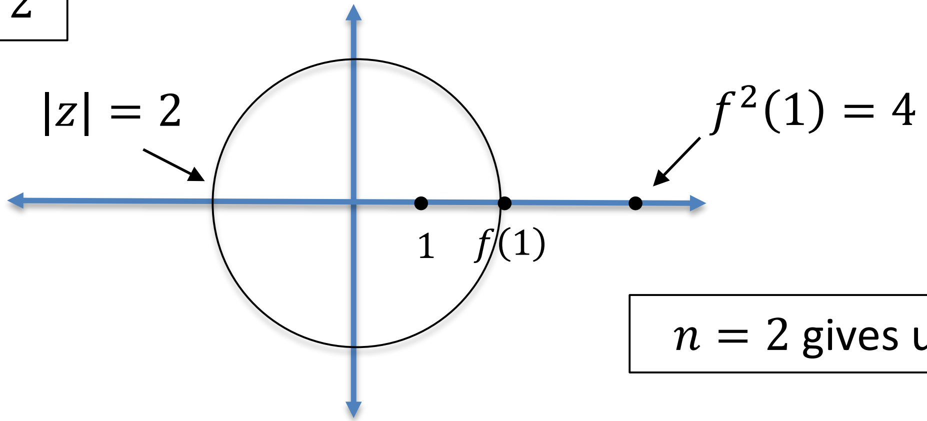
Example: Unbounded Orbit

$$f(z) = 2z$$

Orbit of $c = 1$:

$$1, 2, 4, 8, 16, \ldots, 2^k, \ldots$$

Pick bound $M = 2$



$n = 2$ gives us $|f^2(1)| > 2$

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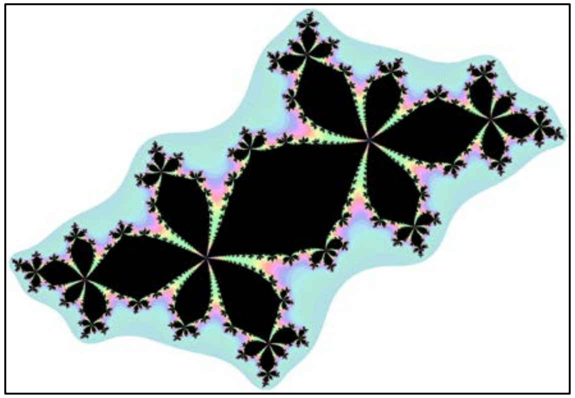
Questions about Orbits

These questions are used to draw fractals:



Points shaded black are the Filled Julia Set for $f(z)$: the set of all points in the complex plane with a bounded orbit

Colored points (including points colored white above): are all points in the complex plane with unbounded orbits.

Iterates: $f(z) = z^2$

$$f(z) = z^2$$
$$f^2(z) = f(z^2) = (z^2)^2 = z^4$$
$$f^3(z) = f(z^4) = (z^4)^2 = z^8$$
$$f^4(z) = f(z^8) = (z^8)^2 = z^{16}$$
$$\vdots$$
$$f^n(z) = z^{2^n}$$

Orbit of a point c:

$$c, c^2, c^4, c^8, c^{16}, \ldots, c^{2^n}, \ldots$$

Orbits: $f(z) = z^2$

We want to understand what the orbit of c $(c, c^2, c^4, c^8, c^{16}, \ldots, c^{2^n}, \ldots)$ behaves like for all c.

- Bounded?
- Unbounded?

To get a representative sample, pick a diverse set of points. Points with:

- small modulus
- large modulus
- only a real part
- only an imaginary part
- both a real and imaginary part.

Orbits: $f(z) = z^2$

Orbit: $c, c^2, c^4, c^8, c^{16}, \dots, c^{2^n}, \dots$

c	Orbit $c, f(c), f^2(c), \dots$	Bounded or Unbounded
0	$0, 0, 0, 0, 0, \dots$	Bounded
$\frac{1}{2}$	$\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \frac{1}{65536}, \dots, \frac{1}{4^n}, \dots$	Bounded
1	$1, 1, 1, 1, 1, \dots$	Bounded
i	$i, -1, 1, 1, 1, \dots$	Bounded
$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$	$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i, i, -1, 1, 1, \dots$	Bounded
$2i$	$2i, -4, 16, 256, \dots, 4^n, \dots$	Unbounded
10	$10, 100, 10000, 100000000, \dots, 100^n, \dots$	Unbounded

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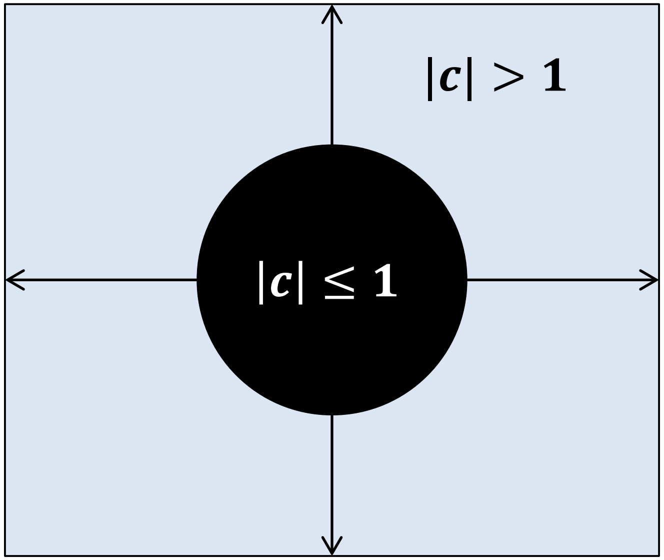
Filled Julia Set: $f(z) = z^2$

Plot c on the complex plane and color c according to whether its orbit is bounded or unbounded:



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Iterates: $f(z) = z^2 - 1$

$$f(z) = z^2 - 1$$

$$f^2(z) = (z^2 - 1)^2 - 1 = z^4 - 2z^2$$

$$f^3(z) = (z^4 - 2z^2)^2 - 1 = z^8 - 4z^6 + 4z^4 - 1$$

$$\begin{aligned} f^4(z) &= (z^8 - 4z^6 + 4z^2 - 1)^2 - 1 \\ &= z^{16} - 8z^{14} + 24z^{12} - 32z^{10} + 14z^8 + 8z^6 \end{aligned}$$

Orbits: $f(z) = z^2 - 1$

c	Orbit $c, f(c), f^2(c), \ldots$ (up to 3 decimal places)	Bounded or Unbounded
-2	$-2, 3, 8, 63, 3968, \ldots$	Unbounded
0	$0, -1, 0, -1, 0, \ldots$	Bounded
$\frac{1}{2}$	$.5, -.75, -.438, -.809, -.346, -.880, -.225, \ldots$	Bounded
1	$1, 0, -1, 0, -1, \ldots$	Bounded
$-\frac{1}{2}i$	$-.5i, .-1.25, .563, -.684, -.533, -.716, -.487, -.763, \ldots$	Bounded
i	$i, -2, 3, 8, 63, 3968, \ldots$	Unbounded
2	$2, 3, 8, 63, 3968, \ldots$	Unbounded
10	$10, 99, 9800, 96039999, \ldots$	Unbounded

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Orbits: $f(z) = z^2 - 1$

c	Orbit $c, f(c), f^2(c), \ldots$ (up to 3 decimal places)	Bounded or Unbounded
-2	$-2, 3, 8, 63, 3968, \ldots$	Unbounded
0	$0, -1, 0, -1, 0, \ldots$	Bounded
$\frac{1}{2}$	$.5, -.75, -.438, -.809, -.346, -.880, -.225, \ldots$	Bounded
1	$1, 0, -1, 0, -1, \ldots$	Bounded
$-\frac{1}{2}i$	$-.5i, . -1.25, .563, -.684, -.533, -.716, -.487, -.763, \ldots$	Bounded
i	$i, -2, 3, 8, 63, 3968, \ldots$	Unbounded
2	$2, 3, 8, 63, 3968, \ldots$	Unbounded
10	$10, 99, 9800, 96039999, \ldots$	Unbounded

What do we notice about points with bounded vs. unbounded orbits?

Filled Julia Set Plot: $f(z) = z^2 - 1$

c	Orbit $c, f(c), f^2(c), \ldots$ (up to 3 decimal places)	Bounded or Unbounded
0	$0, -1, 0, -1, 0, \ldots$	Bounded
$\frac{1}{2}$	$.5, -.75, -.438, -.809, -.346, -.880, -.225, \ldots$	Bounded
1	$1, 0, -1, 0, -1, \ldots$	Bounded
$-\frac{1}{2}i$	$-.5i, .-1.25, .563, -.684, -.533, -.716, -.487, -.763, \ldots$	Bounded

- Plot c on the complex plane
- Color c black if orbit is bounded

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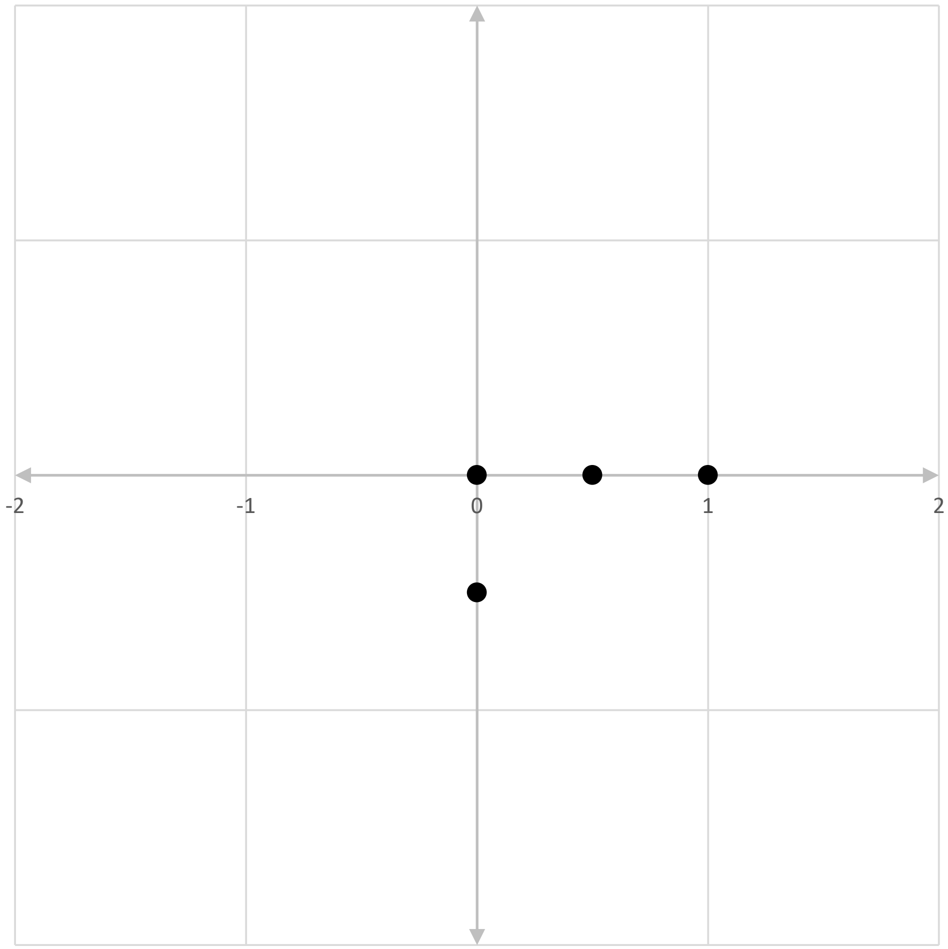
Filled Julia Set Plot: $f(z) = z^2 - 1$



Orbit of c bounded:

c ●

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Orbits: $f(z) = z^2 - 1$

c	Orbit $c, f(c), f^2(c), \ldots$ (up to 3 decimal places)	Bounded or Unbounded
-2	$-2, 3, 8, 63, 3968, \ldots$	Unbounded
0	$0, -1, 0, -1, 0, \ldots$	Bounded
$\frac{1}{2}$	$.5, -.75, -.438, -.809, -.346, -.880, -.225, \ldots$	Bounded
1	$1, 0, -1, 0, -1, \ldots$	Bounded
$-\frac{1}{2}i$	$-.5i, .-1.25, .563, -.684, -.533, -.716, -.487, -.763, \ldots$	Bounded
i	$i, -2, 3, 8, 63, 3968, \ldots$	Unbounded
2	$2, 3, 8, 63, 3968, \ldots$	Unbounded
10	$10, 99, 9800, 96039999, \ldots$	Unbounded

How big does a term in the orbit need to be until we are convinced that the orbit is unbounded?

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Filled Julia Set Plot: $f(z) = z^2 - 1$

What is the smallest n so $|f^n(c)| > 2$?

c	Orbit $c, f(c), f^2(c), \ldots$	Bounded or Unbounded	$\lvert f^n(c)\rvert > 2$ when $n =$
-2	$-2, 3, 8, 63, 3968, \ldots$	Unbounded	1
i	$i, -2, 3, 8, 63, 3968, \ldots$	Unbounded	2
2	$2, 3, 8, 63, 3968, \ldots$	Unbounded	1

Plot c on the complex plane and color c to differentiate between the number of iterations of f it takes until $|f^n(c)| > 2$.

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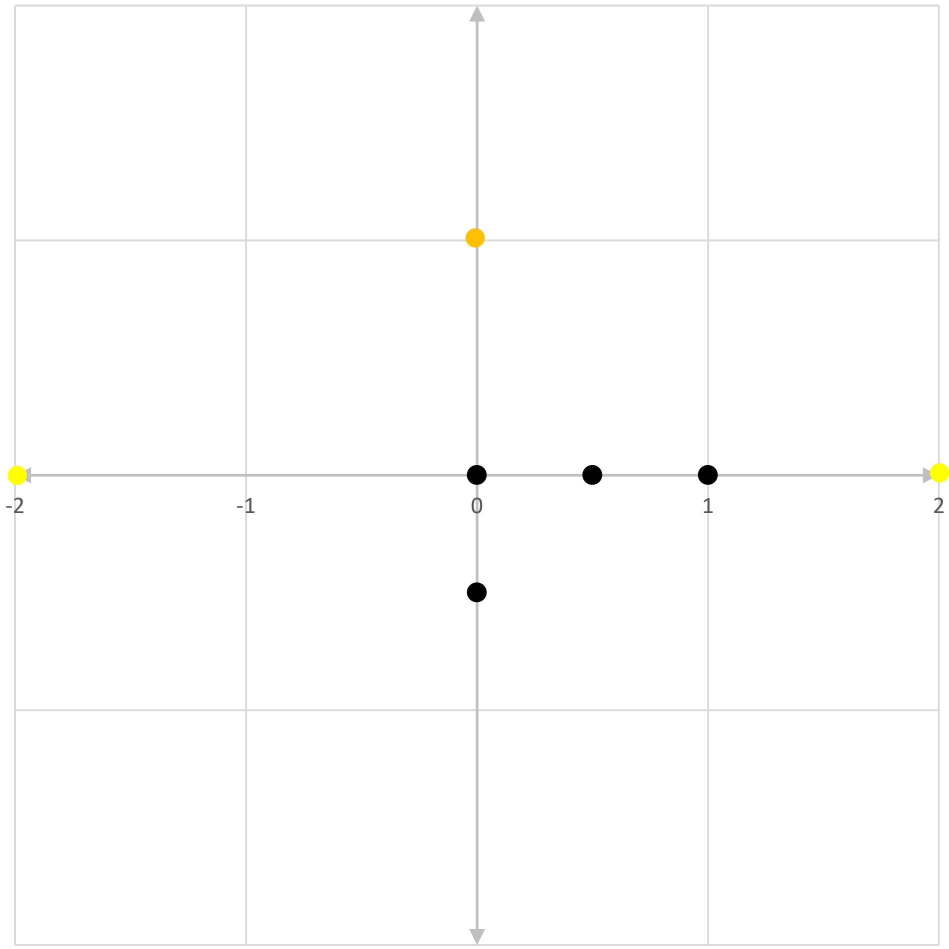
Filled Julia Set Plot: $f(z) = z^2 - 1$



Orbit of c bounded:

c ●

c with $|f^n(c)| > 2$:

n	c
1	●
2	●

Orbits: $f(z) = z^2 - 1$

c	Bounded or Unbounded
$\frac{1}{10} + \frac{1}{10}i$	Bounded
$\frac{1}{4}i$	Bounded
$\frac{1}{2} + \frac{1}{2}i$	Unbounded
$1 + i$	Unbounded

When is $\left|f^n\left(\frac{1}{2} + \frac{1}{2}i\right)\right| > 2$? For $n = 3$: $|-1.9375 - i| \approx 2.009$

When is $|f^n(1 + i)| > 2$? For $n = 1$: $|-1 + 2i| = \sqrt{5}$

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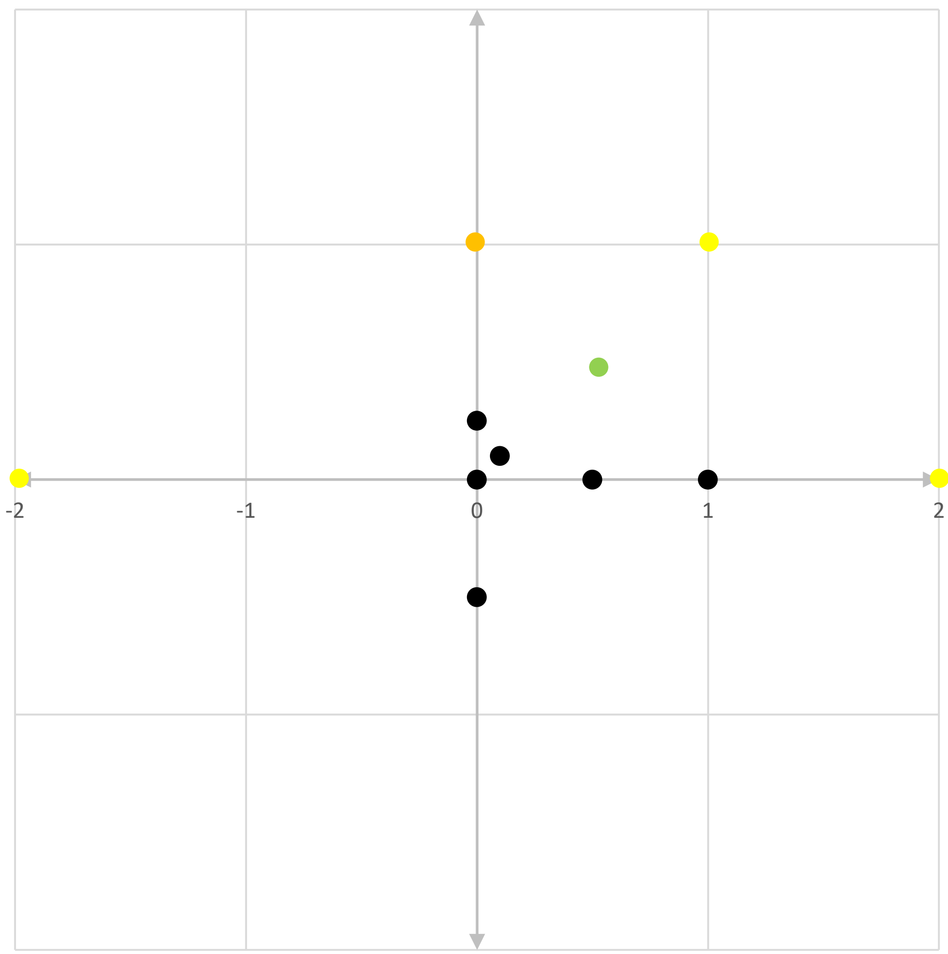
Filled Julia Set Plot: $f(z) = z^2 - 1$



Orbit of c bounded:

c ●

c with $|f^n(c)| > 2$:

n	c
1	(yellow dot)
2	(orange dot)
3	(green dot)

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Student Task: Computing Orbits

I5 | =COMPLEX(G5,H5)

	A	B	C	D	E	F	G	H	I	J	O	P	U
1	initial values a+bi				computing f(a+bi)								
2	a+bi	Real	Imaginary		(a+bi)^2	real part	real part - 1	imaginary part	f(a + bi)	|f(a + bi)|	f^2 (a + bi)	|f^2(a + bi)|	f^3(a + bi)
3	0	0	0		0	0	-1	0	-1	1	0	0	-1
4	1	1	0		1	1	0	0	0	0	-1	1	0
5	-1-i	-1	-1		2i	0	-1	2	-1+2i	2.236067977	-4-4i	5.656854249	-1+32i
6	1+i	1	1		2i	0	-1	2	-1+2i	2.236067977	-4-4i	5.656854249	-1+32i
7	1-i	1	-1		-2i	0	-1	-2	-1-2i	2.236067977	-4+4i	5.656854249	-1-32i
8	-1+i	-1	1		-2i	0	-1	-2	-1-2i	2.236067977	-4+4i	5.656854249	-1-32i
9	0.01+0.01i	0.01	0.01		0.0002i	0	-1	0.0002	-1+0.0002i	1.00000002	-3.99999999789458E-08-0	0.0004	-1.00000016+3.19999999831566E-11
10	0.1+0.1i	0.1	0.1		0.02i	0	-1	0.02	-1+0.02i	1.00019998	-0.000399999999999956-0	0.040002	-1.00159984+0.00003199999999999
11	0.25+0.25i	0.25	0.25		0.125i	0	-1	0.125	-1+0.125i	1.007782219	-0.015625-0.25i	0.250487805	-1.06225585937+0.0078125i
12	0.498+0.336i	0.498	0.336		0.135108+0.334656i	0.135108	-0.864892	0.334656	-0.864892+0.334656i	0.927379539	-0.363956466672-0.57888	0.683790441	-1.2026407483557б+0.42137612728
13	0.498+0.398i	0.498	0.398		0.0896+0.396408i	0.0896	-0.9104	0.396408	-0.9104+0.396408i	0.992958943	-0.328311142464-0.72177	0.792940176	-1.4131777094336 7+0.4739366268 9
14	0.5+0.3i	0.5	0.3		0.16+0.3i	0.16	-0.84	0.3	-0.84+0.3i	0.891964125	-0.3844-0.504i	0.633860679	-1.10625264+0.3874752i
15	0.5+0.4i	0.5	0.4		0.09+0.4i	0.09	-0.91	0.4	-0.91+0.4i	0.994032193	-0.3319-0.728i	0.800088501	-1.41982639+0.4832464i
16	0.5+0.5i	0.5	0.5		0.5i	0	-1	0.5	-1+0.5i	1.118033989	-0.25-i	1.030776406	-1.9375+0.5i
17	0.6+0.2i	0.6	0.2		0.32+0.24i	0.32	-0.68	0.24	-0.68+0.24i	0.721110255	-0.5952-0.3264i	0.67882251	-0.75227392+0.38854656i
18	0.6+0.4i	0.6	0.4		0.2+0.48i	0.2	-0.8	0.48	-0.8+0.48i	0.932952303	-0.5904-0.768i	0.968708501	-1.24125184+0.9068544i
19	0.75+0.75i	0.75	0.75		1.125i	0	-1	1.125	-1+1.125i	1.505199322	-1.265625-2.25i	2.58153184	-4.460693359375+5.6953125i
20	-0.0222222222-0.9i	-0.022222	-0.9		-0.809506172840494+(	-0.809506173	-1.809506173	0.04	-1.80950617284049+0.03	1.809948228	2.27271258955104-0.1447	2.277318185	4.1442669141726-0.657997992923 5
21	0.5+i	0.5	1		-0.75+i	-0.75	-1.75	1	-1.75+i	2.015564437	1.0625-3.5i	3.657718722	-12.12109375-7.4375i
22	1+0.25i	1	0.25		0.9375+0.5i	0.9375	-0.0625	0.5	-0.0625+0.5i	0.503891109	-1.24609375-0.0625i	1.247660164	0.5488433837890 6+0.15576171875i
23	1+0.5i	1	0.5		0.75+i	0.75	-0.25	1	-0.25+i	1.030776406	-1.9375-0.5i	2.000976324	2.50390625+1.9375i
24	0.25i	0	0.25		-0.0625	-0.0625	-1.0625	0	-1.0625	1.0625	0.12890625	0.12890625	-0.983383178710938
25	0.5i	0	0.5		-0.25	-0.25	-1.25	0	-1.25	1.25	0.5625	0.5625	-0.68359375
26	0.75i	0	0.75		-0.5625	-0.5625	-1.5625	0	-1.5625	1.5625	1.44140625	1.44140625	1.07765197753906
27	i	0	1		-1	-1	-2	0	-2	2	3	3	8
28	1.61803398874989	1.618034	0		2.61803398874988	2.618033989	1.618033989	0	1.61803398874988	1.618033989	1.61803398874985	1.618033989	1.61803398874975

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Programming Task

Write a program to check whether a point has an unbounded or bounded orbit

Define function realF as accepting two real values a_0, a_1 and returning $a_0^2 - a_1^2 - 1$

Define function imagineF as accepting two real values a_0, a_1 and returning $2a_0a_1$

Define the complex modulus function calcModulus as accepting two real values b_0, b_1 and returning $\sqrt{}(b_0^2 + b_1^2)$

Programming Task

Input real part of initial point c_0

Input imaginary part of initial point c_1

Initialize break variable to true.

Initialize index to 1.

While break variable is true

    Apply realF to point c_0, c_1 and return point $d_0 = c_0^2 - c_1^2 - 1$

    Apply imagineF to point c_0, c_1 and return point $d_1 = 2c_0c_1$

    Set $c_0 = d_0$

    Set $c_1 = d_1$

    If calcModulus applied to c_0, c_1 is greater than 2

        Print “Your initial point has an unbounded orbit, and it is not in the filled Julia set.”

        Set break variable to false

    Else if index is greater than or equal to 200

        Print “Your initial point looks to have a bounded orbit and be in the filled Julia set.”

        Set break variable to false

    Add 1 to the index.

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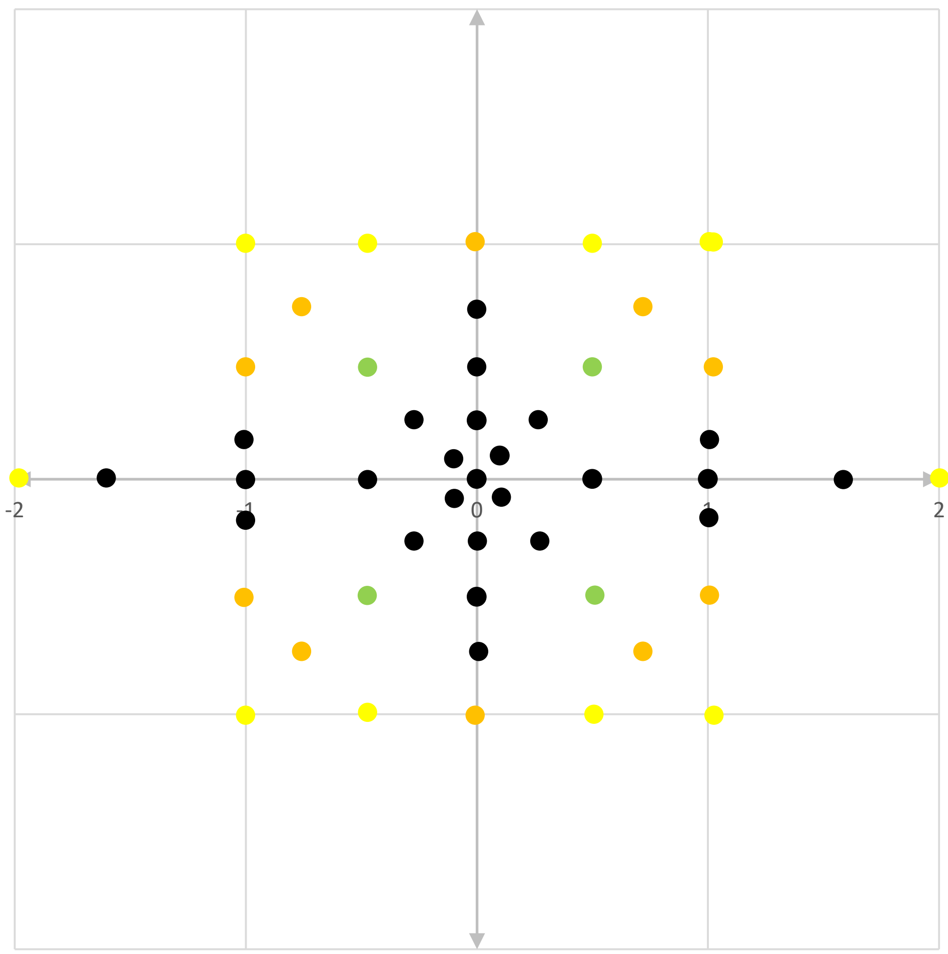
Filled Julia Set Plot: $f(z) = z^2 - 1$



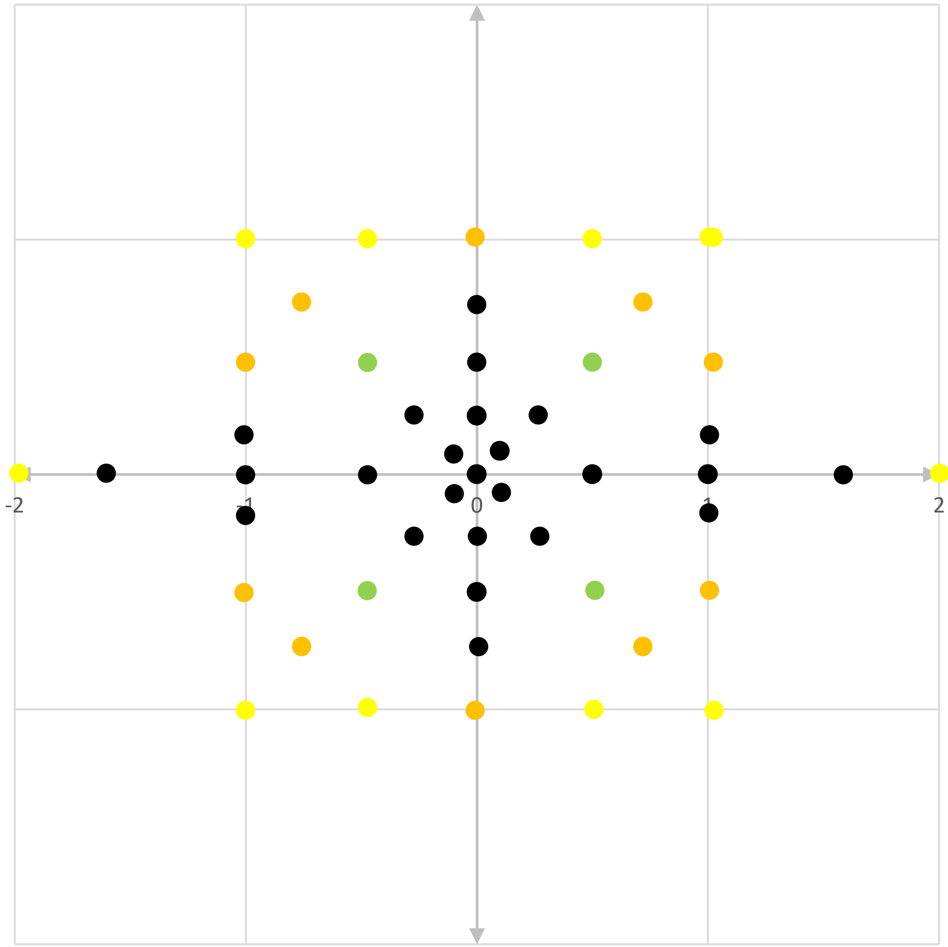
Orbit of c bounded:

c ●

c with $|f^n(c)| > 2$:

n	c
1	(yellow)
2	(orange)
3	(green)

Student Task



Explain why the shading of points in the filled Julia Set plot is symmetric around each axis.

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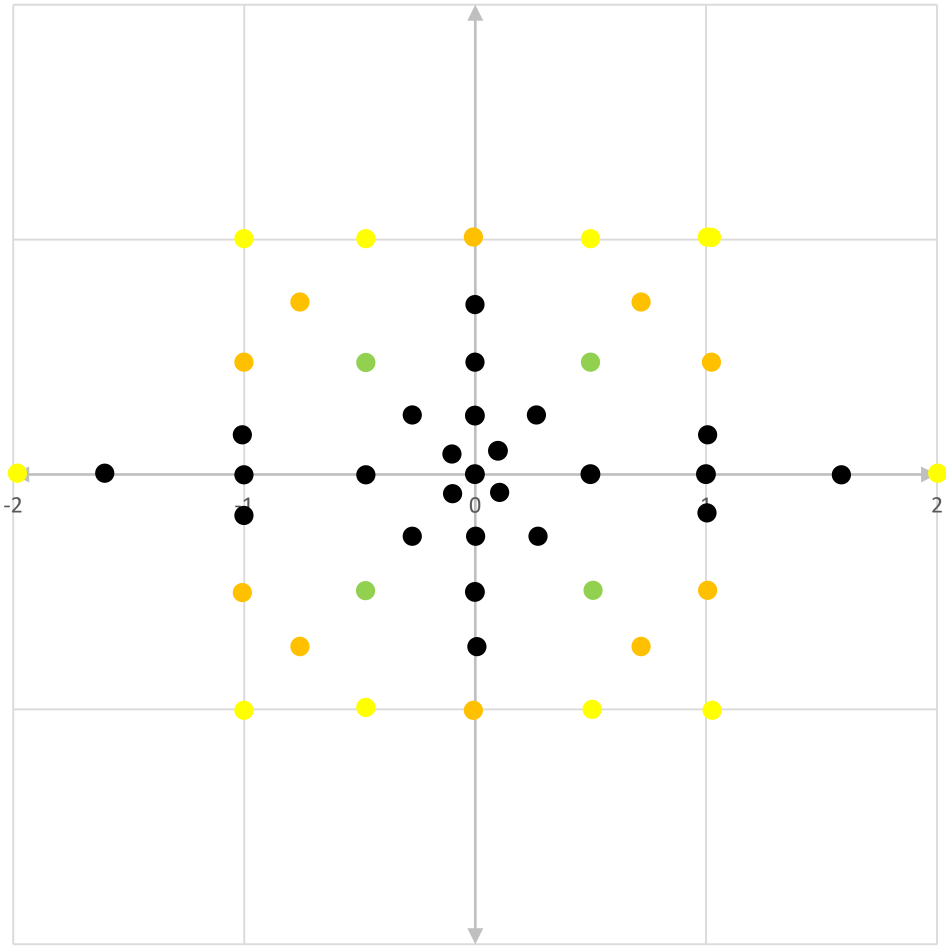
Filled Julia Set Plot: $f(z) = z^2 - 1$



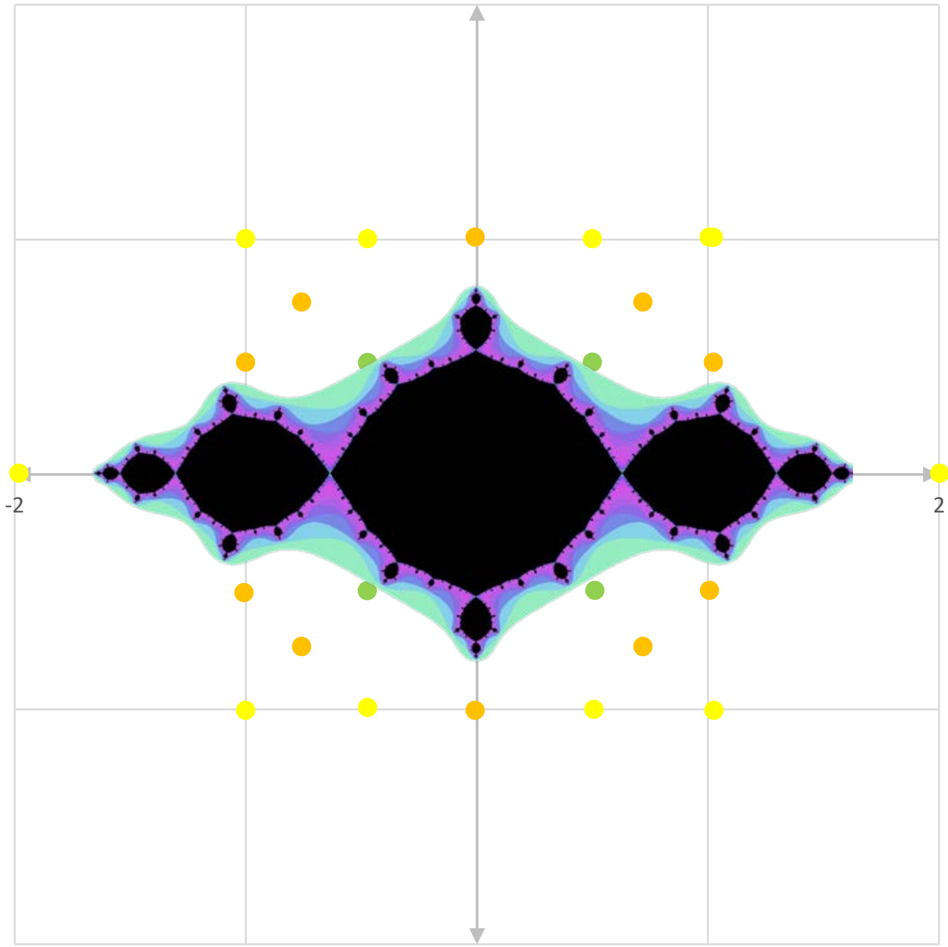
Orbit of c bounded:

c

c with $|f^n(c)| > 2$:

n	c
1	
2	
3	

Filled Julia Set Plot: $f(z) = z^2 - 1$



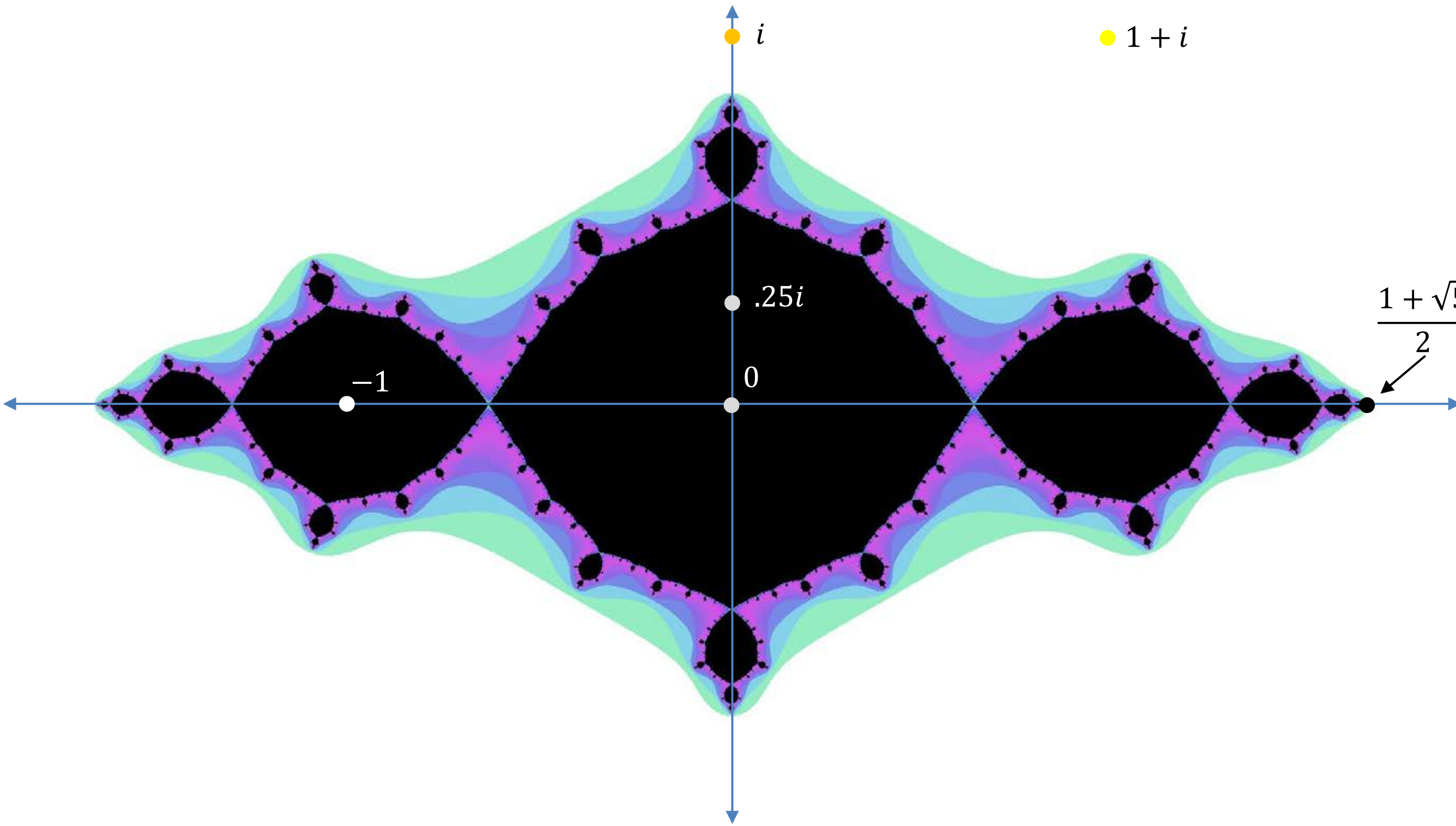
Orbit of c bounded:

c ●

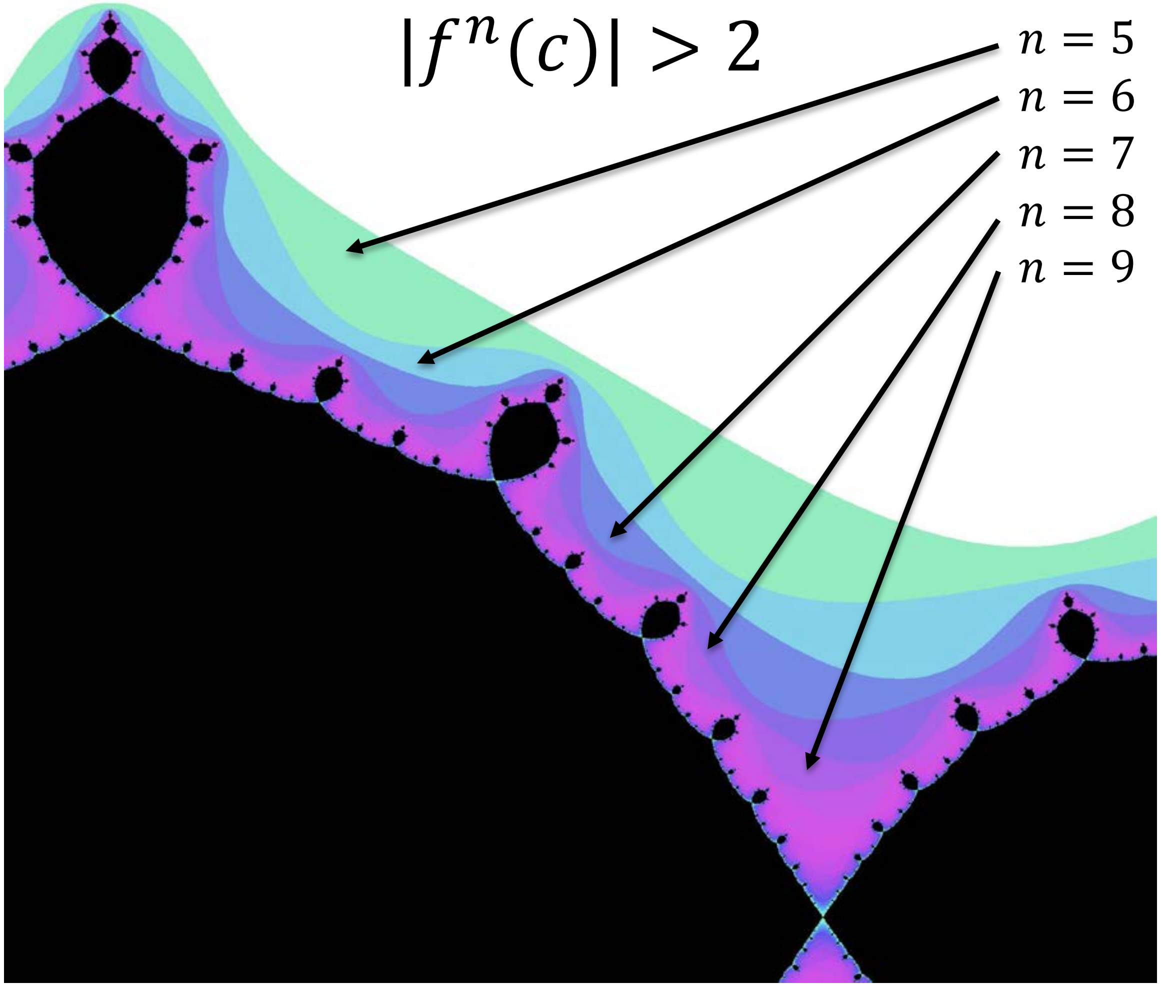
c with $|f^n(c)| > 2$:

n	c
1	● (yellow)
2	● (orange)
3	● (green)

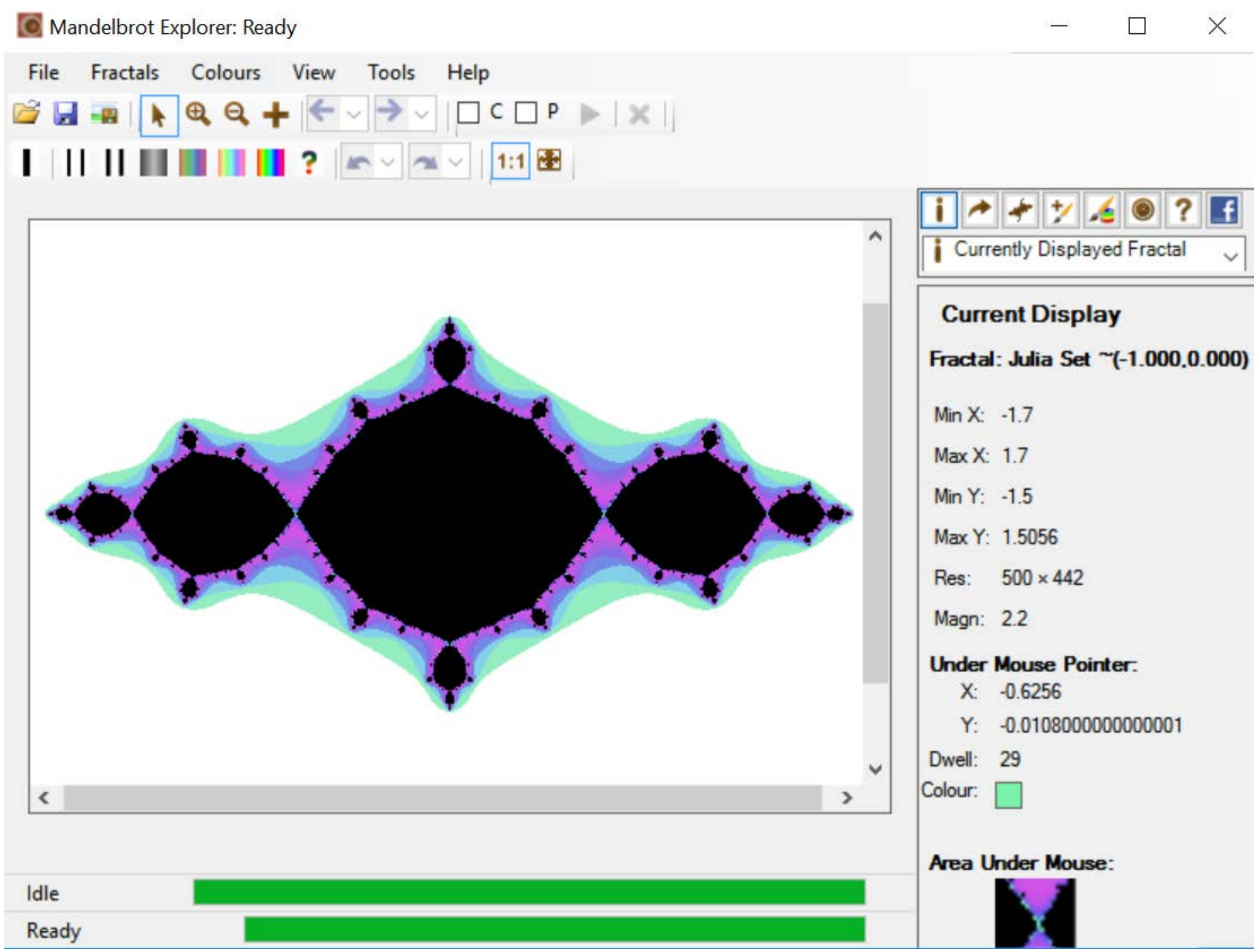
Filled Julia Set: $f(z) = z^2 - 1$



Filled Julia Set: $f(z) = z^2 - 1$



Filled Julia Set: Technology



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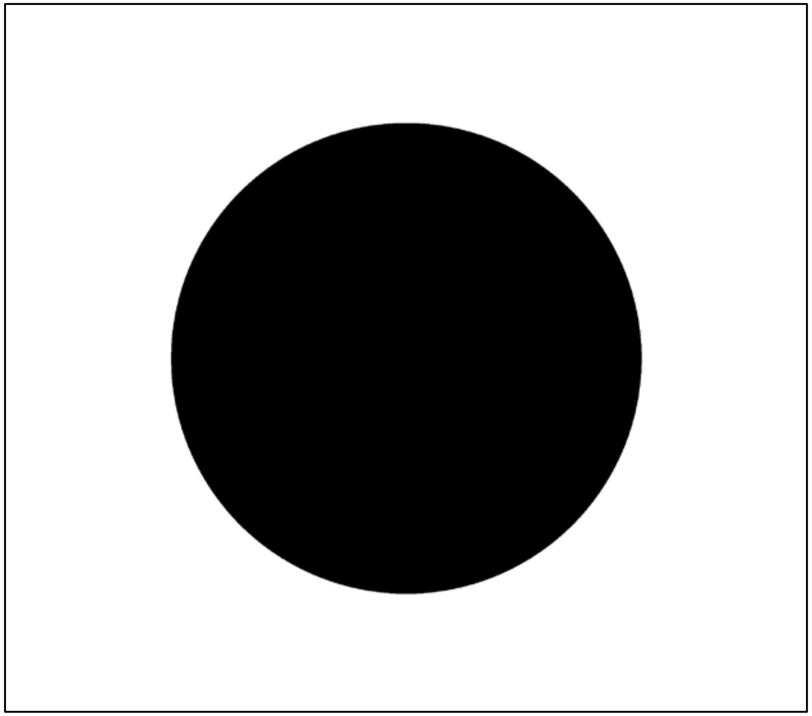
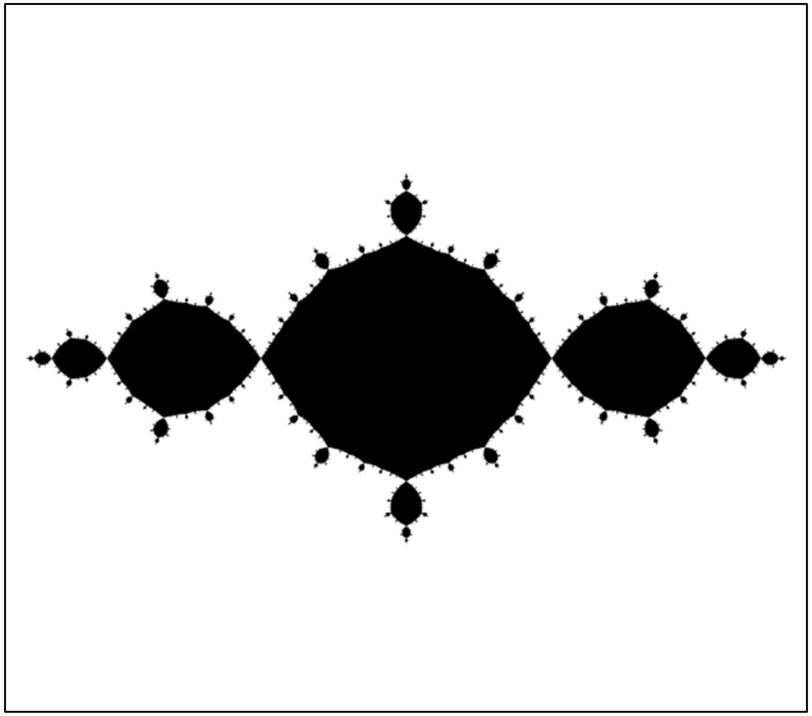
Comparison of Filled Julia Sets



Filled Julia Set: $f(z) = z^2$



Filled Julia Set: $f(z) = z^2 - 1$

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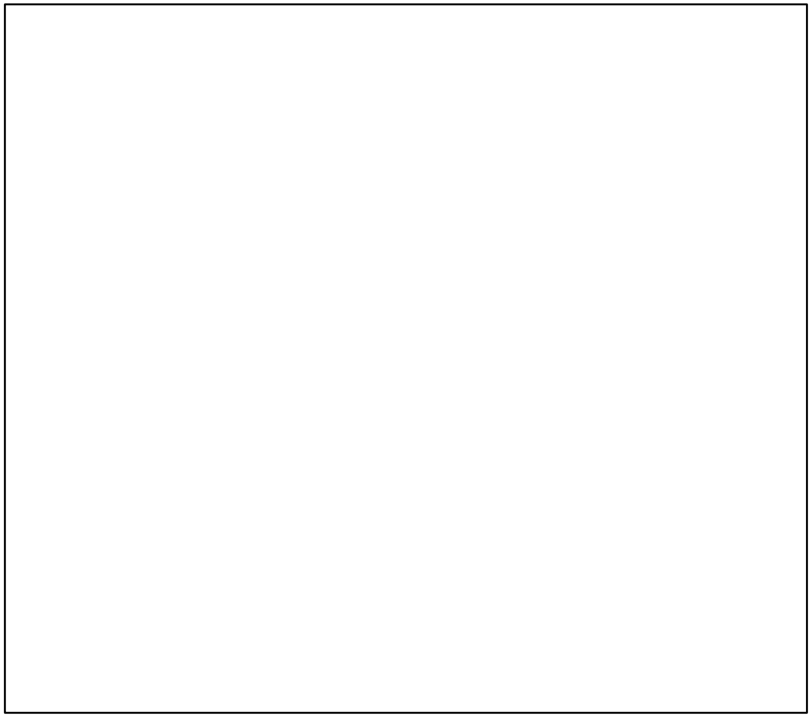
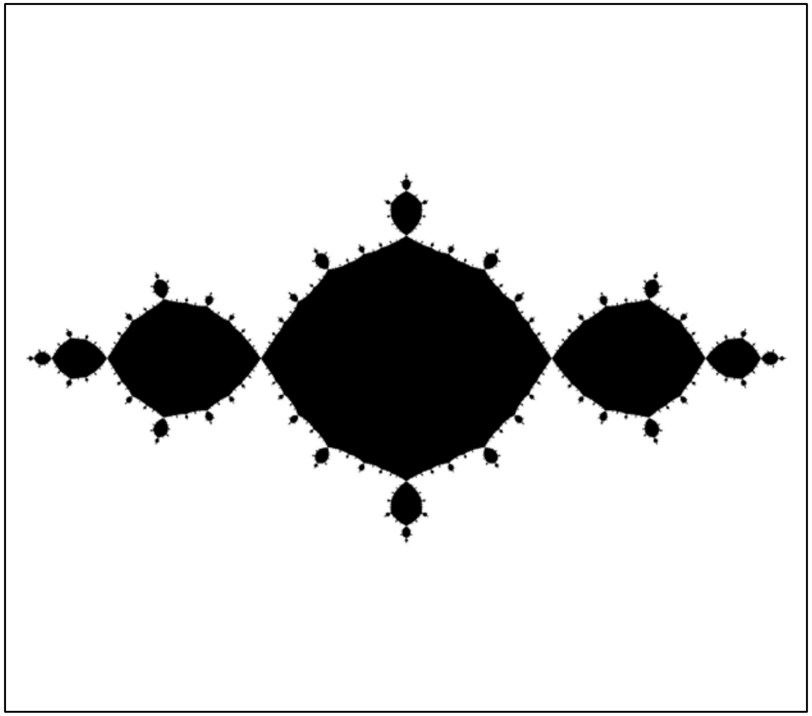
Comparison of Filled Julia Sets



Filled Julia Set: $f(z) = z^2 + 1$



Filled Julia Set: $f(z) = z^2 - 1$

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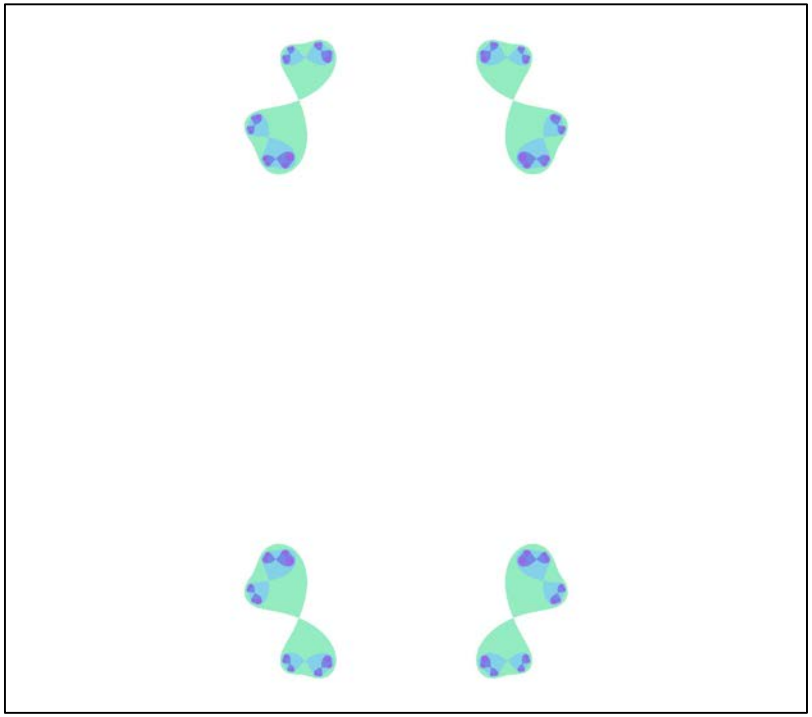
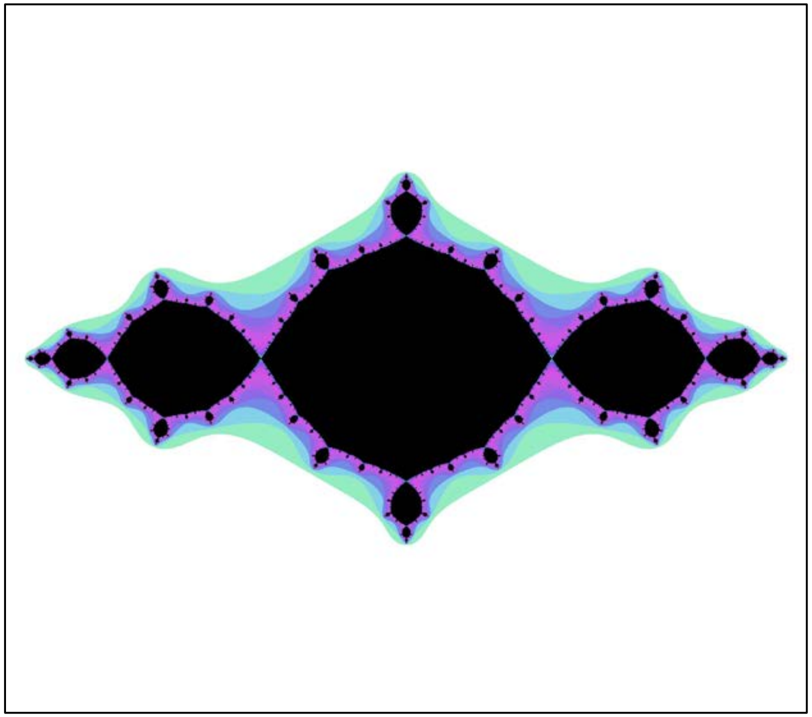
Comparison of Filled Julia Sets



Filled Julia Set: $f(z) = z^2 + 1$



Filled Julia Set: $f(z) = z^2 - 1$

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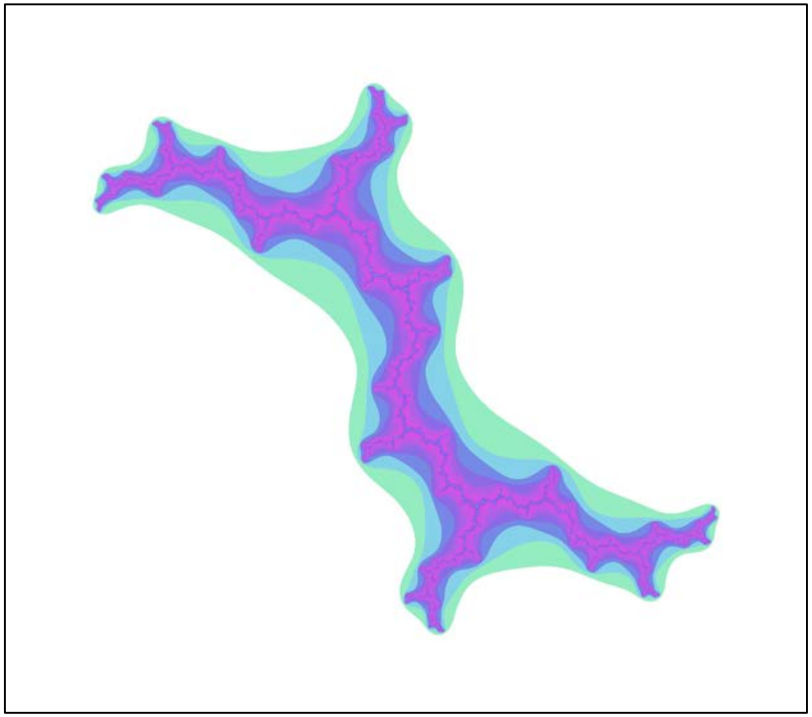
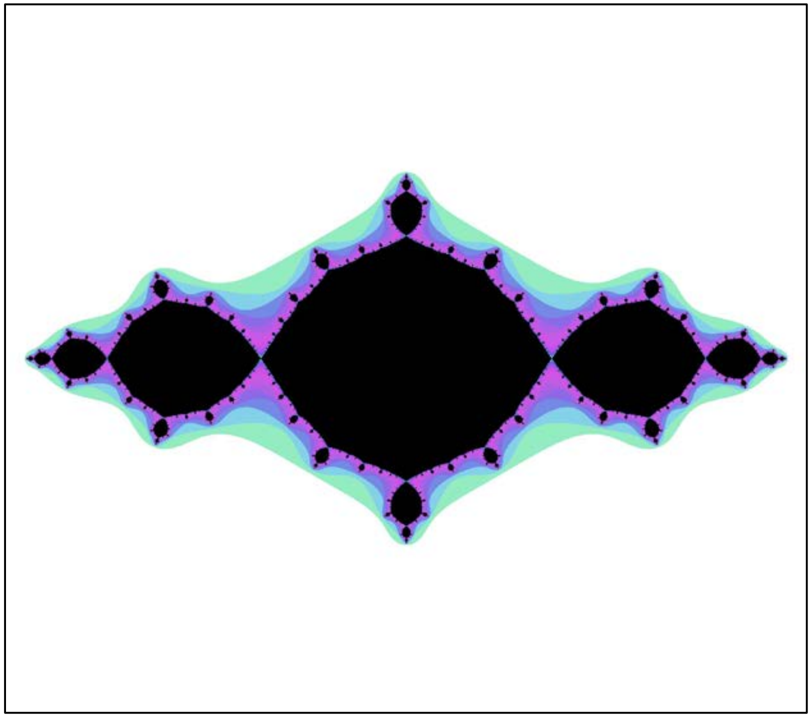
Comparison of Filled Julia Sets



Filled Julia Set: $f(z) = z^2 - i$



Filled Julia Set: $f(z) = z^2 - 1$

Programming Task

Plot the filled Julia set of function

- Java: *Java Number Cruncher: The Java Programmer's Guide to Numerical Computing* by Ronald Mak
- Python: https://www.linuxvoice.com/issues/010/julia.pdf
- Various languages: https://rosettacode.org/wiki/Julia_set

Further Explorations

- Compute orbits and graph filled Julia Sets of other noteworthy functions—a nice list of functions to try can be found at
  http://www.math.uni-bonn.de/people/karcher/Julia_Sets.pdf.

- The Mandelbrot Set:

  The set of a such that
  the orbit of 0 for $f(z) = z^2 + a$ is bounded.

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