Numbers are Numbers. Why Treat Fractions and Decimals Differently?

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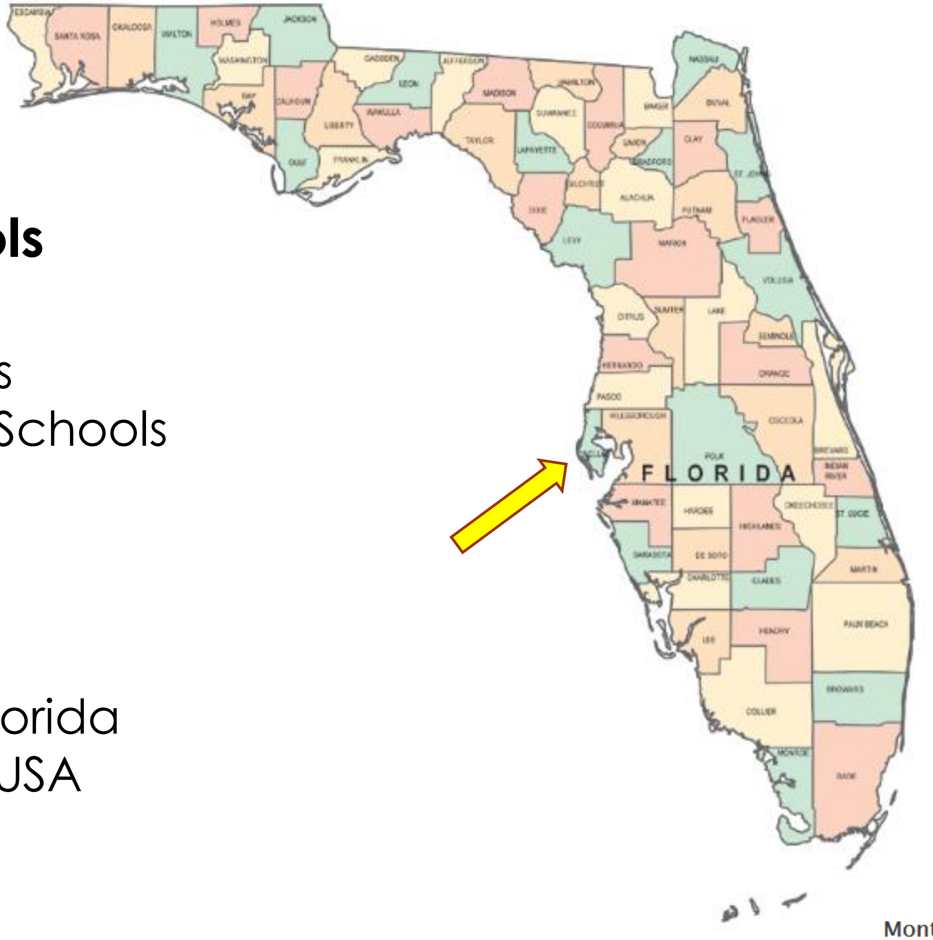
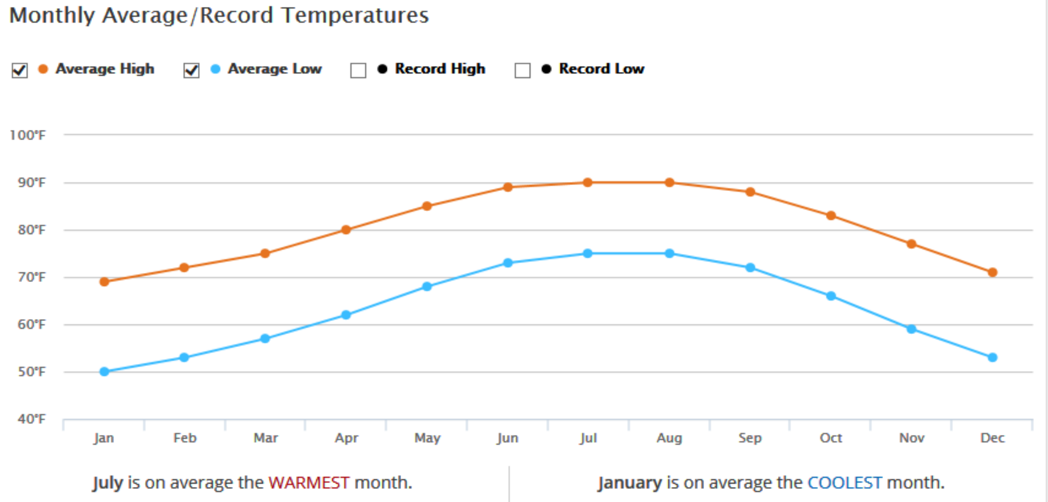


Pinellas County Schools

- Over 102,000 students
- 74 Elementary Schools
- 3 Elementary/Middle Schools
- 21 Middle Schools
- 18 High Schools
- 5 ESE Schools
- 23 Charter Schools
- 8th Largest District in Florida
- 26th Largest District in USA

Pinellas County, Florida

Almost 1,000,000 people
Most densely populated county in Florida
588 miles of coastline
35 miles of white sandy beaches
11 barrier islands



Goals

- Become familiar with the models for whole number multiplication.

- Follow a learning progression for multiplication of fractions and multiplication of decimals that builds procedural fluency from conceptual understanding.

- Learn how to apply whole number multiplication models to fraction and decimal multiplication.

What's happening with multiplication instruction

- Many teachers teach procedurally because that is what they are comfortable with…knowing "how," not "why."
- Many teachers don't solve the problems using the various models themselves.
- Many teachers are still developing an understanding of models used for multiplication of whole numbers.

Principles to Action: Effective Mathematics Teaching Practices

- Establish mathematics goals to focus learning.
- Implement tasks that promote reasoning and problem solving.
- Use and connect mathematical representations.
- Facilitate meaningful discourse.
- Pose purposeful questions.
- Build procedural fluency from conceptual understanding.
- Support productive struggle in learning mathematics.
- Elicit and use evidence of student thinking.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

How would you explain…

4×3

Third Grade

$4 \times \frac{3}{4}$

Fourth Grade

4×0.3

Fifth Grade

A "go to" explanation for multiplication

Whole numbers $4 \times 3 = 3 + 3 + 3 + 3 = 12$

What about fractions? $4 \times \frac{3}{4} = \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = \frac{12}{4}$

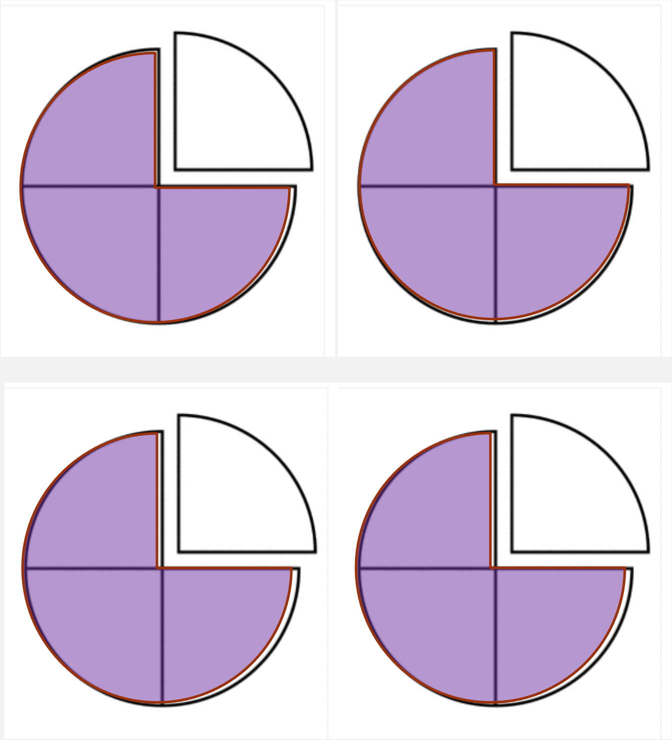
What about decimals? $4 \times 0.3 = 0.3 + 0.3 + 0.3 + 0.3 = 1.2$

Is repeated addition really the best model to convey $\frac{3}{4}$ x 4?

Students should recognize that **4 x 3 means adding 4 equal-sized groups of 3**, rather than as 3 added 4 times (that is, 3 + 3 + 3 + 3).

As students move toward developing an understanding of the meaning of fractions and of multiplication of fractions, they will discover that **finding $\frac{3}{4}$ of a group of 4 makes sense**, but that **adding 4 three-fourths of a time does not.**

Finding $\frac{3}{4}$ of a group of 4

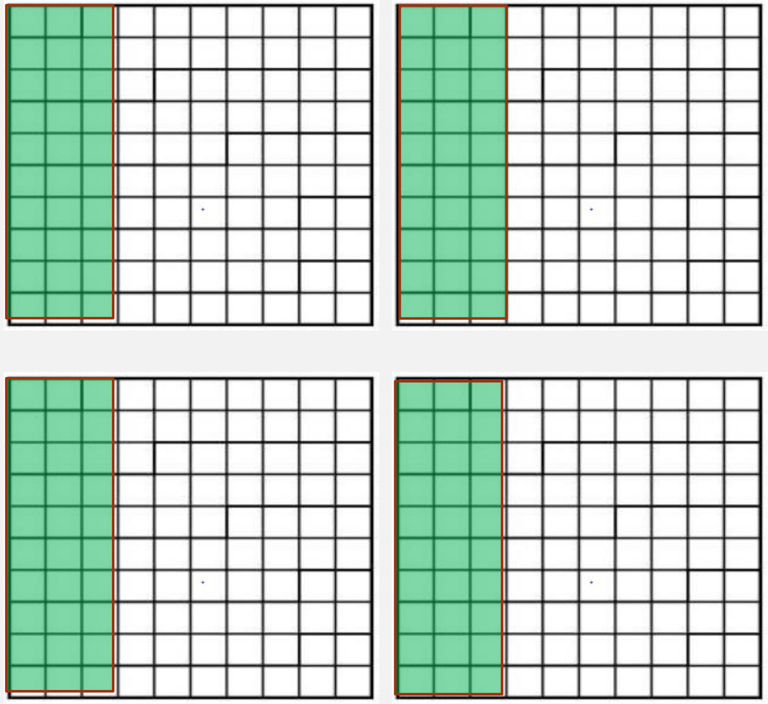


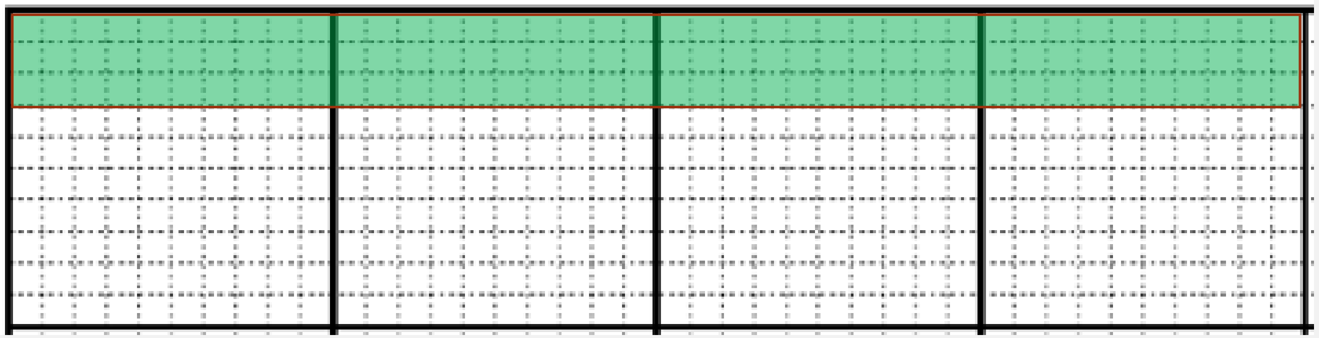
If I can find $\frac{3}{4}$ of a group of 1,

I can use that to find $\frac{3}{4}$ of a group of 4,

which is $\frac{12}{4}$ or 3.

Finding 0.3 of a group of 4





If I can find 0.3 of a group of 1.

I can use that to find 0.3 of a group of 4, which is 12 tenths or 1.2.

Multiplying Whole Numbers

What models and tools build conceptual understanding and lead to procedural fluency?

What models would your students use to evaluate each expression?

6 x 8

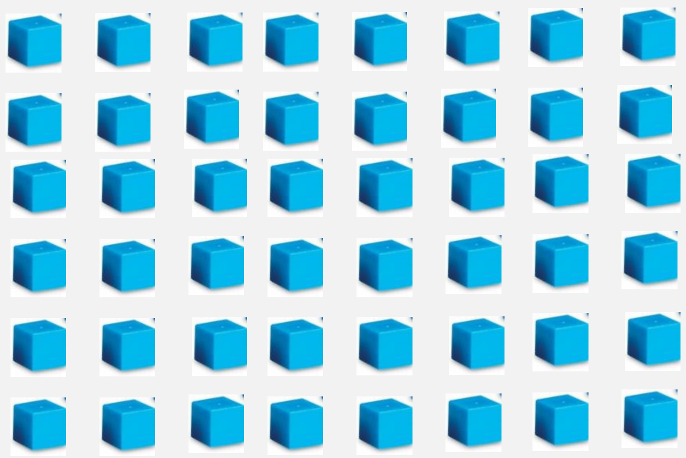
6 x 16

How does the expression 6 x 8 help students understand 6 x 16?

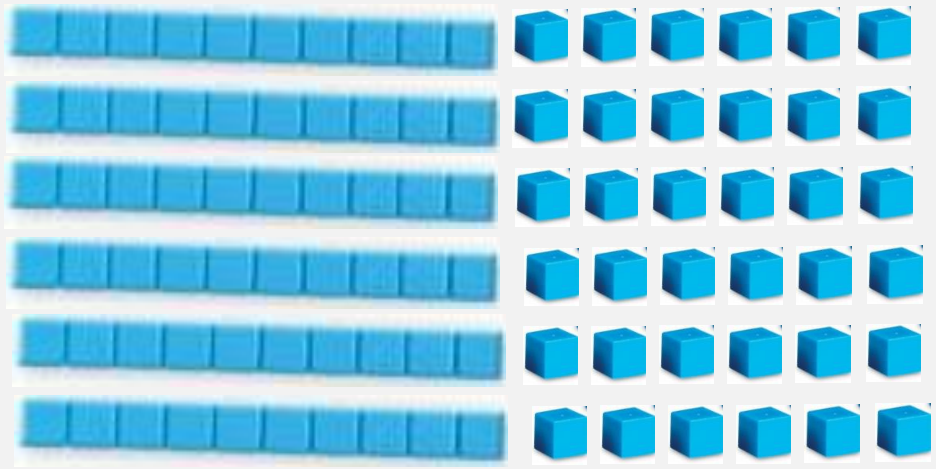
Hint: Think of Number Talks.

Building whole number arrays with base ten blocks

6 x 8



6 x 16



Notice how well the base ten blocks show 6 x 16 as (6 x 10) + (6 x 6).

How do we transition away from base ten blocks, two-color counters, or color tiles?

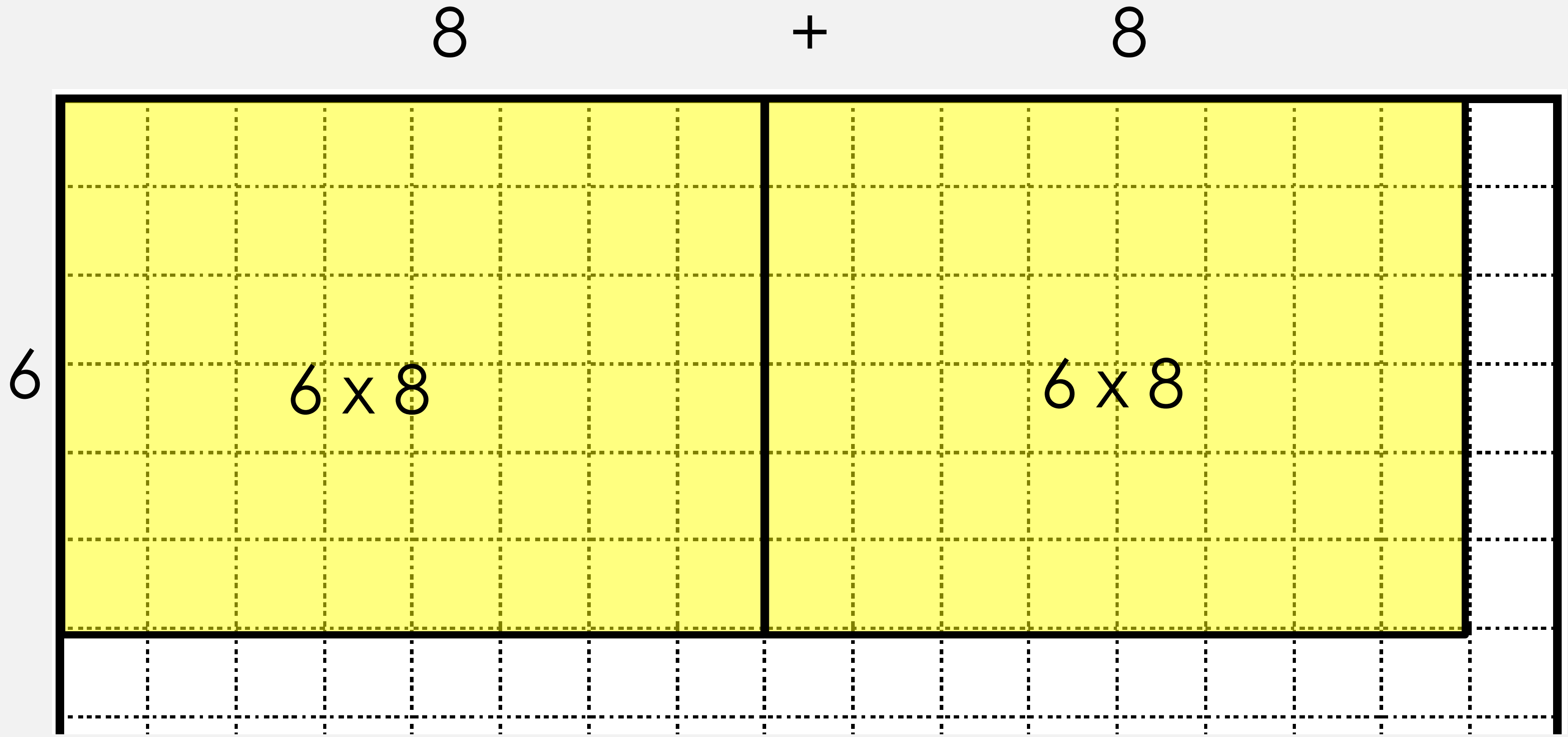
Switch to grid paper.

We find our third grade teachers don't think to use grid paper when teaching arrays in multiplication, but they use grid paper whey they teach area and perimeter. Why is this?

Decomposing:

Breaking apart a factor using the distributive property

6 x 16 = ?



6 x (8 + 8) = ?

6 x 8 = 48

(6 x 8) + (6 x 8) = 96

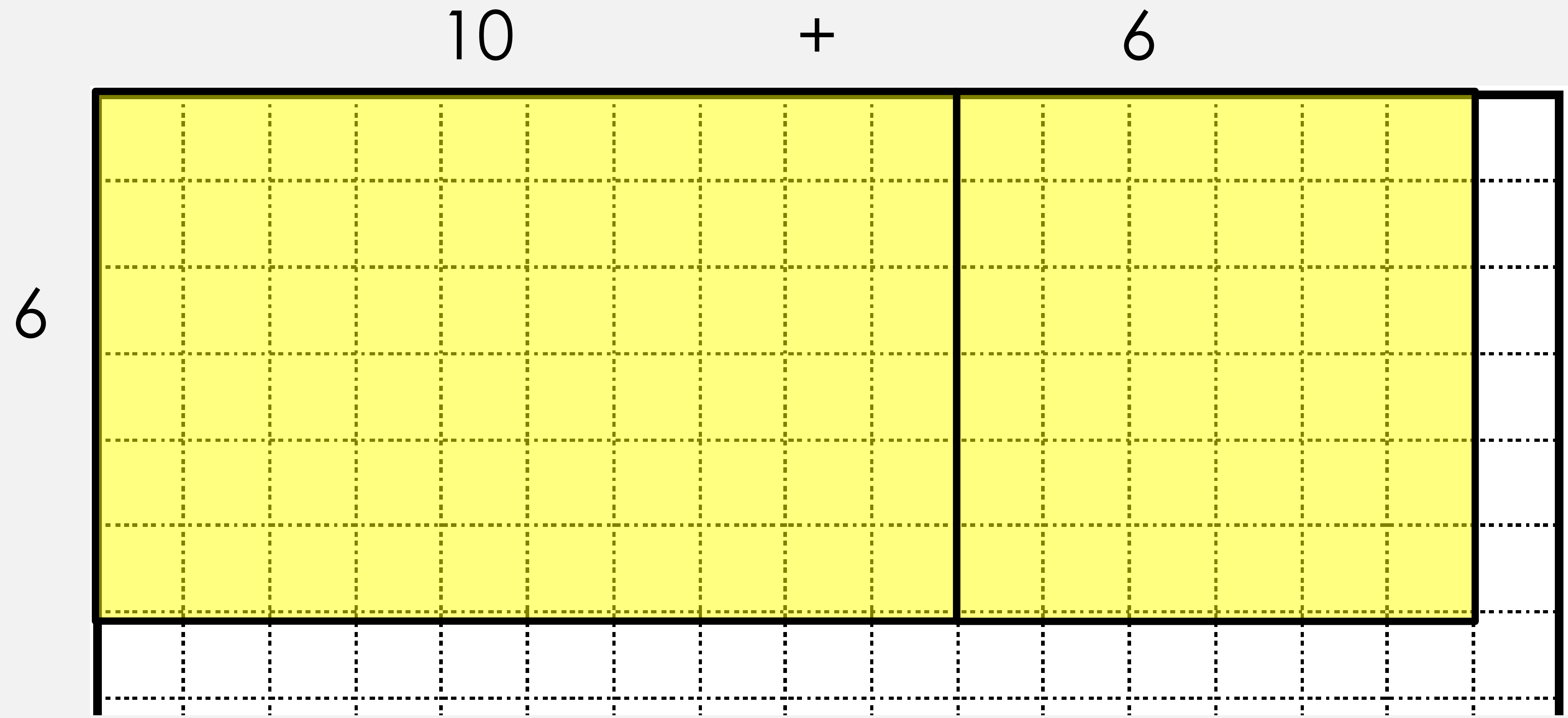
(6 x 8) x 2 = 96

6 x 16 = 96

Distributive property:

Breaking apart a factor based on place value
(Developing an understanding of partial products)

6 x 16 = ?



6 x (10 + 6)= 96

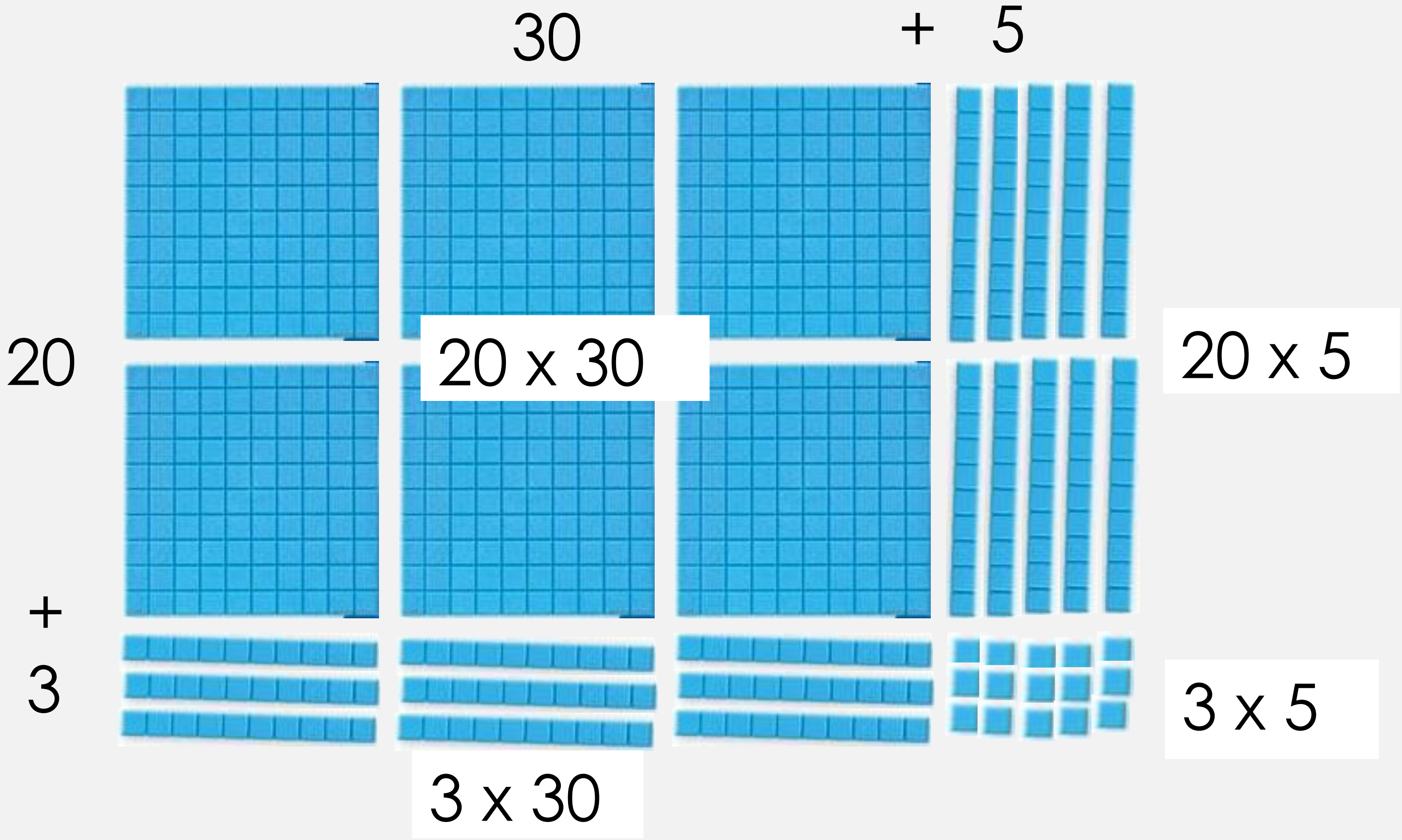
(6 x 10) + (6 x 6) = 96

6 x 16 = 96

What about this model with larger numbers?

Building arrays with base ten blocks for larger whole numbers

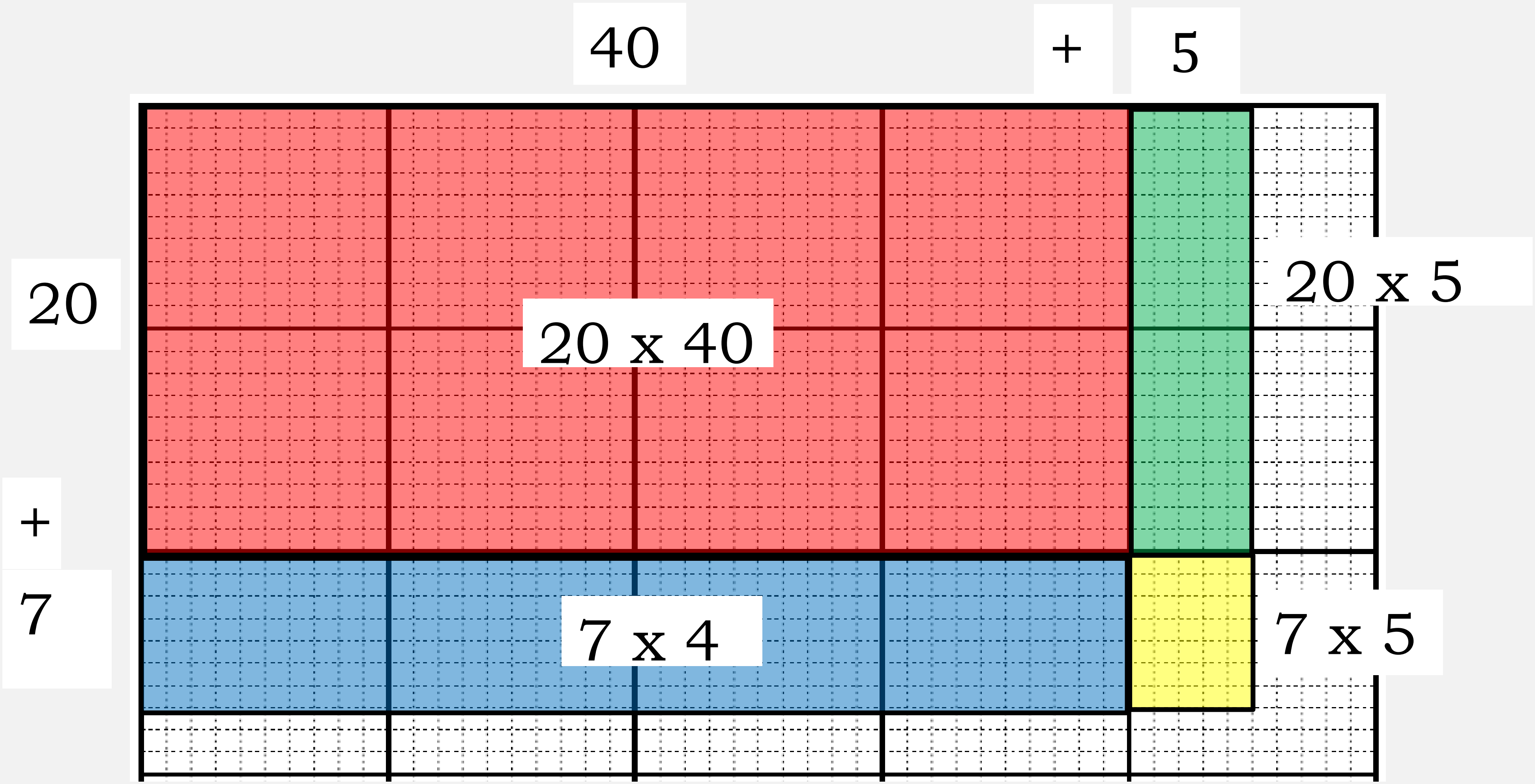
23 x 35 = ?




```
   35
 x 23
 ----
   15
   90
  100
+ 600
 ----
  805
```

Extending use of grid paper to build understanding: Using base ten grid paper

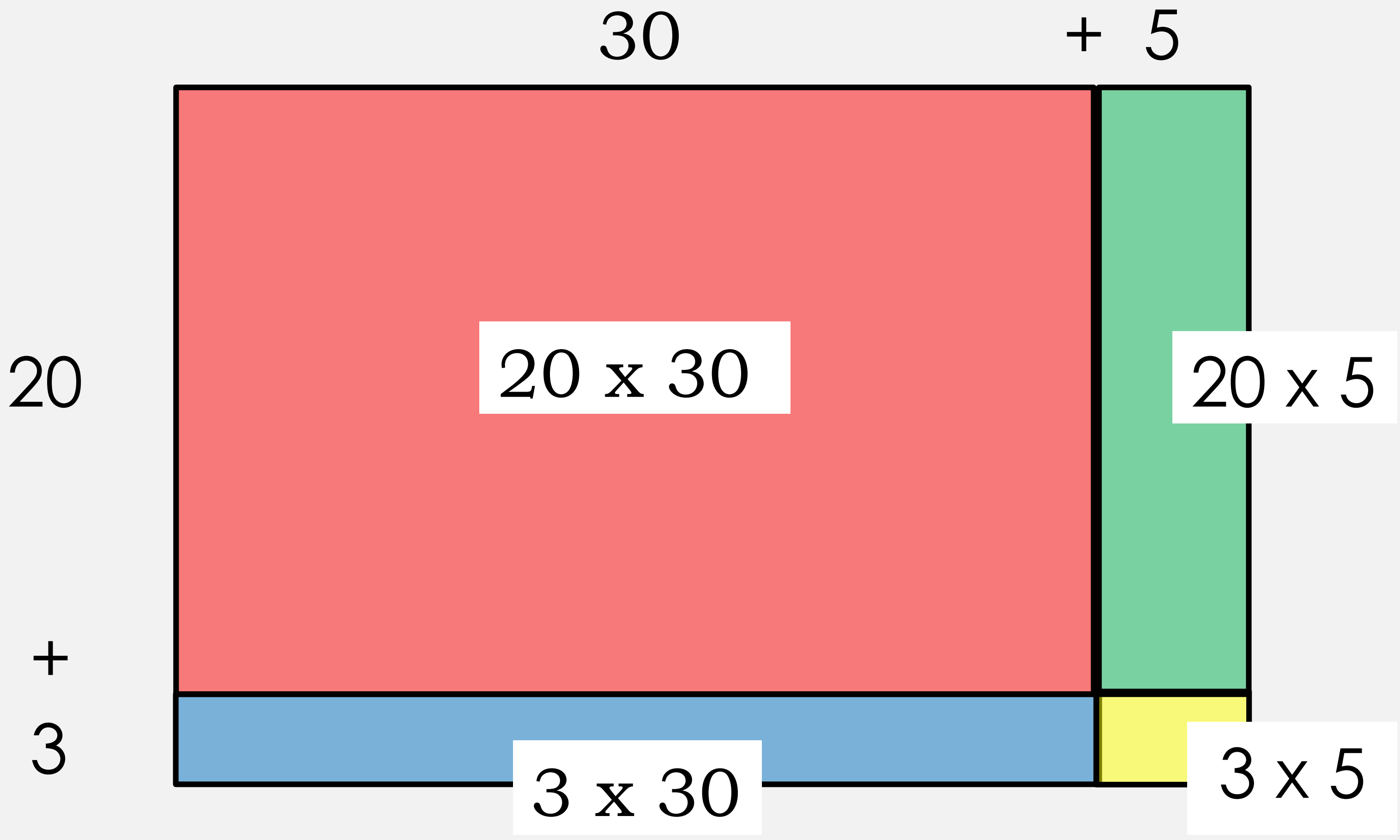
27 x 45




```
   35
 x 23
   15
   90
  100
+ 600
  805
```

Area model with no grid paper

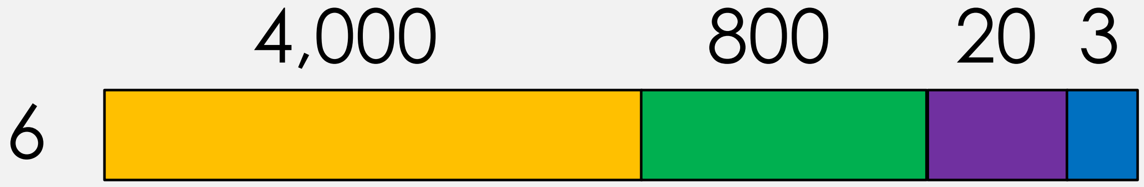
23 x 35 = ?




```
   35
 x 23
 ----
   15
   90
  100
+ 600
 ----
  805
```

We use the area model to expand our understanding of multiplication with larger numbers.

6 x 4,823 = ?



6 x 4,000 = 24,000
6 x 800 = 4,800
6 x 20 = 120
6 x 3 = 18
28,938

There is some attempt to show proportion between the boxes.

Box multiplication:
Is this an appropriate model?

23 x 35 = ?

	30 +	5
20 +	20 x 30	20 x 5
3	3 x 30	3 x 5

This model is very procedural. It is almost an algorithm.

But, aren't we moving towards a more abstract representation?

If you are going to use this model, don't rush to it too quickly.

Let’s revisit 6 x 4,823.
Does box multiplication make more sense for 6 x 4,823?

	4,000 +	800 +	20 +	6
6	6 x 4,000	6 x 800	6 x 20	6 x 3

6 x 4,000 = 24,000
6 x 800 = 4,800
6 x 20 = 120
6 x 3 = 18
28,938

Is it useful for multiplying even larger numbers?

23 x 535 = ?

	500	+ 30	+ 5
20	20 x 500	20 x 30	20 x 5
+ 3	3 x 500	3 x 30	3 x 5

Box multiplication clearly identifies the partial products for the students who have trouble identifying all partial products.

Viewing multiplication as a relationship between the number of groups and the number in each group allows for a smooth transition to the meaning of fractions and of fraction multiplication.

Multiplying Fractions

How do the whole number multiplication models work with fractions?

How would you model these problems with base ten blocks?

$2 \times \frac{5}{10}$

$\frac{5}{10} \times 2$

one whole

one tenth

one hundredth

We have to change the value of the base ten blocks.
This adds value to naming them a flat, a rod, and a cube.

Multiplication of decimal fractions

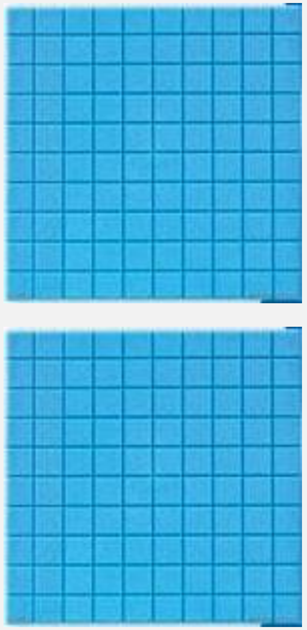
2×1

$2 \times \frac{5}{10}$

$\frac{5}{10} \times 2$

1

2

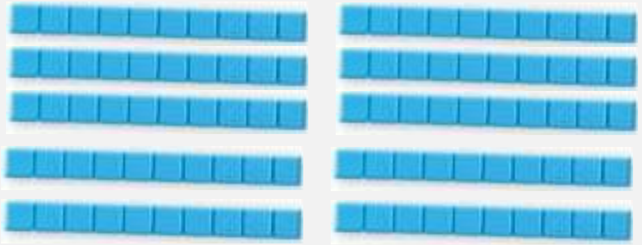


$\frac{5}{10}$

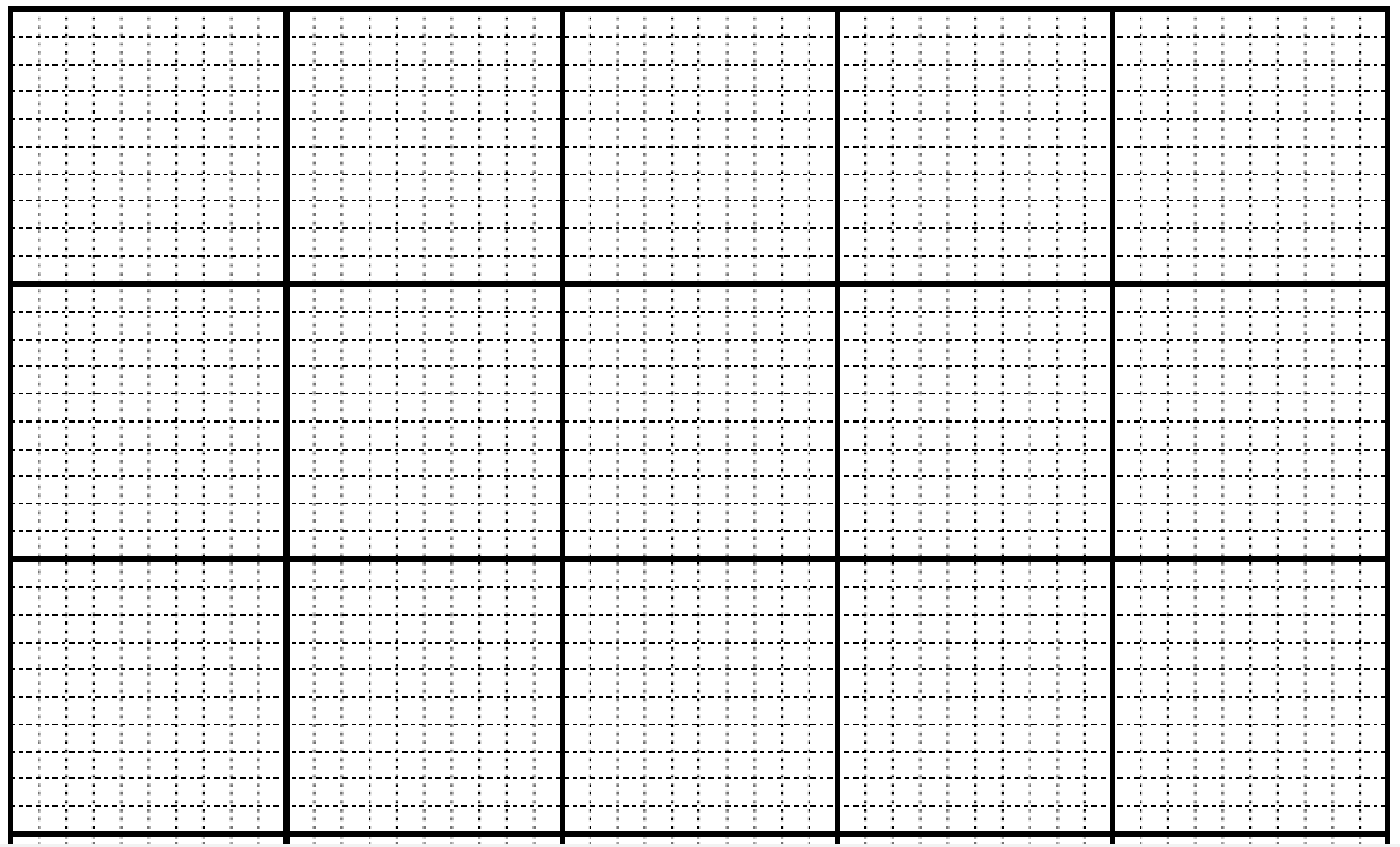
2

2

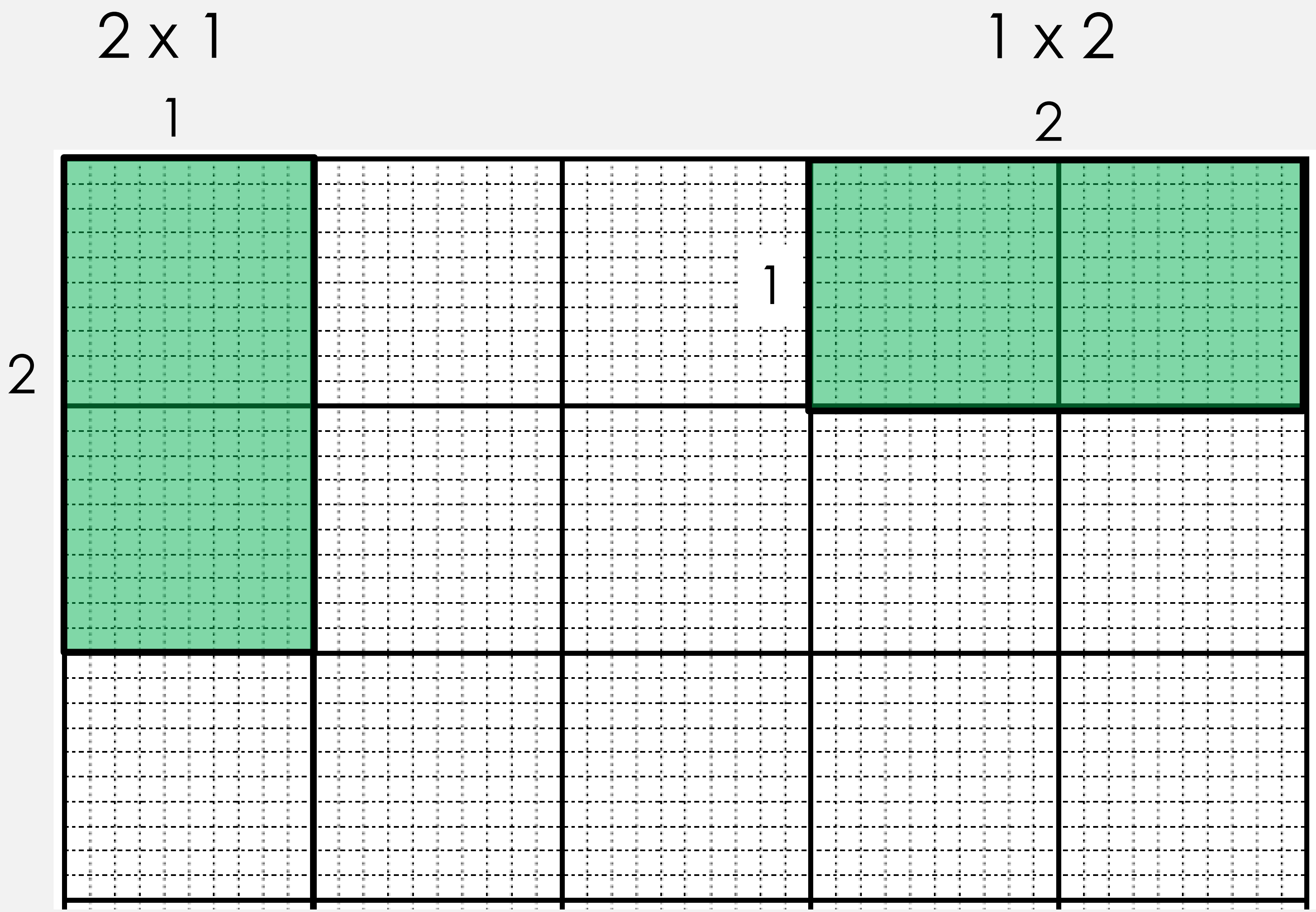
$\frac{5}{10}$



How do you use base ten grid paper to model multiplication of fractions?



Base ten grid paper: Rename the unit.
Each box of 100 squares equals one whole.

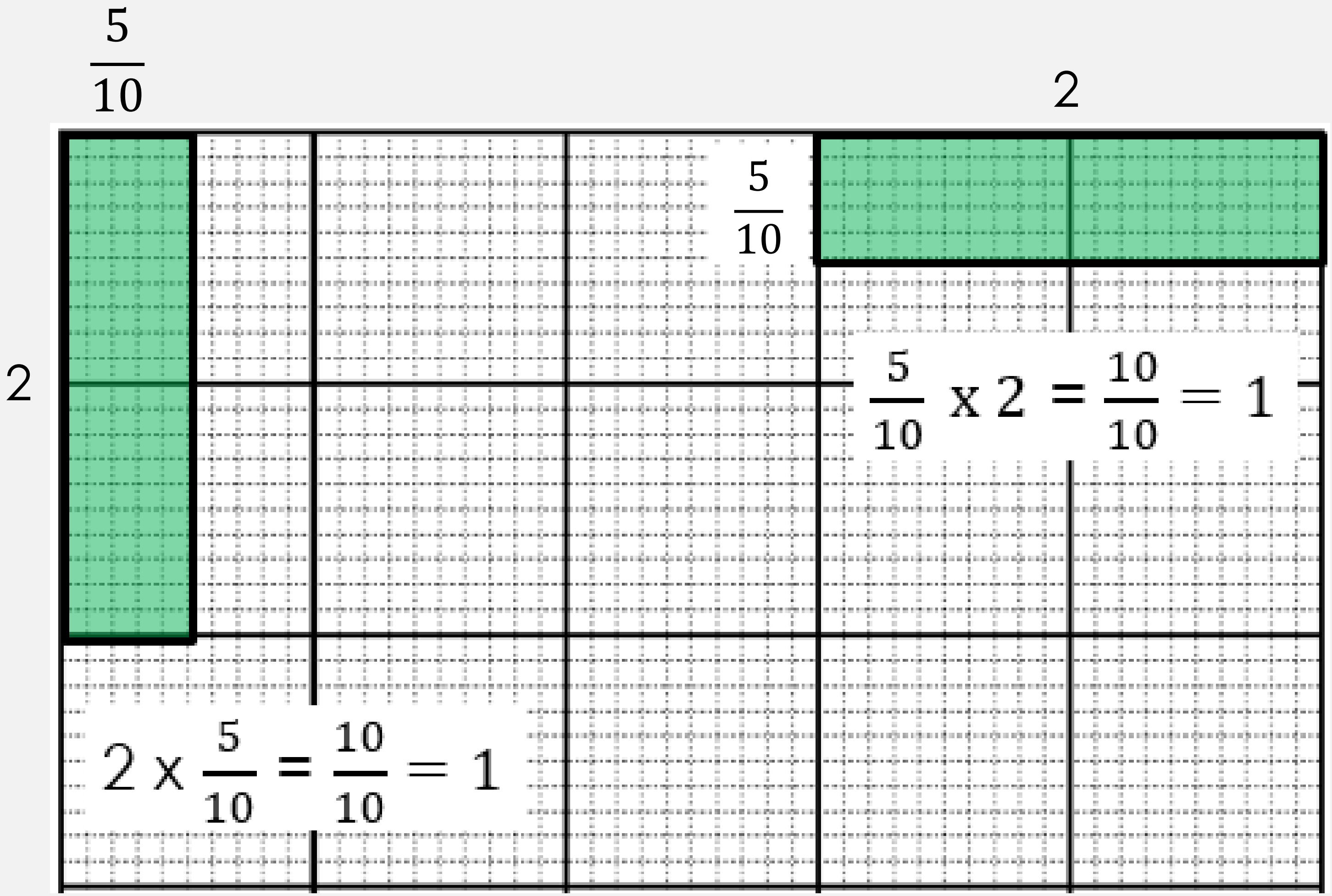


How will you model this problem?

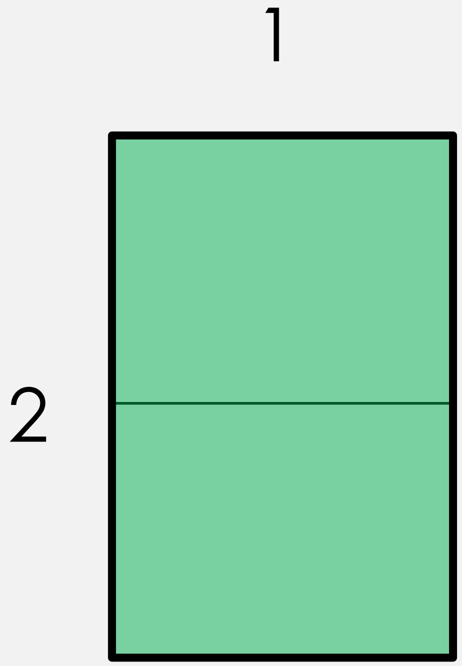
Now we can apply this model to multiplying decimal fractions: Whole number times a fraction or a fraction times a whole number.

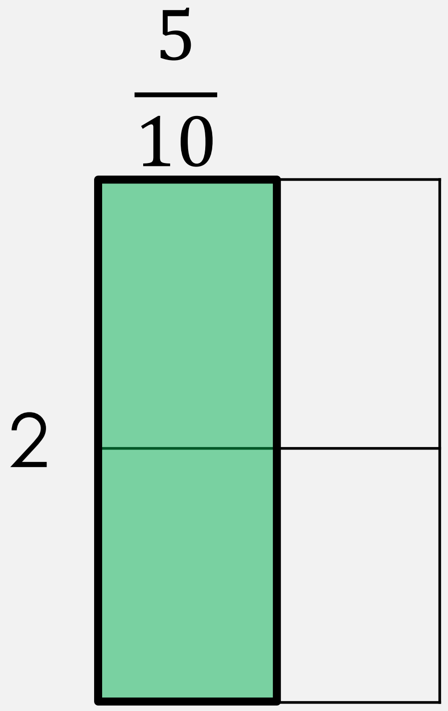
$2 \times \frac{5}{10}$

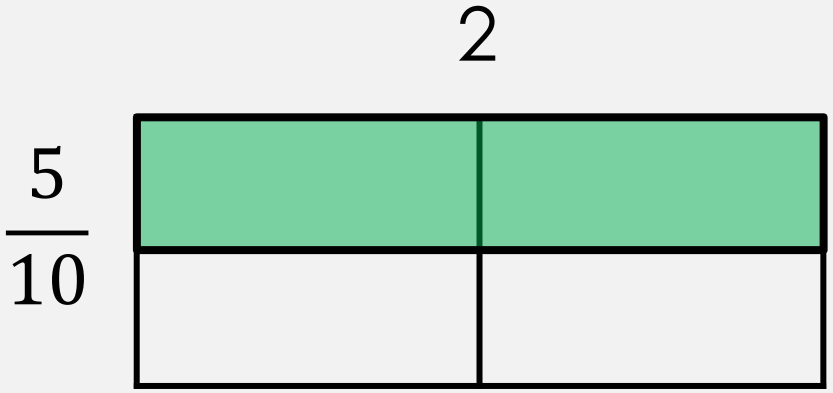
$\frac{5}{10} \times 2$



2×1 $\qquad 2 \times \frac{5}{10} = \frac{10}{10}$ or 1 $\qquad \frac{5}{10} \times 2 = \frac{10}{10}$ or 1



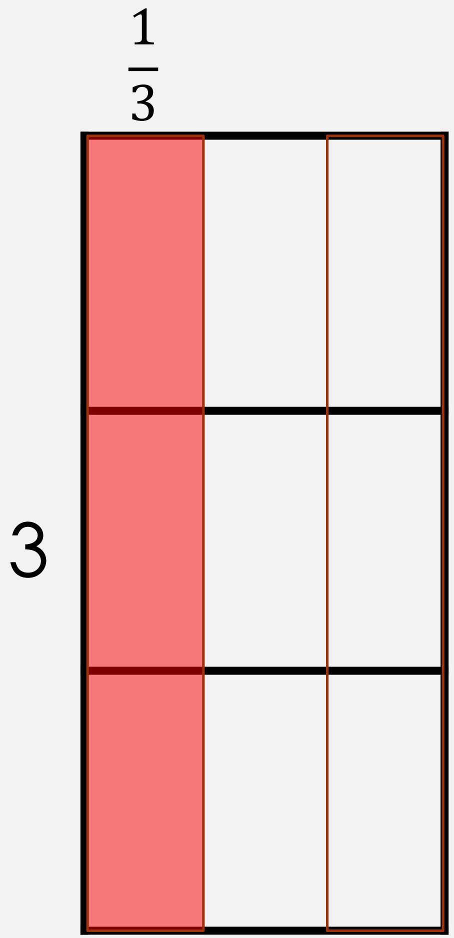




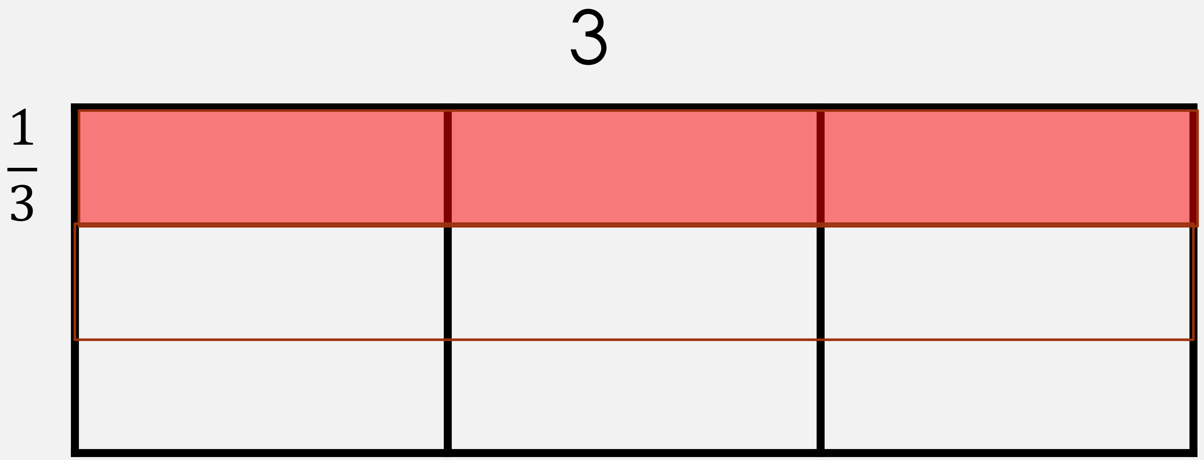
The area model also works well when multiplying decimal fractions.

We can use this model when multiplying common fractions.

$3 \times \frac{1}{3} = \frac{3}{3}$ or 1



$\frac{1}{3} \times 3 = \frac{3}{3}$ or 1



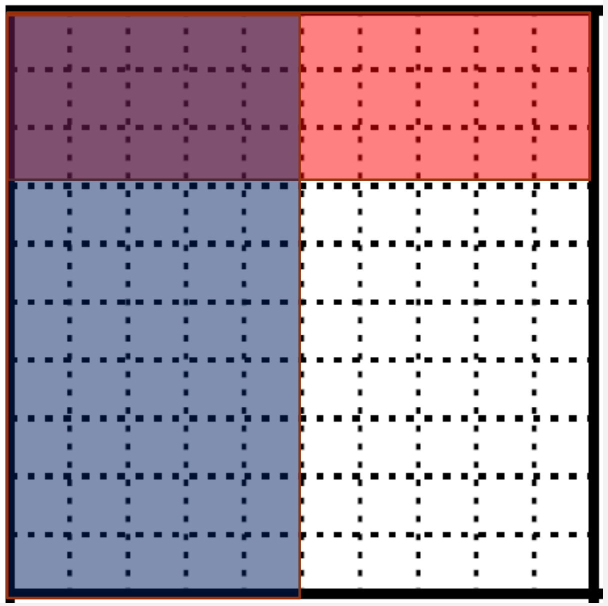
Decimal fraction multiplied by decimal fraction

$$\frac{3}{10} \times \frac{5}{10} = \frac{15}{100}$$

Another way to represent this.

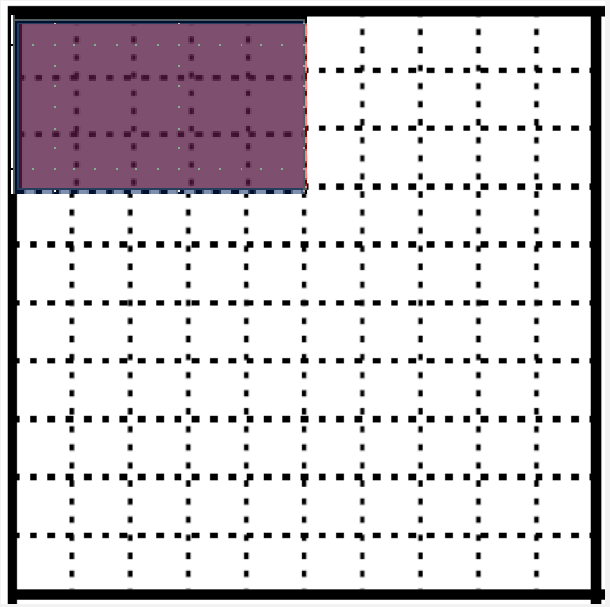
$\frac{5}{10}$

$\frac{3}{10}$



$\frac{5}{10}$

$\frac{3}{10}$



Common fraction multiplied by common fraction

What is $\frac{1}{2}$ of $\frac{3}{4}$?

Why do teachers race to the standard algorithm when working with fractions? Does the traditional algorithm build conceptual understanding?

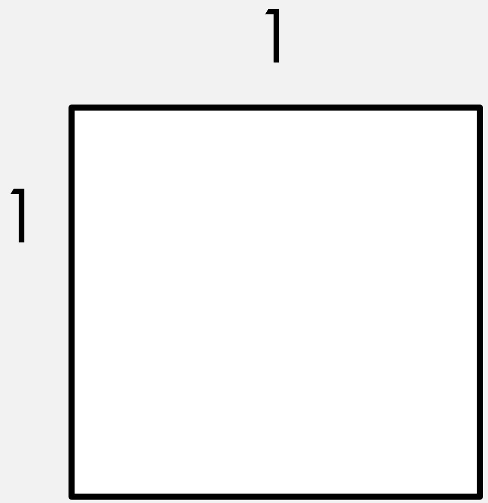
$\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$

But why is the product $\frac{3}{8}$?

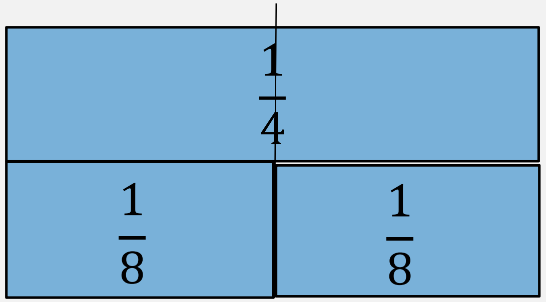
Common fraction multiplied by common fraction

What is $\frac{1}{2}$ of $\frac{3}{4}$?

1 x 1



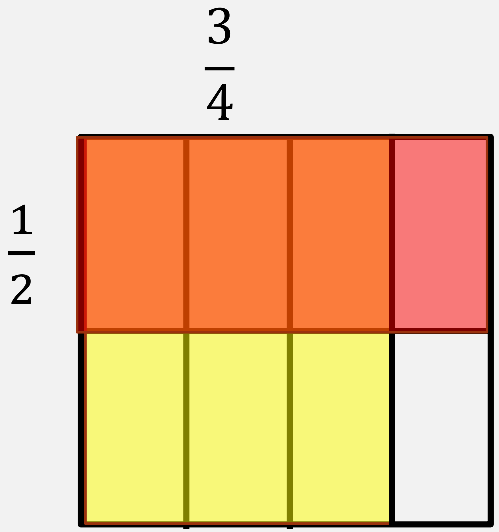
First, let's make sense of $\frac{1}{2}$ of $\frac{1}{4}$.



$\frac{1}{2}$ of $\frac{1}{4} = \frac{1}{8}$

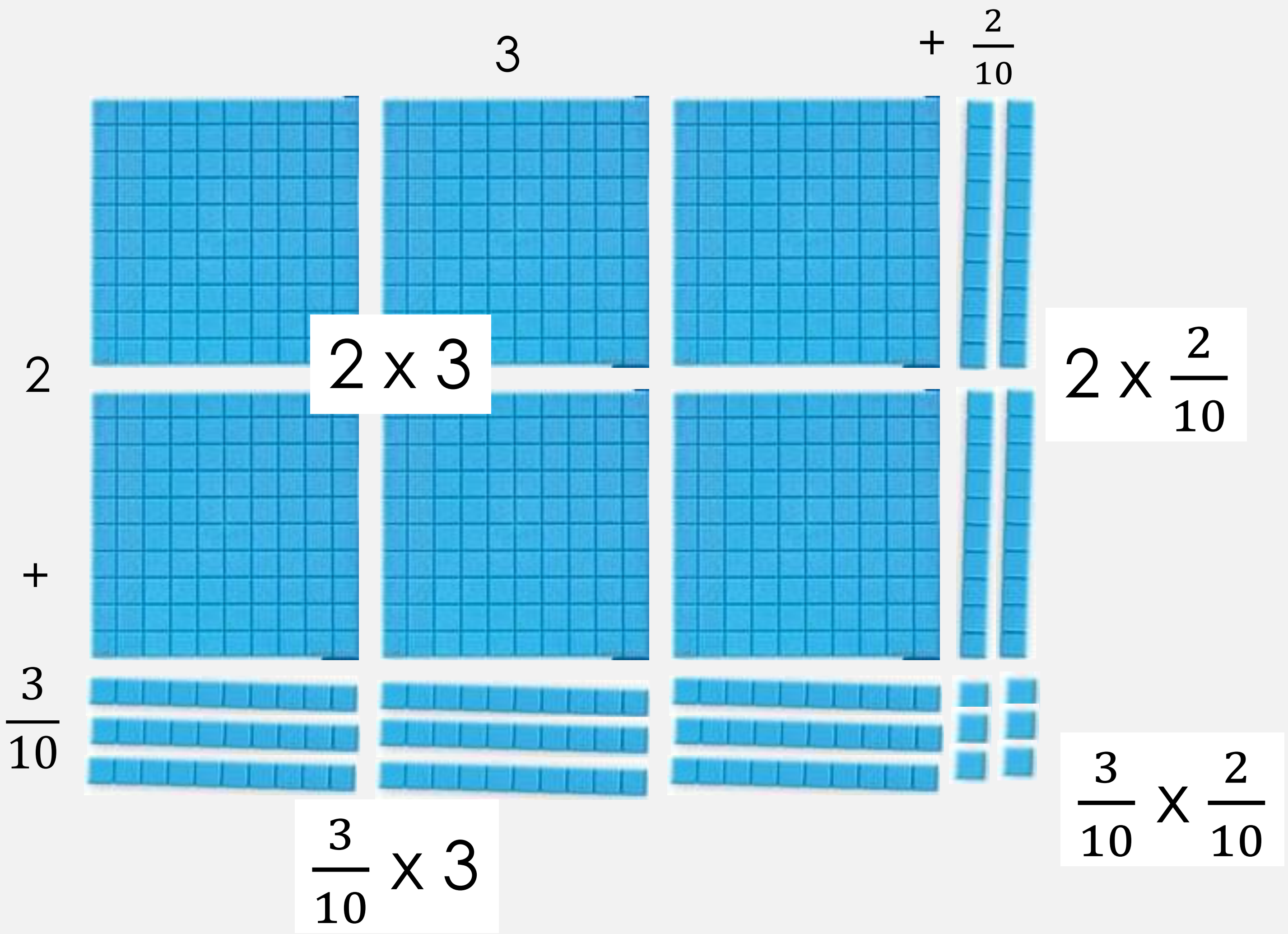
So, what is $\frac{1}{2}$ of $\frac{3}{4}$?

$\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$



Building arrays for mixed numbers with base ten blocks

$$2\frac{3}{10} \times 3\frac{2}{10} = ?$$



$$6 + \frac{4}{10} + \frac{9}{10} + \frac{6}{100} =$$

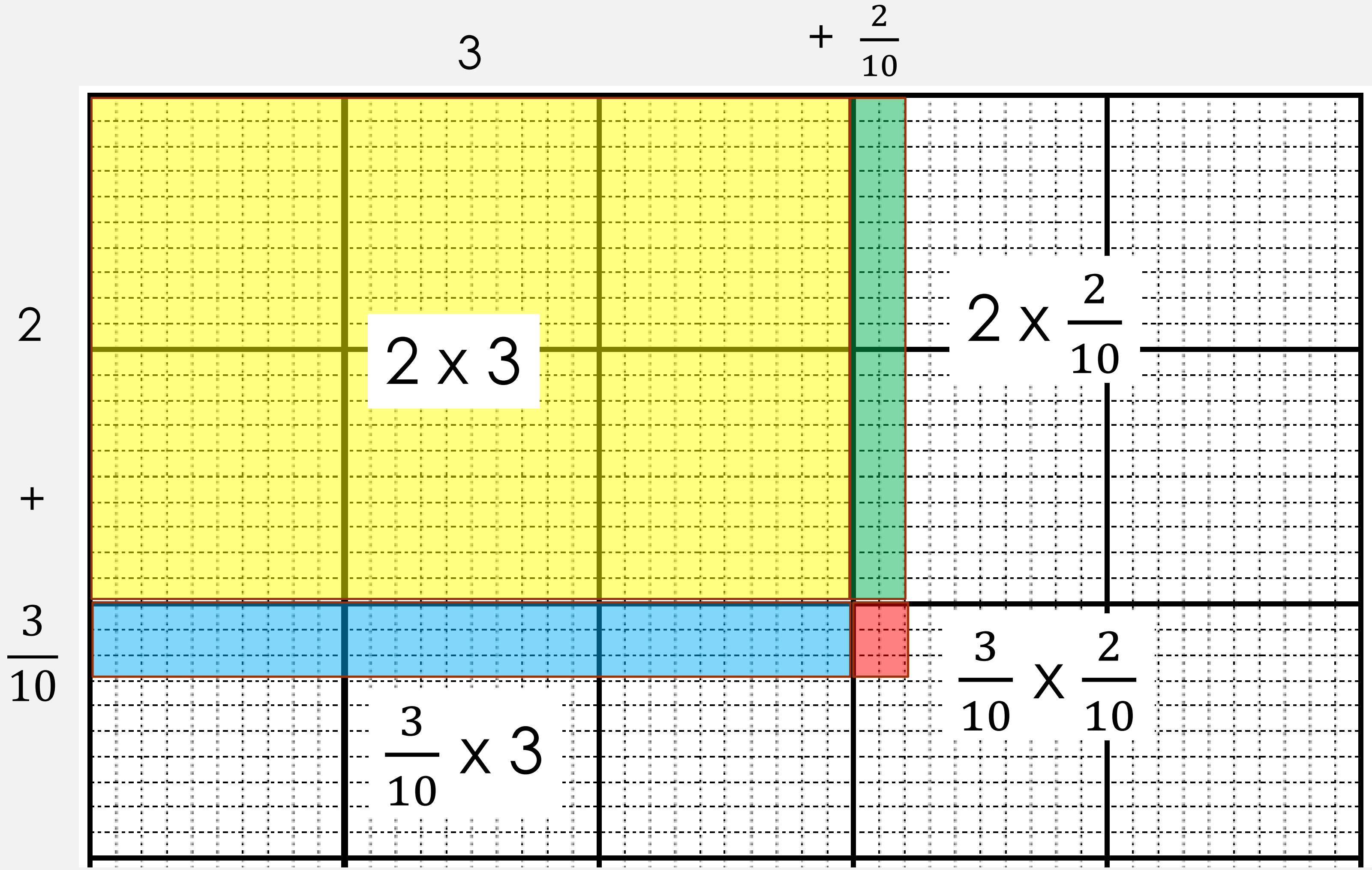
$$6 + \frac{13}{10} + \frac{6}{100} =$$

$$6 + 1 + \frac{3}{10} + \frac{6}{100} =$$

$$7 + \frac{30}{100} + \frac{6}{100} = 7\frac{36}{100}$$

Building arrays for mixed numbers with base ten grid paper

$$2\frac{3}{10} \times 3\frac{2}{10} = ?$$



$$6 + \frac{4}{10} + \frac{9}{10} + \frac{6}{100} =$$

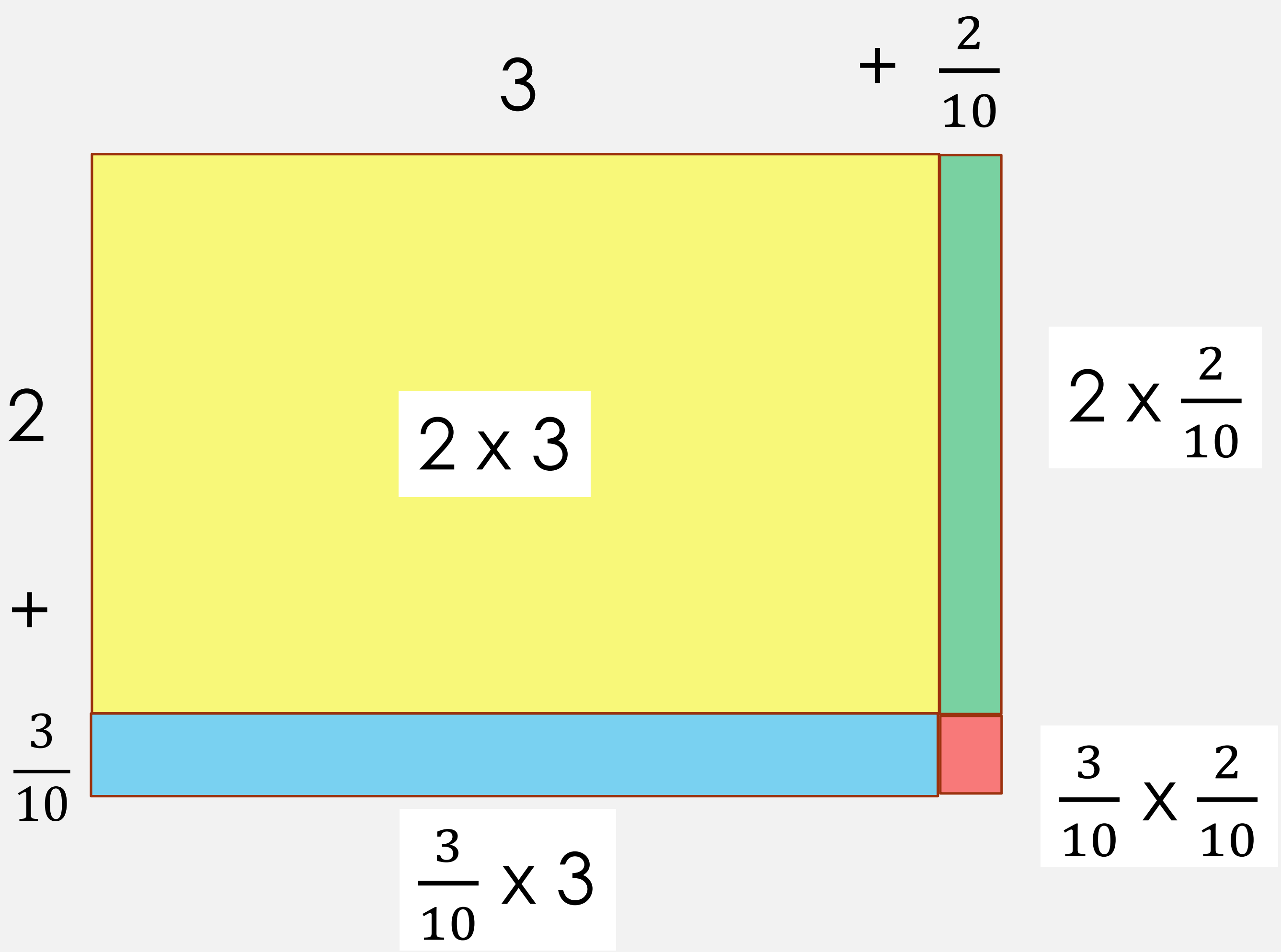
$$6 + \frac{13}{10} + \frac{6}{100} =$$

$$6 + 1 + \frac{3}{10} + \frac{6}{100} =$$

$$7 + \frac{30}{100} + \frac{6}{100} = 7\frac{36}{100}$$

Building arrays for mixed numbers without grid paper

$$2\frac{3}{10} \times 3\frac{2}{10} = ?$$



$$6 + \frac{4}{10} + \frac{9}{10} + \frac{6}{100} =$$

$$6 + \frac{13}{10} + \frac{6}{100} =$$

$$6 + 1 + \frac{3}{10} + \frac{6}{100} =$$

$$7 + \frac{30}{100} + \frac{6}{100} = 7\frac{36}{100}$$

Box multiplication with mixed numbers

$$2\frac{3}{10} \times 3\frac{2}{10} = ?$$

	3	$+$	$\frac{2}{10}$
2	$2 \times 3 = 6$		$2 \times \frac{2}{10} = \frac{4}{10}$
$+$			
$\frac{3}{10}$	$\frac{3}{10} \times 3 = \frac{9}{10}$		$\frac{3}{10} \times \frac{2}{10} = \frac{6}{100}$

$$6 + \frac{4}{10} + \frac{9}{10} + \frac{6}{100} =$$

$$6 + \frac{13}{10} + \frac{6}{100} =$$

$$6 + 1 + \frac{3}{10} + \frac{6}{100} =$$

$$7 + \frac{30}{100} + \frac{6}{100} = 7\frac{36}{100}$$

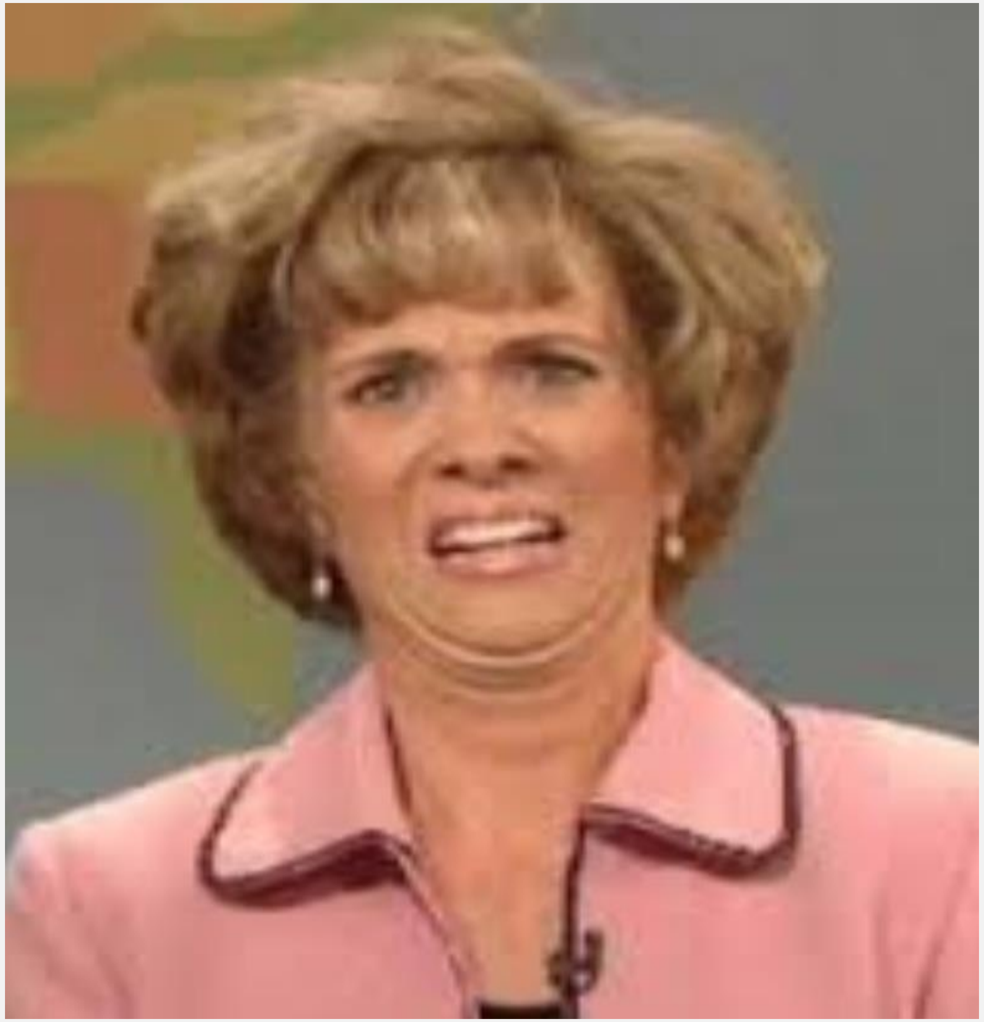
What would this look like if we used the traditional algorithm?

$$2\frac{3}{10} \text{ x } 3\frac{2}{10} = ?$$

$$2\frac{3}{10} \text{ x } 3\frac{2}{10} =$$

$$\frac{23}{10} \text{ x } \frac{32}{10} =$$

$$\frac{736}{100} = 7\frac{36}{100}$$



21 what? I got $\frac{420}{20}$.

How did you get 21?

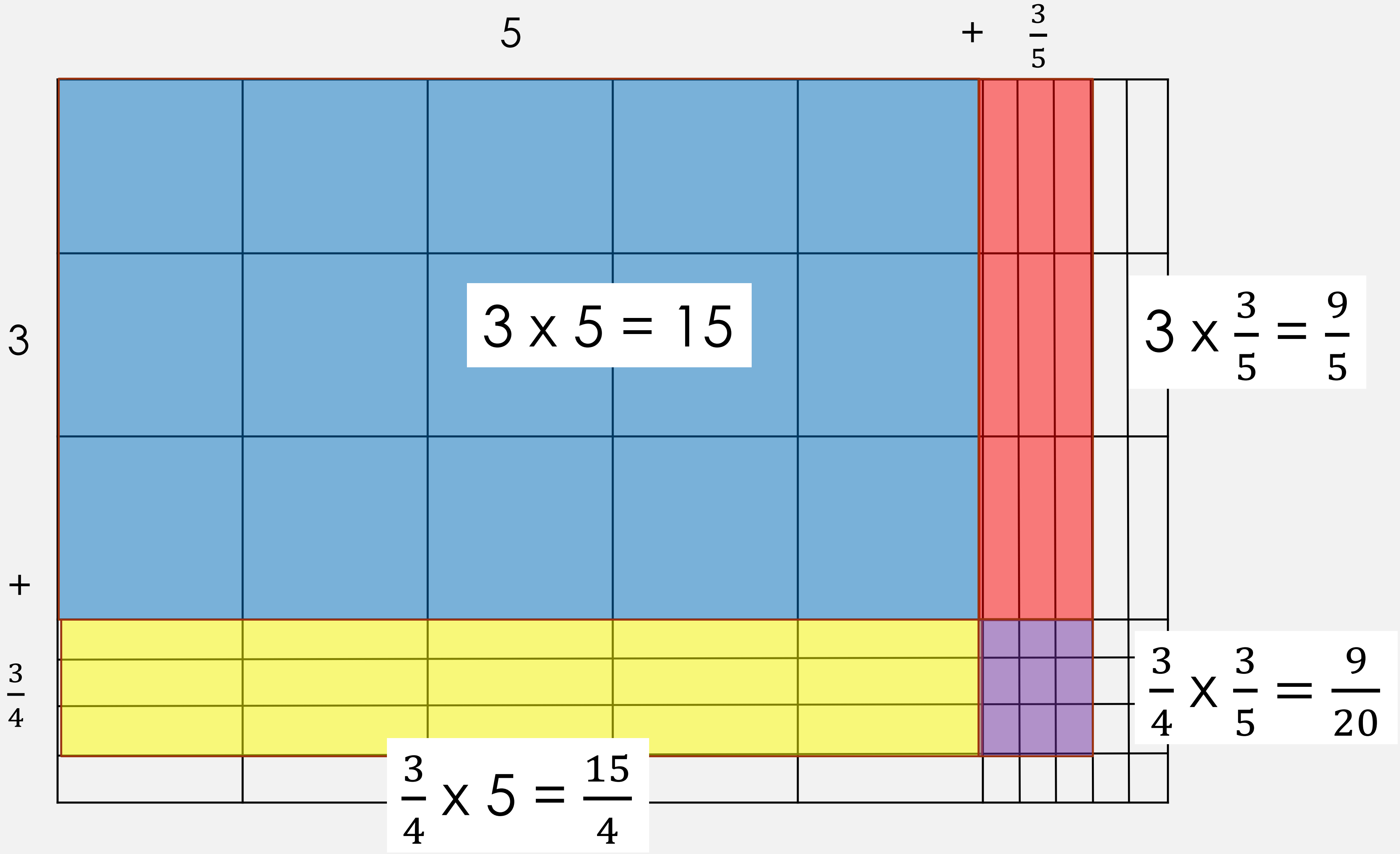
$$3\tfrac{3}{4} \times 5\tfrac{3}{5} =$$

$$\frac{15}{4} \times \frac{28}{5} = \frac{420}{20}$$

$$\frac{\overset{3}{\cancel{15}}}{\underset{1}{\cancel{4}}} \times \frac{\overset{7}{\cancel{28}}}{\underset{1}{\cancel{5}}} = \frac{21}{1} = 21$$

Building arrays for mixed numbers with grid paper

$$3\frac{3}{4} \times 5\frac{3}{5} = ?$$



Identifying the partial products for $3\frac{3}{4} \times 5\frac{3}{5} =$

$3 \times 5 = 15$

$3 \times \frac{3}{5} = \frac{9}{5} = 1\frac{4}{5}$

$\frac{3}{4} \times 5 = \frac{15}{4} = 3\frac{3}{4}$

$\frac{3}{4} \times \frac{3}{5} = \frac{9}{20}$

Adding the partial products together

$15 + 1\frac{4}{5} + 3\frac{3}{4} + \frac{9}{20}$

$15 + 1 + 3 = 19$

$\frac{4}{5} + \frac{3}{4} + \frac{9}{20} =$

$\frac{16}{20} + \frac{15}{20} + \frac{9}{20} = \frac{40}{20} = 2$

$19 + 2 = 21$

Multiplying Decimals

How do the whole number multiplication models work with decimals?

Building arrays for decimals with base ten blocks

We have to change the value of the base ten blocks.

one whole

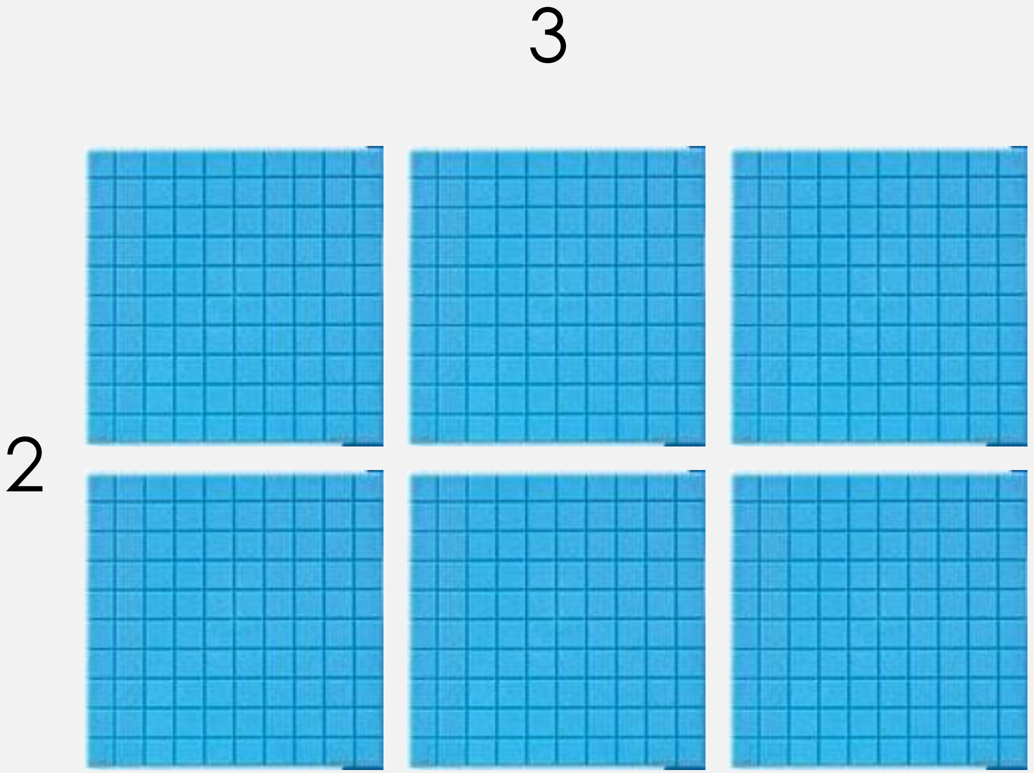
one tenth

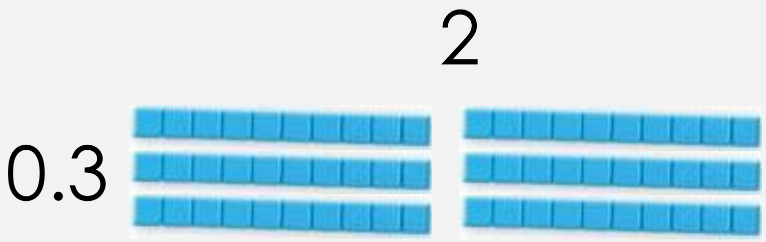
one hundredth

2 x 3 = 6

2 x 0.3 = 0.6

0.3 x 2 = 0.6





Nowhere am I telling the students to multiply 2 x 3 and move the decimal one place to the left.

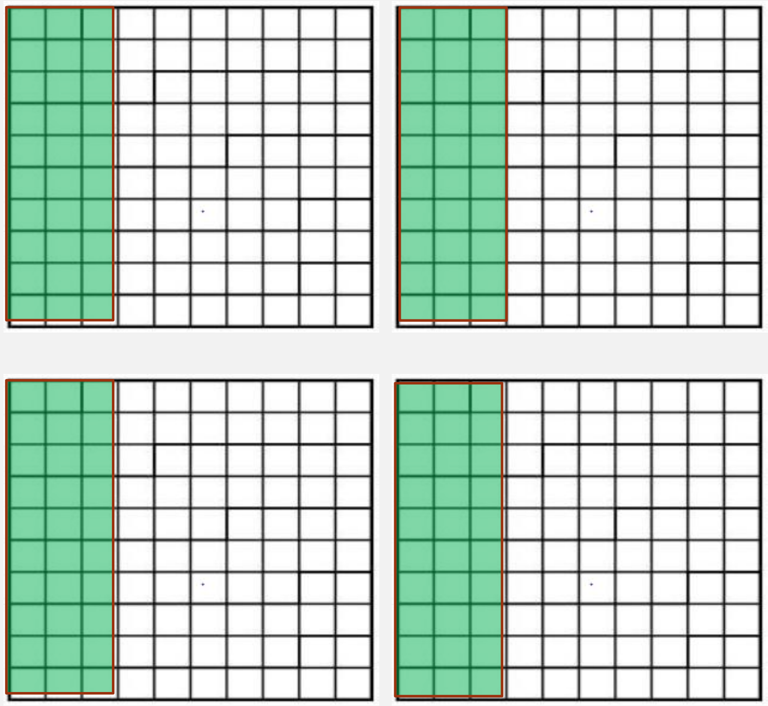
The visual models help the students see why the answer is 0.6.

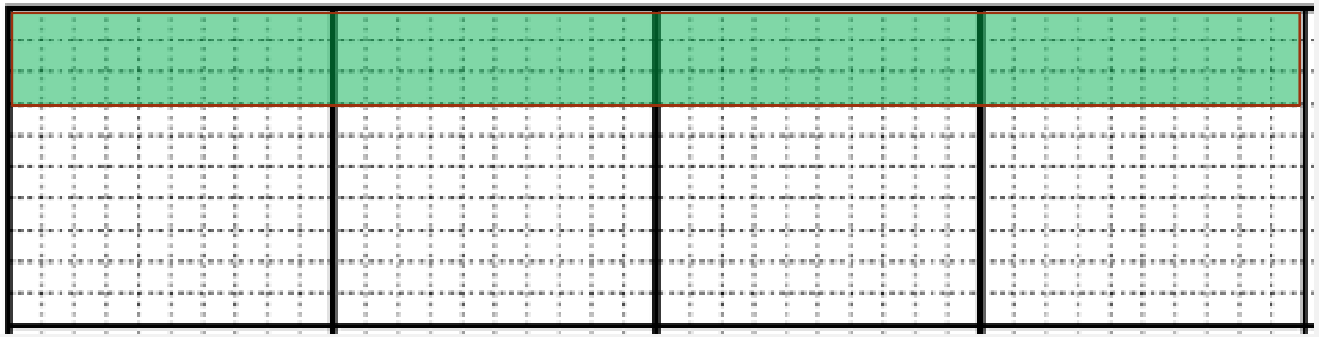
We should not to tell the students to multiply 2 x 3 and move the decimal point one place to the left.

We do not move the decimal point. It has a fixed position in our place value system.

Instead, we place the decimal point where it will assign proper place value to all of the digits in the product.

Finding 0.3 of a group of 4



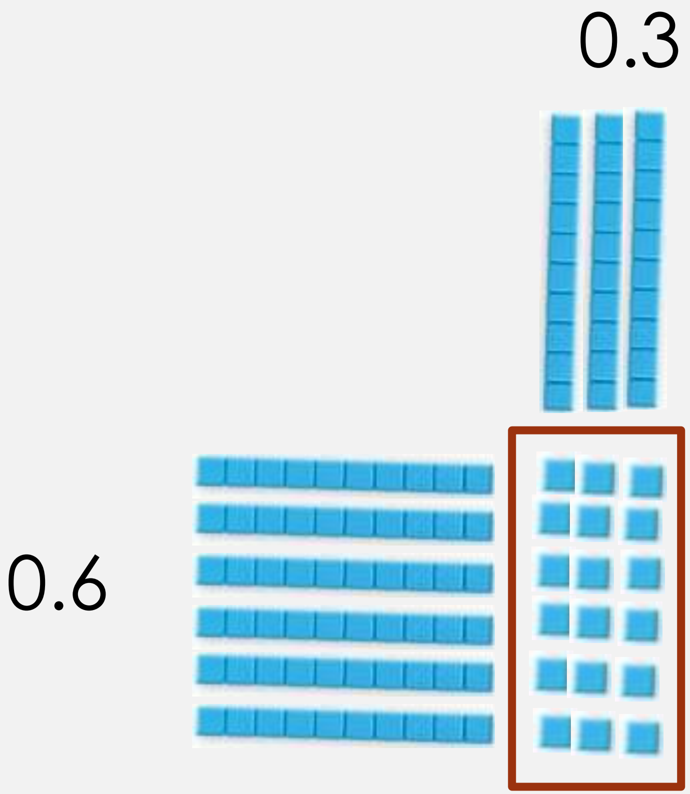


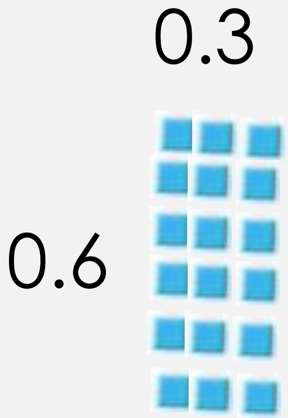
If I can find 0.3 of a group of 1.

I can use that to find 0.3 of a group of 4, which is 12 tenths or 1.2.

Using base ten blocks to multiply decimals can get messy.

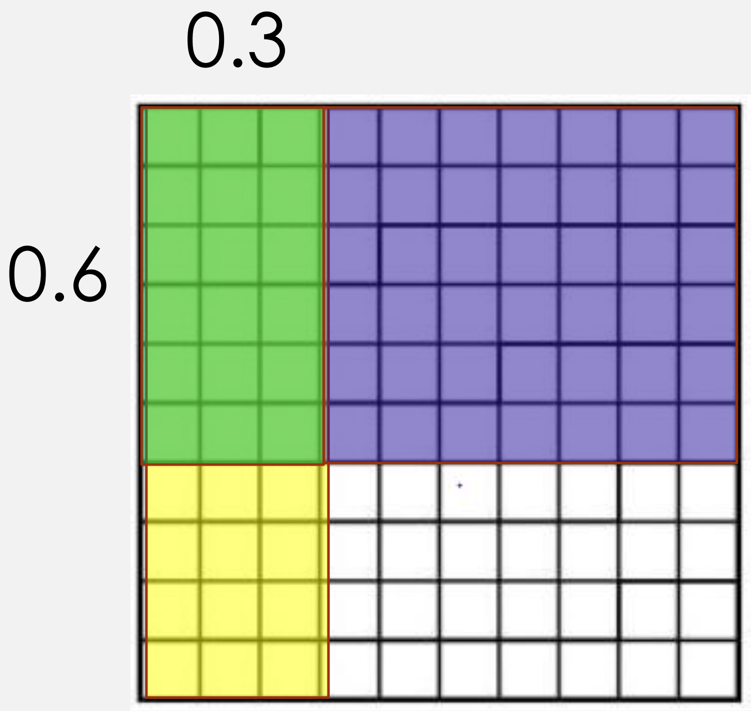
0.6 x 0.3 =0.18

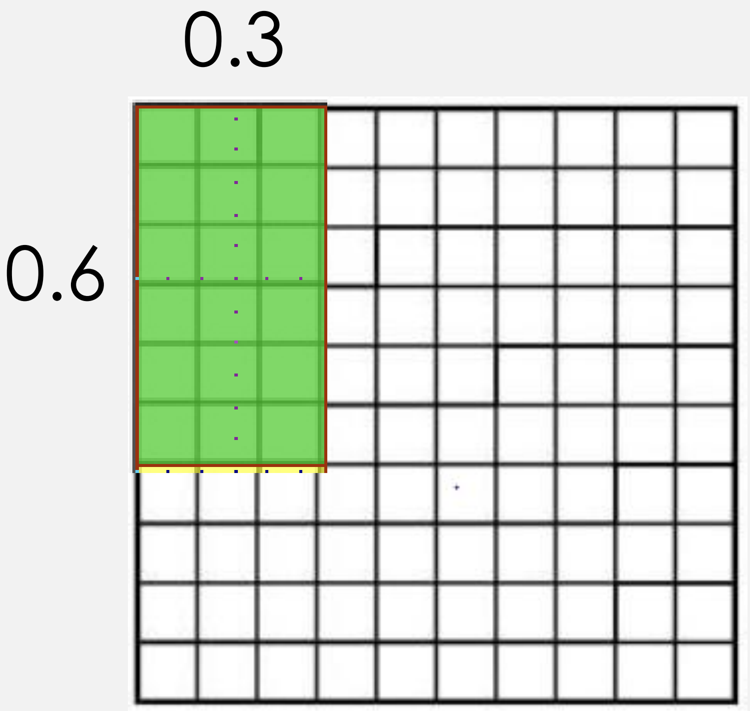




Base ten grid paper tends to be easier to follow.

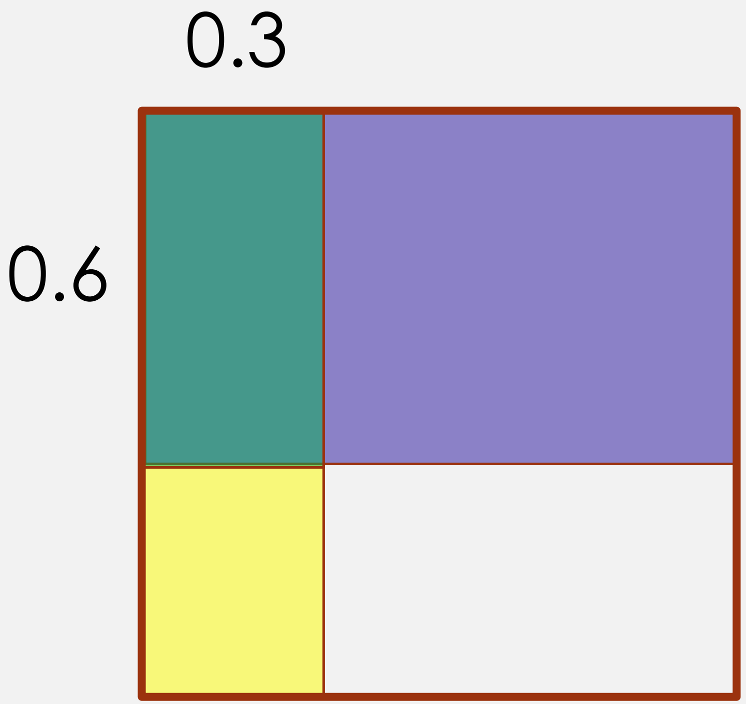
0.6 x 0.3 = 0.18

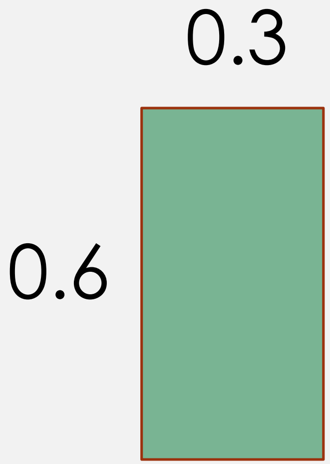




This loses some of the meaning.

0.6 x 0.3 = 0.18

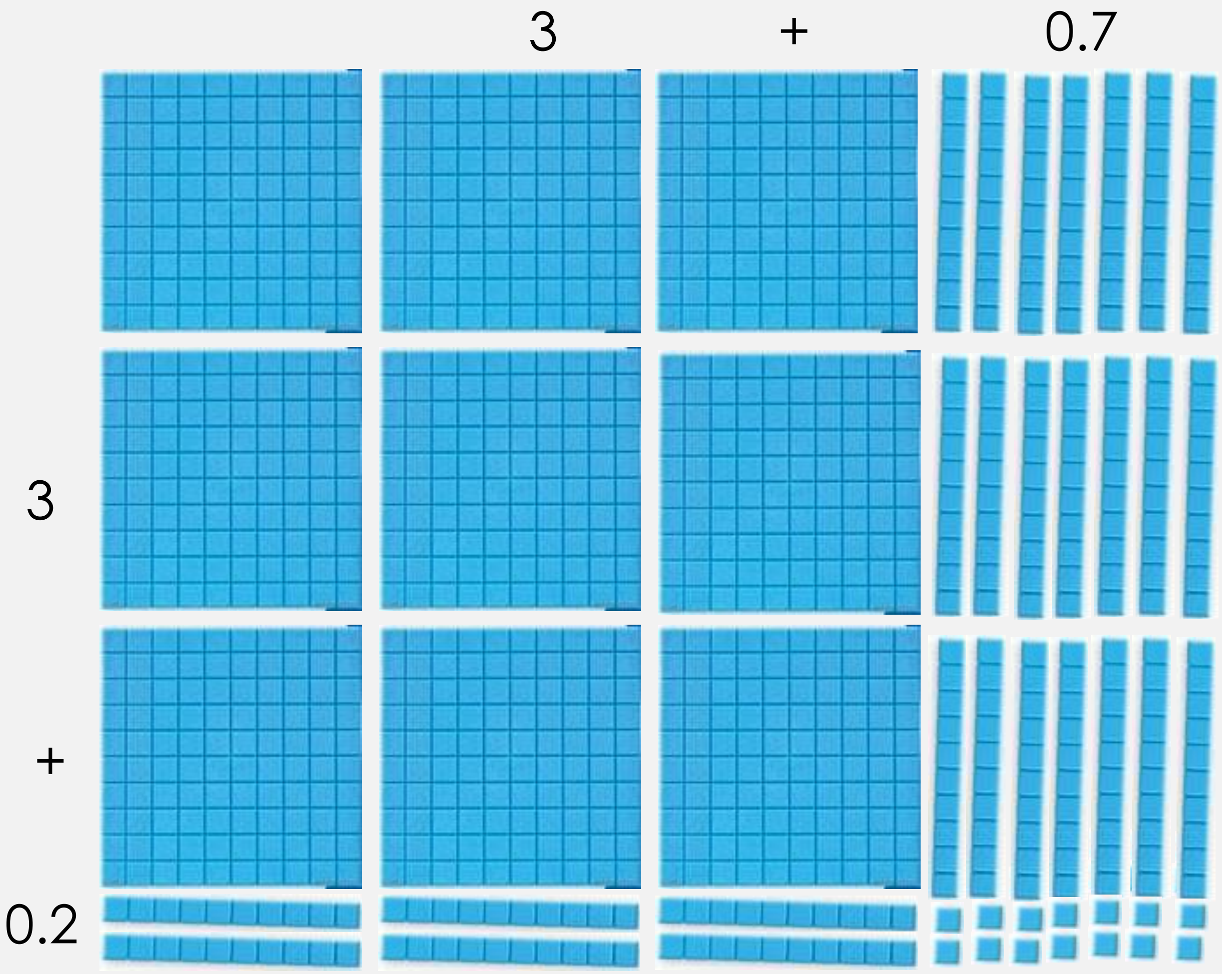




Unless I already know 0.6 x 0.3 = 0.18 this provides no help.

Building arrays for decimals using base ten blocks

3.2 x 3.7= ?

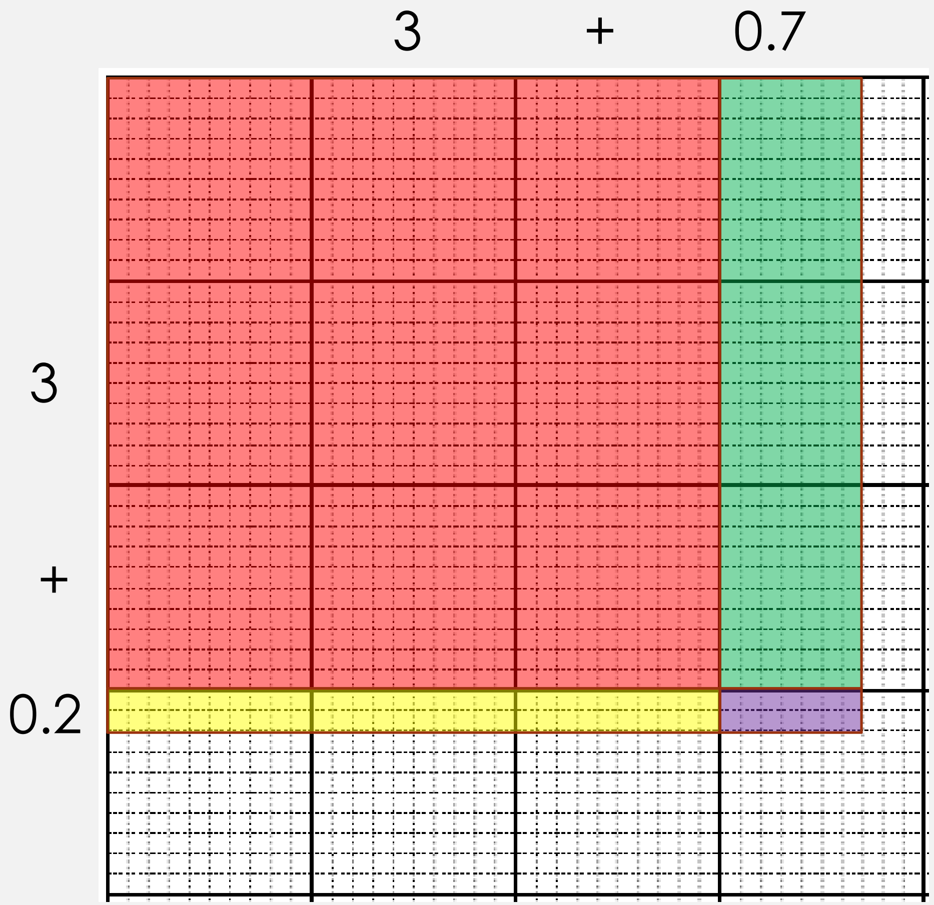


3.2 x 3.7

3 x 3	= 9.00
3 x 0.7	= 2.10
0.2 x 3	= 0.60
0.2 x 0.7	= 0.14
	11.84

Building arrays for decimals using base ten grid paper

3.2 x 3.7= ?

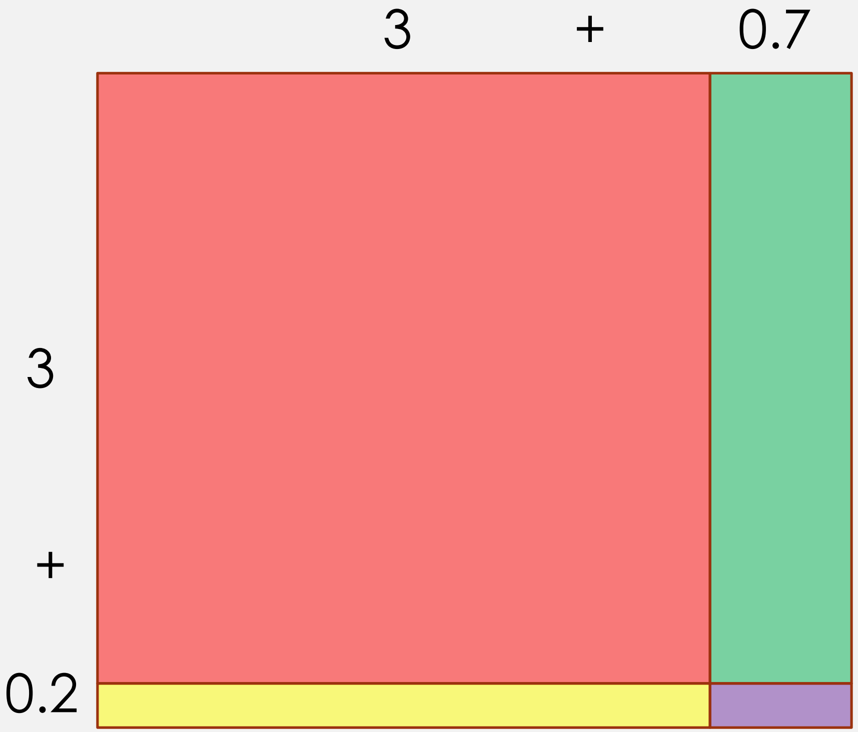


3.2 x 3.7

3 x 3	= 9.00
3 x 0.7	= 2.10
0.2 x 3	= 0.60
0.2 x 0.7	= <u>0.14</u>
	11.84

Building area models for decimals

3.2 x 3.7= ?



3.2 x 3.7

3 x 3	= 9.00
3 x 0.7	= 2.10
0.2 x 3	= 0.60
0.2 x 0.7	= 0.14
	11.84

Box multiplication or just record the partial products?

	3	+	0.7
3	3 x 3		3 x 0.7
+			
0.2	0.2 x 3		0.2 x 0.7

3.2 x 3.7

3 x 3 = 9.00
3 x 0.7 = 2.10
0.2 x 3 = 0.60
0.2 x 0.7 = <u>0.14</u>
11.84

Let's revisit our goals

- Become familiar with the models for whole number multiplication.
- Follow a learning progression for multiplication of fractions and multiplication of decimals that builds procedural fluency from conceptual understanding.
- Learn how to apply whole number multiplication models to fraction and decimal multiplication

Principles to Action: Effective Mathematics Teaching Practices

- Establish mathematics goals to focus learning.
- Implement tasks that promote reasoning and problem solving.
- Use and connect mathematical representations.
- Facilitate meaningful discourse.
- Pose purposeful questions.
- Build procedural fluency from conceptual understanding.
- Support productive struggle in learning mathematics.
- Elicit and use evidence of student thinking.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Resources

NCTM. Developing Essential Understanding of Rational Numbers for Teaching Mathematics Grades 3-5. Reston, VA; NCTM 2010

NCTM. Putting Essential Understanding of Fraction into Practice 3-5. Reston, VA; NCTM 2013

NCTM. Putting Essential Understanding of Multiplication and Division into Practice 3-5. Reston, VA; NCTM 2013

NCTM. Principles to Actions; Ensuring Mathematical Success for All. Reston, VA; NCTM 2014

Insert NCTM page