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What are Trapezoids? A Debate on the Inclusive vs. Exclusive Definition

Thursday, April 26, 2018 1:30-2:30
NCTM National Convention, Washington DC

Will Rose and John Chase

[1:30]

[Have slide up that gives instructions to access the poll so that audience members can get their technology up and running and pull up the website.] [Encourage people to do the poll!]

[Will]

Welcome everyone, we're here today to talk about the most important thing I can possibly think of… trapezoids. I'm Will Rose and I've been teaching for 12 years at Montgomery Blair High School in nearby Silver Spring, MD. A fun fact about me is that I live one block away from here.

[John]

And I'm John Chase, and I teach in the same county as Will Rose and a fun fact about me is that I love to juggle. I can juggle 7 balls at a time!

This afternoon, we will be having a lively dialogue around something that lies at the very heart of our geometry curriculum. We will be discussing the definition of a trapezoid.

Some of you are very familiar with the debate, but others may have no idea what we're talking about. You didn't know that the definition of trapezoid was contentious and you didn't know there even *were* two possible definitions! This debate will be for everyone in this audience.

I will be arguing for the Inclusive Definition as my position today and my slides will be BLUE.

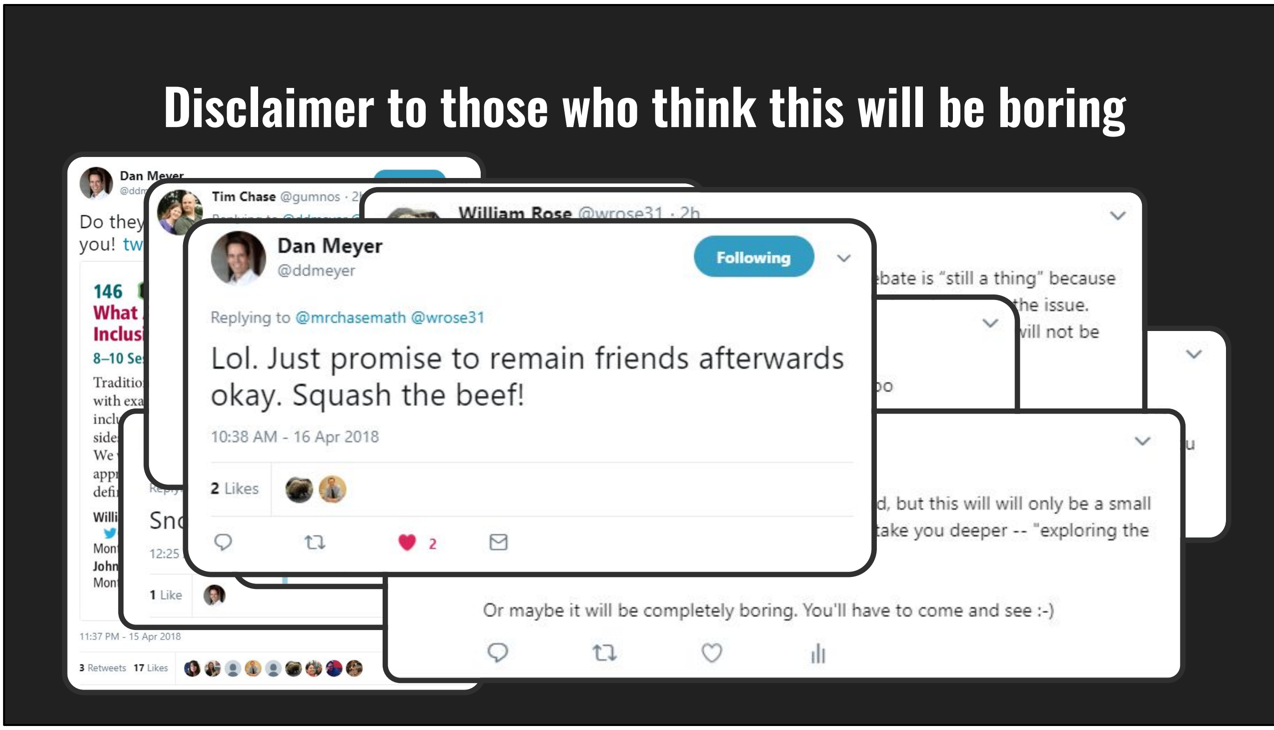
[Will]

And I will be arguing for the Exclusive Definition and my slides will be MAROON.

[John]

We will soon launch into a full-on discussion of today’s topic, but first let’s first take the temperature of the room with a little poll. For those of you who are aware of this controversy and have given some thought to it already, Review the options and then be prepared to raise your hand on one of these four options. Be bold and unashamed! Everyone has to vote.

1. I prefer the exclusive definition
2. I prefer the inclusive definition
3. I’m indifferent
4. I’ve never heard of this debate



[Will]

Now, before we begin, I need to say that we got a little flack on twitter for trying to "rehash" this debate and I thought we needed to give a little disclaimer about the talk.

A colleague of mine heard that I was giving a talk on the definition of trapezoids and he accused me of giving a caricature of an NCTM session.
My wife thinks I'm crazy. She told her entire office yesterday that I was giving a talk on trapezoids and I think they all laughed at me.

But it's actually not a joke. We were inspired to give this talk based on the many many math teacher blog posts we've read of teachers all around the country struggling to work this out. Often this is a teacher who was raised with one approach finding themselves teaching in a district that has adopted a different standard. There is a log of dialogue back and forth on this issue and many people are very angry and very convinced that they are right.

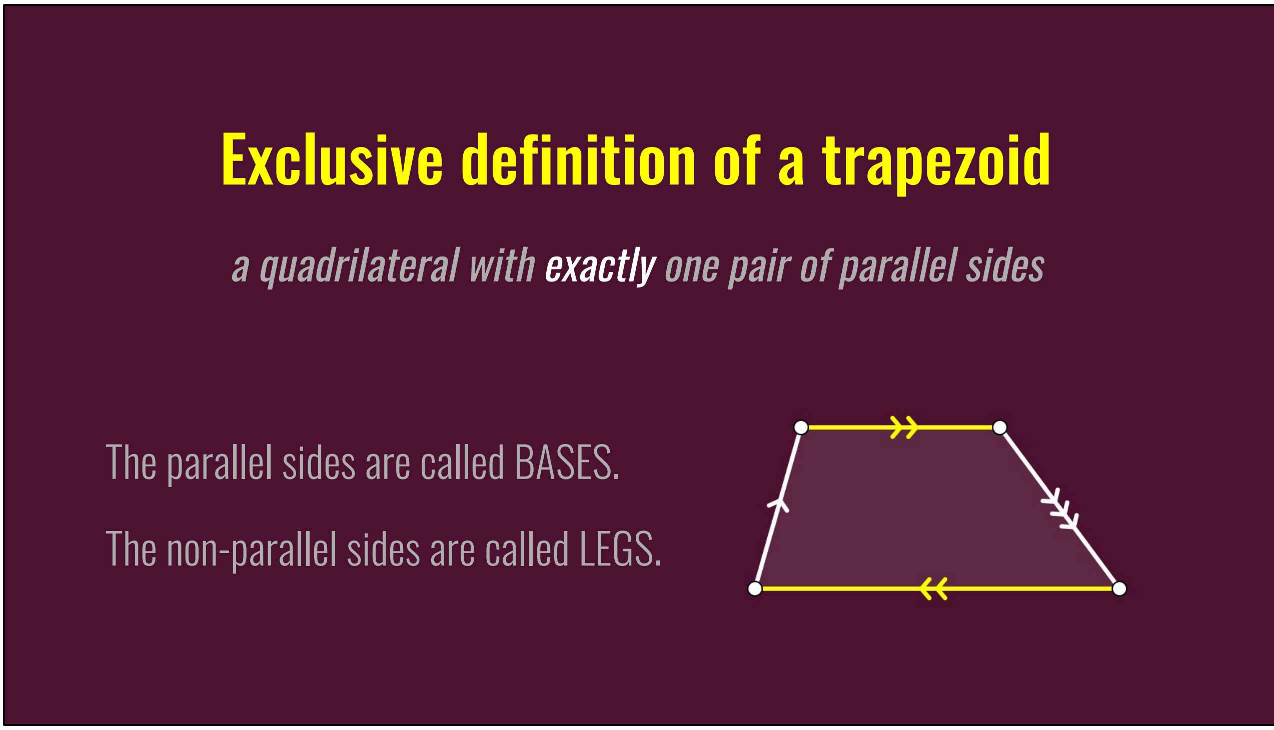
We think that people have been mostly talking past each other and we'd like to demystify what's going on and try to explain some of the intransigence. This is a divisive issue, so we'd like to teach the controversy by first trying to fully understand the controversy. And we really that not everyone has the 200 hours or whatever that we had to devote to this topic.

[Will]

John and I have actually been discussing the definition of a trapezoid for as long as we've been friends.

And here are some memories from some of the infancy of our debate, way back in 2015.

Let the debate begin!



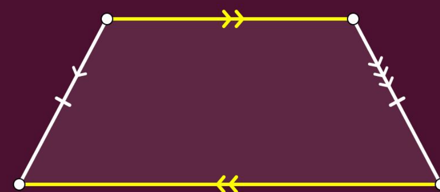
[1:35] [Will]

Traditionally, a trapezoid is defined as a quadrilateral with exactly one pair of parallel sides.
Define bases, legs, isosceles, etc.

Definition of isosceles trapezoid

a trapezoid with congruent legs

Just as a trapezoid is fundamentally a truncated triangle, an isosceles trapezoid is fundamentally a **truncated isosceles triangle**.



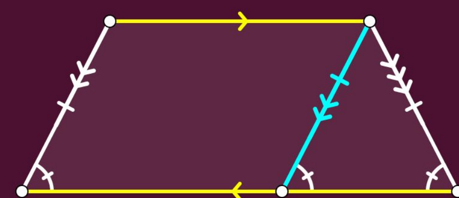
"isosceles" via late Latin from Greek isoskelēs, from isos 'equal' + skelos 'leg.'

[Will]

A necessary property of isosceles trapezoids

If a trapezoid is isosceles (legs are congruent), then the base angles are congruent.

- Since the right leg isn't parallel to the left leg, construct the line that is.
- By parallelogram properties, this new segment is equal to the left leg.
- The triangle on the right is therefore isosceles, with congruent base angles.



[Will]

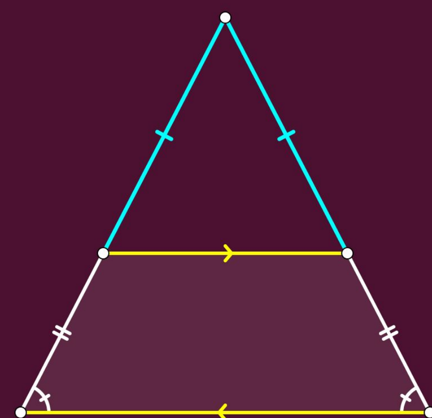
So those are the basic traditional definitions you know and love. And just to complete the whole picture of the traditional approach, allow me to show a few select proofs to showcase this classic, timeless approach to trapezoids.

Since the leg on the right ISN'T PARALLEL to the leg on the left, construct the line through the top-right vertex that IS PARALLEL to the leg on the left.

A sufficient condition for a trapezoid to be isosceles

If the base angles are congruent, then the trapezoid is isosceles.

- Since the legs are not parallel, extend them until they intersect.
- Since the base angles are equal, the small triangle and large triangle are both isosceles.
- The trapezoid has equal legs by segment subtraction.

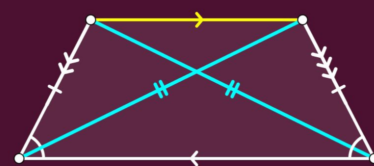


[Will]

Isosceles trapezoids have congruent diagonals

If a trapezoid is isosceles, the diagonals are congruent.

- The base angles of the isosceles trapezoid are equal by an already proved theorem.
- The large overlapping triangles are congruent by SAS.
- Congruent diagonals follow.

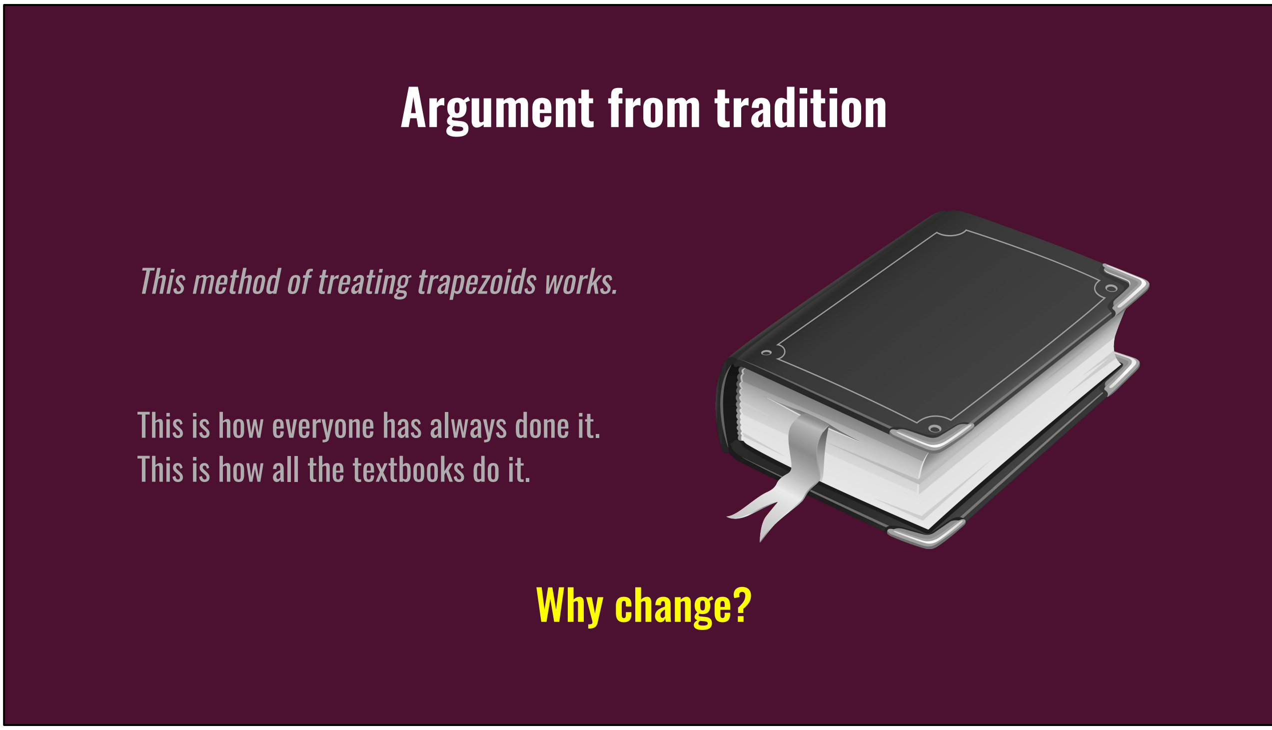


[Will]

A sufficient condition for a trapezoid to be isosceles

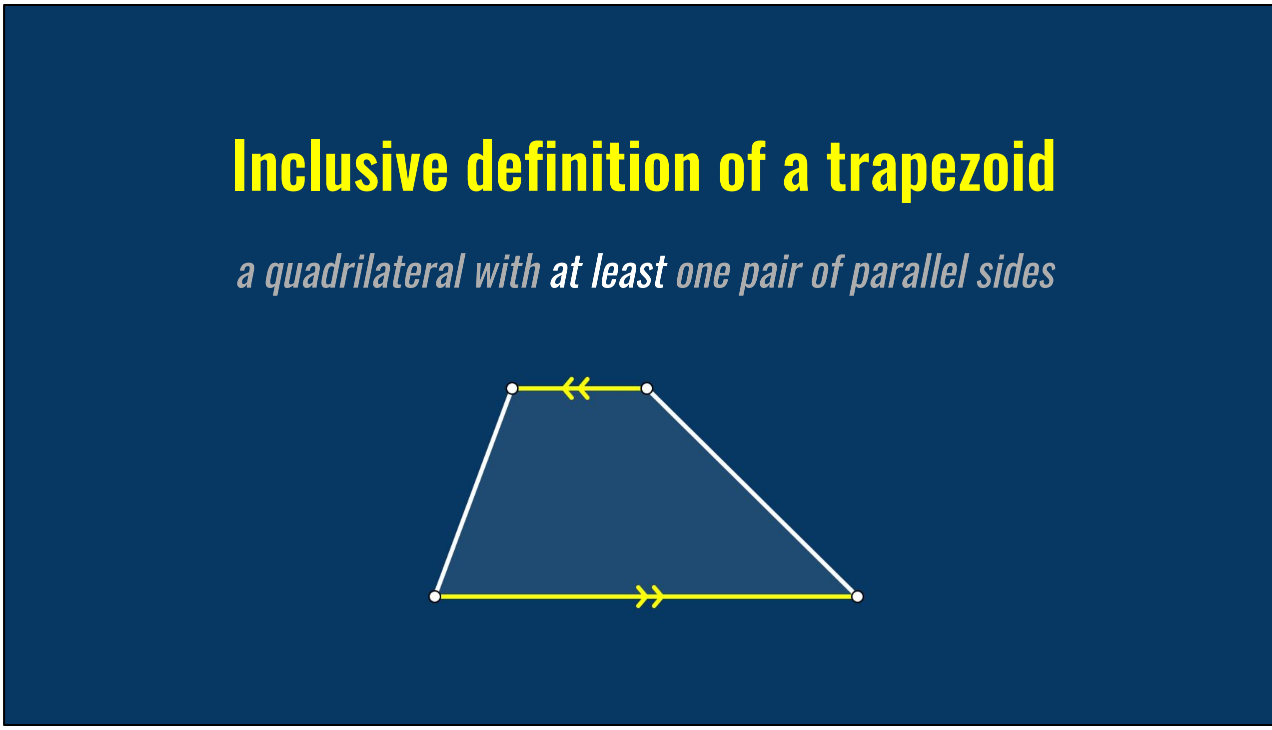
If the diagonals of a trapezoid are congruent, then it is isosceles.

- Drop perpendiculars to the bottom base, which are equal since the bases are parallel
- Each diagonal is the hypotenuse of a large right triangle. These overlapping large right triangles are congruent by HL
- The overlapping segments along the bottom base are equal
- By segment subtraction and then SAS, the small right triangles are congruent



[Will]

I’ve recently heard that there are some rebels proposing another definition of trapezoid. The onus is on them to show they can do a better job than what I’ve done here.

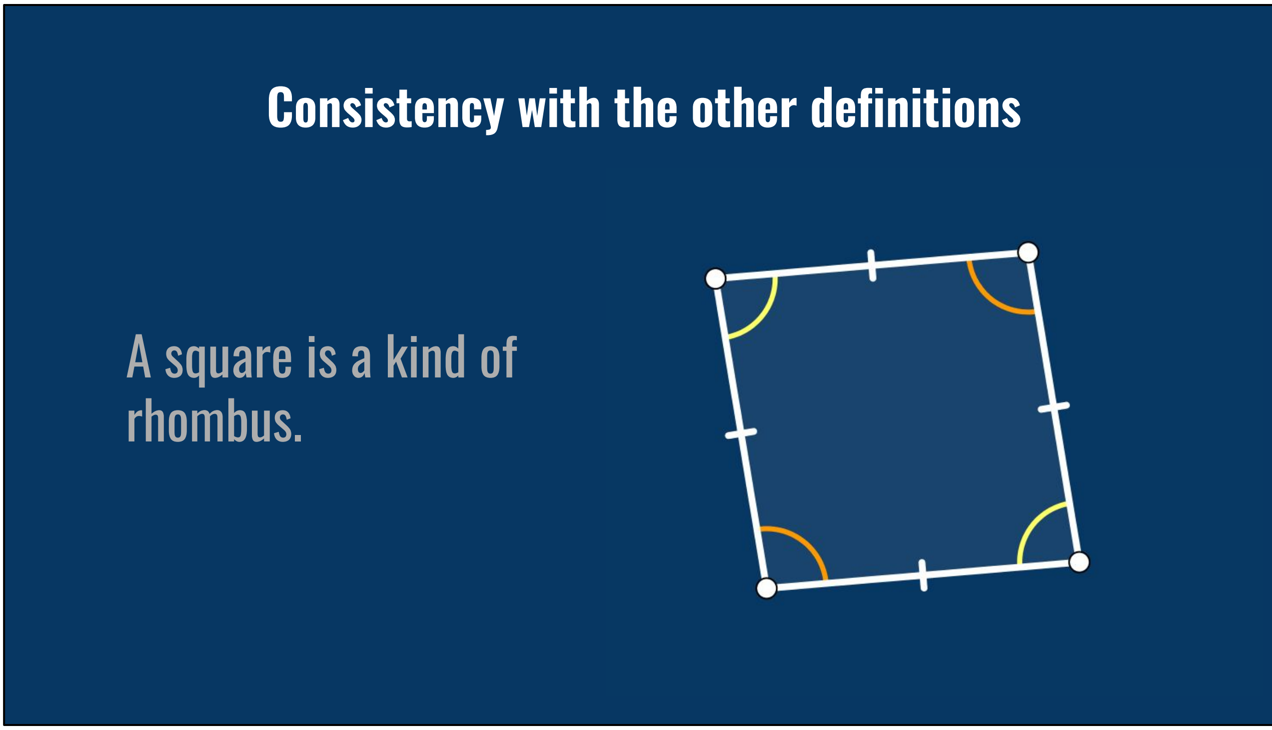


[1:45] [John]

Thank you, Will. Let me now present the inclusive definition:

*"A quadrilateral with **at least** one pair of parallel sides."*

What makes the inclusive definition more elegant and powerful? The same thing that makes the inclusive definition attractive in every other case.



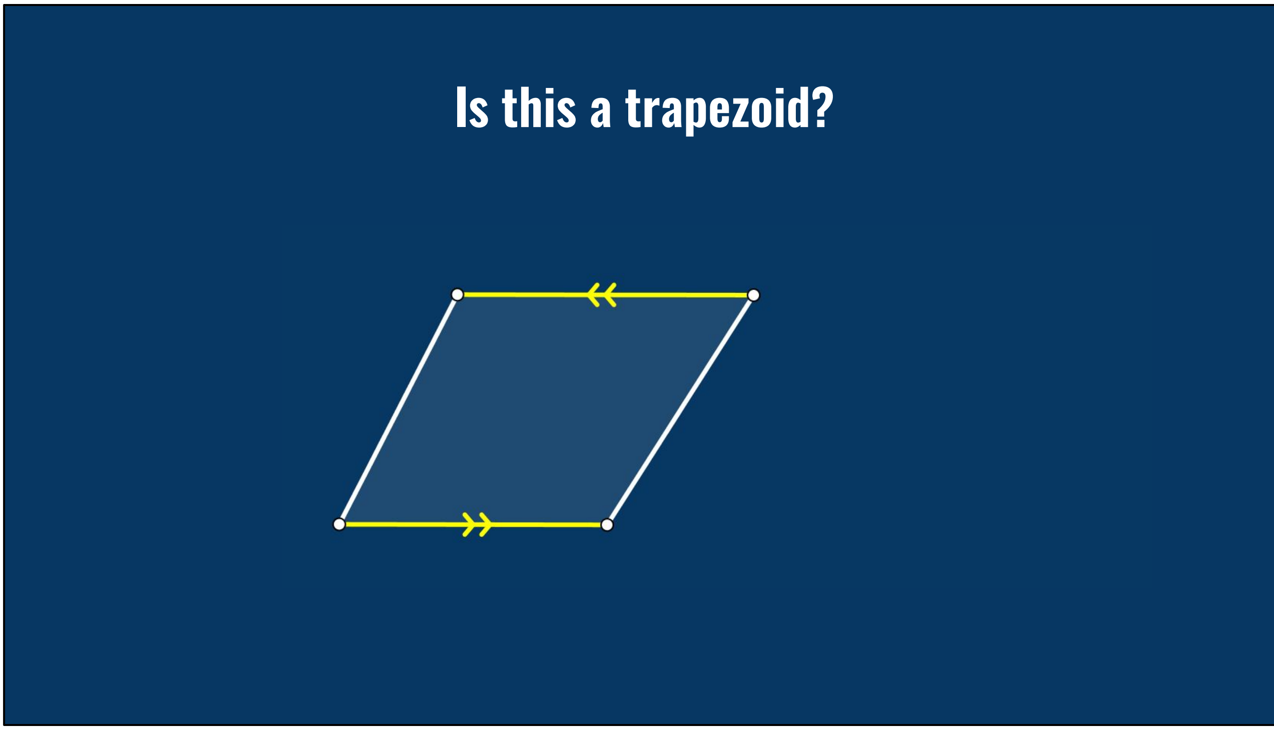
[John]

Shouldn't trapezoids enjoy the same treatment as the other quadrilaterals? We say *a square is a kind of rhombus*, for example. A rhombus has opposite sides that are congruent. If ALL the angles are congruent then it just happens to be a square.

[click on rhombus applet if time allows]

Suppose I construct using dynamic geometry software a quadrilateral with all four sides congruent. We will all agree that this is a rhombus. As we drag it around so that the angles gradually change, there is a brief moment when all the angles are right angles and the shape becomes a square. But is it still a rhombus? Yes, unquestionably, it is still a rhombus because it meets the definition.

There's just no debate about this in the mathematical community -- we all agree this is the best, most useful, and most powerful understanding of rhombi. The square meets the condition for being a rhombus and every property that's true of rhombi is true of squares also.



[John]

So now imagine we construct another quadrilateral using dynamic geometry software, this time forcing two sides to always be parallel. Would you say this is a trapezoid?

As I drag around one of the vertices, there is a moment when the trapezoid becomes a parallelogram. According to ***you*** there's one moment, as you're dragging it around, that it stops being a trapezoid and for that one second is exclusively a parallelogram.

But this is wildly inconsistent with the way we treat the other quadrilaterals!

How did we let this happen?

We don’t do this with the definition of any other quadrilateral.

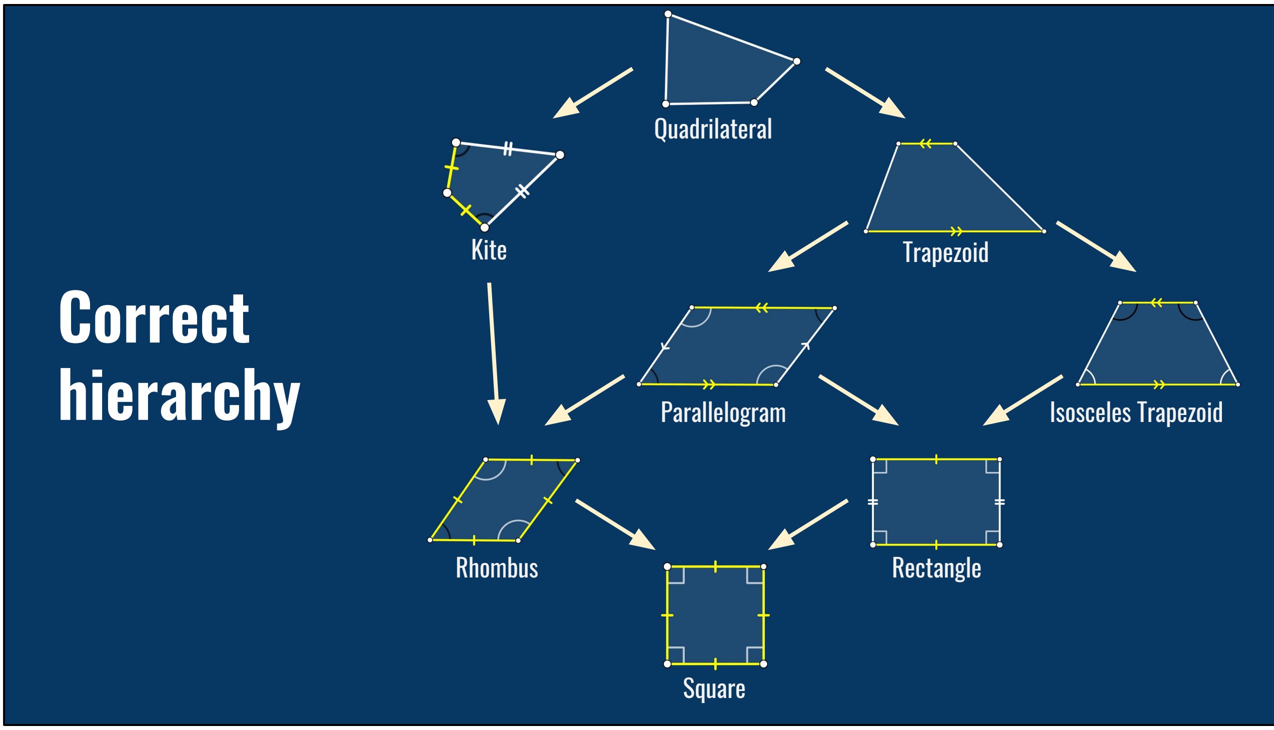
[John]

At this point, the mathematical listener should be crying, *“Foul! How did we ever let this happen? This definition of a trapezoid is so inelegant!!”* And I couldn’t agree more.

We don’t do this with the definition of any other quadrilateral. Why do it with a trapezoid?

[Will]

So you’re saying a parallelogram is a kind of trapezoid? Ugh, that’s terrible.

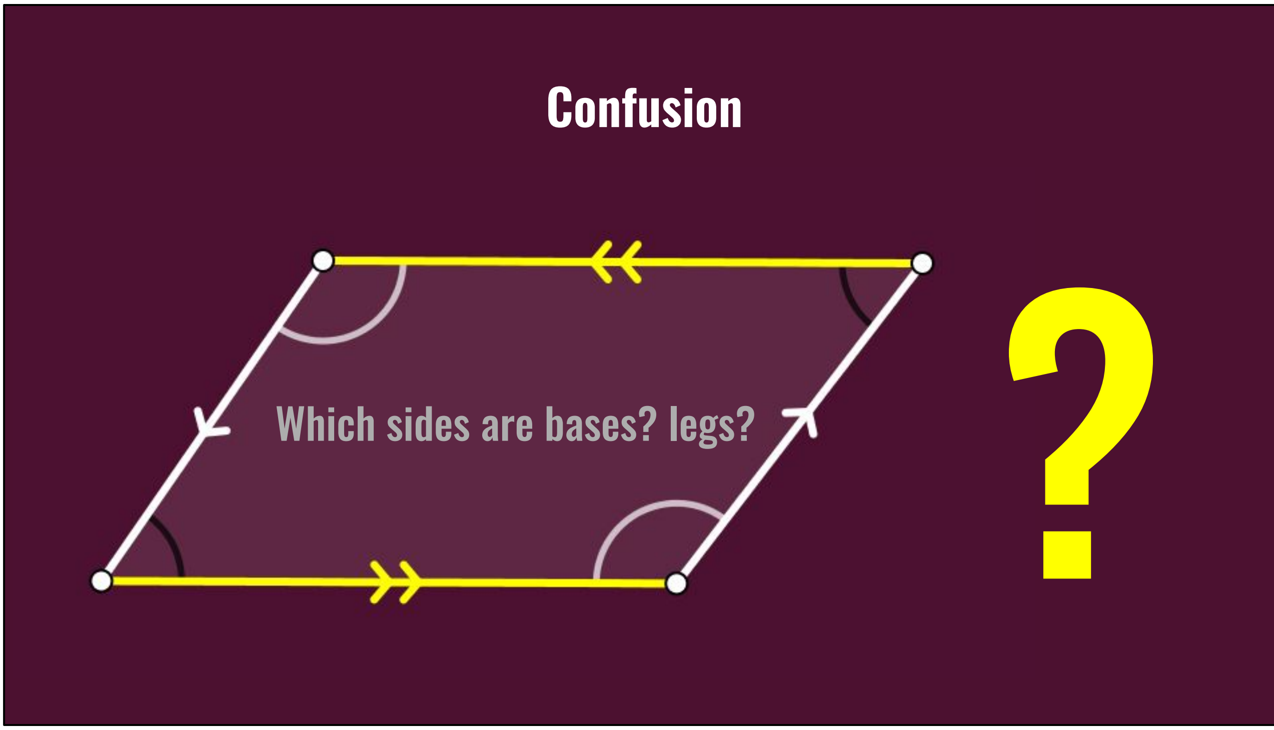


[John]

Yes, absolutely! And this is exactly what we *want*. You wouldn't hate me for saying that a *square* is a kind of rectangle. I'm actually proposing a completely different family tree for quadrilaterals than you.

[Will]

Parallelograms are trapezoids. Parallelograms are trapezoids. I hate it. I hate everything about it.



[1:50] [Will]

I'm not even sure that this works. In the traditional definition of trapezoid, the bases are the parallel sides. When I refer to the bases of a trapezoid, everyone knows which sides I'm referring to. And the legs are the non-parallel sides. Under your definition, I have to constantly worry that the other sides might also be parallel. If you talk about the bases of a trapezoid and that trapezoid happens to be a parallelogram, which sides are you referring to?

[John]

Given a quadrilateral with a pair of parallel sides, those sides are the bases of the trapezoid. If the other sides turn out to be parallel also, then congratulations, you have a parallelogram. If this happens, no "wrench" is thrown into the works -- nothing you could have said about the trapezoid before *stops* being true when it is revealed that the *other* two sides happen to also be parallel.

[Will]

Ugh. That's confusing. Under your approach, "leg" and "base" cease to have any meaning. They are dependent on the knowledge of the speaker.

[John]

Yes. But we do this all the time. The same issue arises in an isosceles triangle. We can say “drop a perpendicular to the base of an isosceles triangle”. The base is just the side that we don’t know anything about, while the legs are the sides that we know or are assuming to be congruent. If the base turns out to be congruent to the legs, then congratulations, your triangle is equilateral. Surely you don’t have any problem with that?!?

If you *know* you have a parallelogram, the bases are whichever pair of parallel sides you want to pick! Once you pick which pair of opposite sides are your bases, the other are the legs. This is not a weakness of the inclusive definition, this is a *strength*!

[Will]

I might be starting to have a problem with that.

Every fact about trapezoids also applies to parallelograms

Area of a trapezoid $$\frac{1}{2}h(b_1 + b_2)$$

Area of a parallelogram $$\frac{1}{2}h(b + b) = bh$$

[John]

Just like every theorem about isosceles triangles is also true of equilateral triangles, every fact about trapezoids is also true about parallelograms! That's why it's natural to consider parallelograms as a special type of trapezoid. Take the *area* formula, for example -- the area of a trapezoid A=½h(b_1+b_2) works even if the trapezoid just so happens to turn out to be a parallelogram.

The Trapezoidal Rule

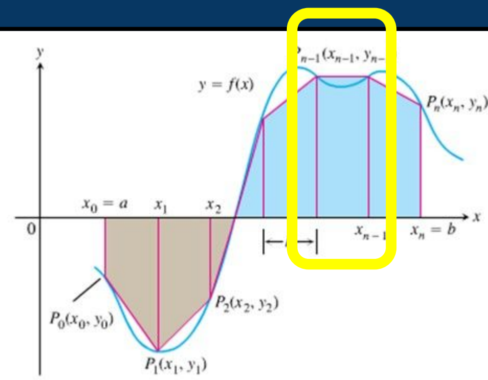


Figure 6.31 The trapezoidal rule approximates short stretches of the curve $y = f(x)$ with line segments. To approximate the integral of f from a to b, we add the "signed" areas of the trapezoids made by joining the ends of the segments to the x-axis.

[John]

And on a related note, the trapezoidal approximation method in Calculus doesn't fail when one of the trapezoids is actually a rectangle.

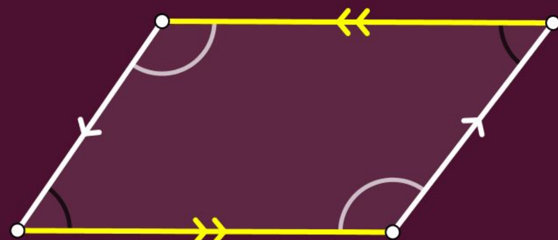
[Will]

I don't believe it. Trapezoids are trapezoids and parallelograms are parallelograms. I need to revisit everything I thought I knew. The constant anxiety about having to worry that my legs are also parallel is too much to bear. In fact, if I examine all my proofs from 5 minutes ago…

A theorem about trapezoids that fails for parallelograms

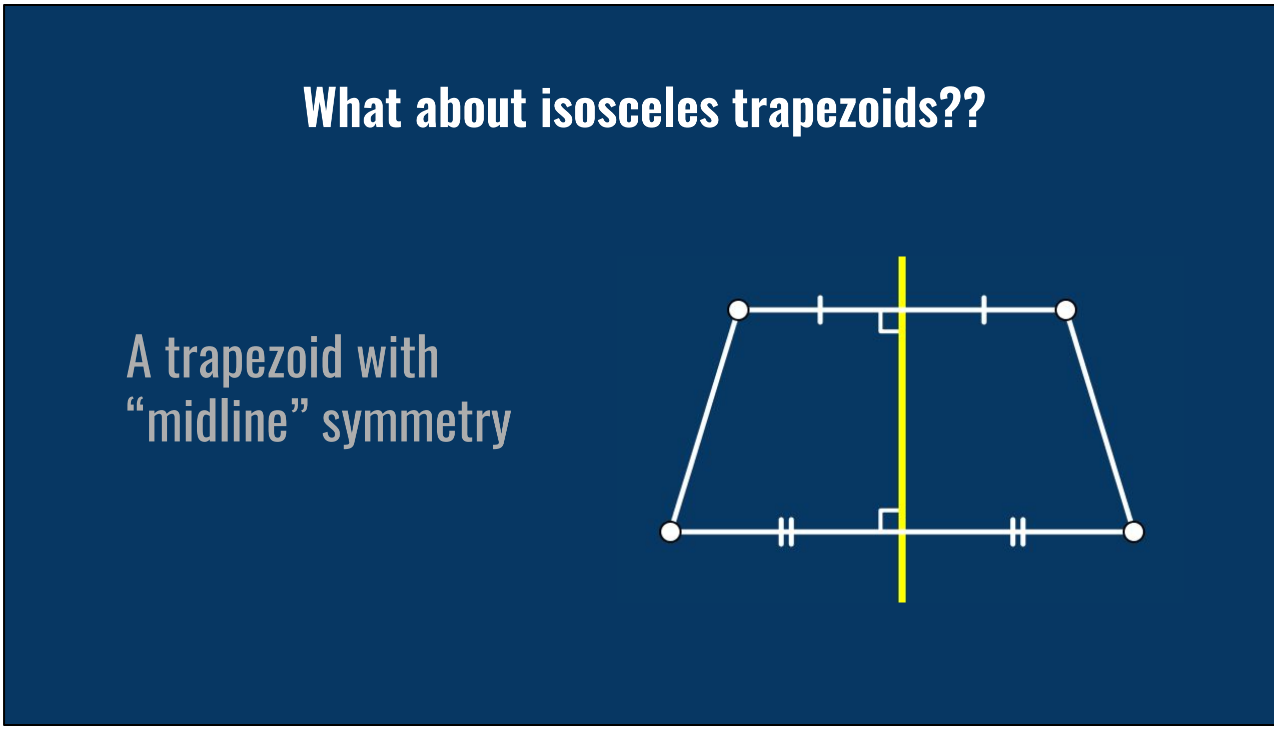
Theorem: The base angles of an isosceles trapezoid are congruent.

Counterexample:



[Will]

Wait a second. Take a parallelogram. According to you, that’s a trapezoid. But the legs are also congruent, so that’s an isosceles trapezoid. Every parallelogram is an isosceles trapezoid! But the base angles just aren’t congruent.



[1:55] [John]

True. I've been holding out on you a little. We also have to redefine an isosceles trapezoid. An isosceles trapezoid is just one that has a line of symmetry through its bases. And you may not like this, but really, isn't this what we've always meant by an isosceles trapezoid?

[Will]

Does this actually work?

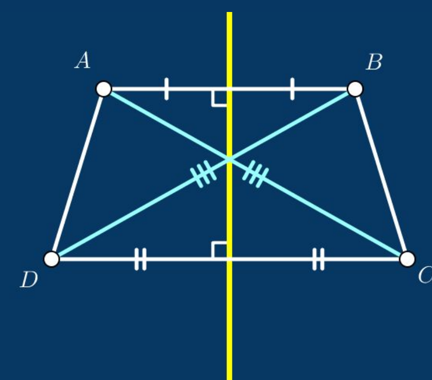
[John]

Yes! Let me prove the same results you did at the beginning of our time.

Immediate from the definition

If a trapezoid is isosceles, then the base angles are congruent, legs are congruent, and diagonals are congruent.

- Isosceles trapezoid has midline symmetry. Reflection over this line maps *A* to *B* and *D* to *C*.
- Base angle ∠*ADC* is superimposed pointwise on ∠*BCD*, thus congruent.
- From superposition, you also get the other base angles congruent, legs congruent, and diagonals congruent.



[John]

It may seem like we're backed into a corner with this redefinition, but it actually gives us a lot of new power. It also gives us a chance to show off the power of transformational geometry. The new approach to geometry called for in the Common Core State Standards is centered around transformations, so I think this is worth mentioning. Symmetry is powerful, and really it's how we should think about *all* the quadrilaterals.

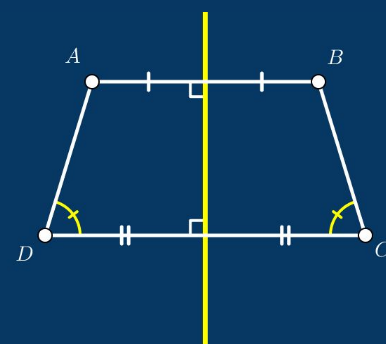
So, taking this approach, if we understand *midline* symmetry as the defining attribute of an isosceles trapezoid, then a reflection over the midline maps A to B and D to C. And thus we automatically get the base angles mapping to one another, the legs mapping to one another, and the diagonals mapping to one another.

You may be uncomfortable with proofs styled in this manner, but the transformational approach really does work and gives power. You can do these proofs with the traditional approach, but the fact that our definition emphasizes symmetry gives us a great deal of power and we should take advantage of that.

A sufficient condition for a trapezoid to be isosceles

If the base angles are congruent, then the trapezoid is isosceles.

- Construct the perpendicular bisector of the bottom base, forming two quadrilaterals.
- This is also perpendicular to the other base because the bases are parallel.
- The two quadrilaterals are congruent by ASASA.
- The perpendicular bisector of the one base also perpendicularly bisects the other. Thus, we have midline symmetry.



[John]

For the converse of the base angles theorem, it’s a little bit messier.

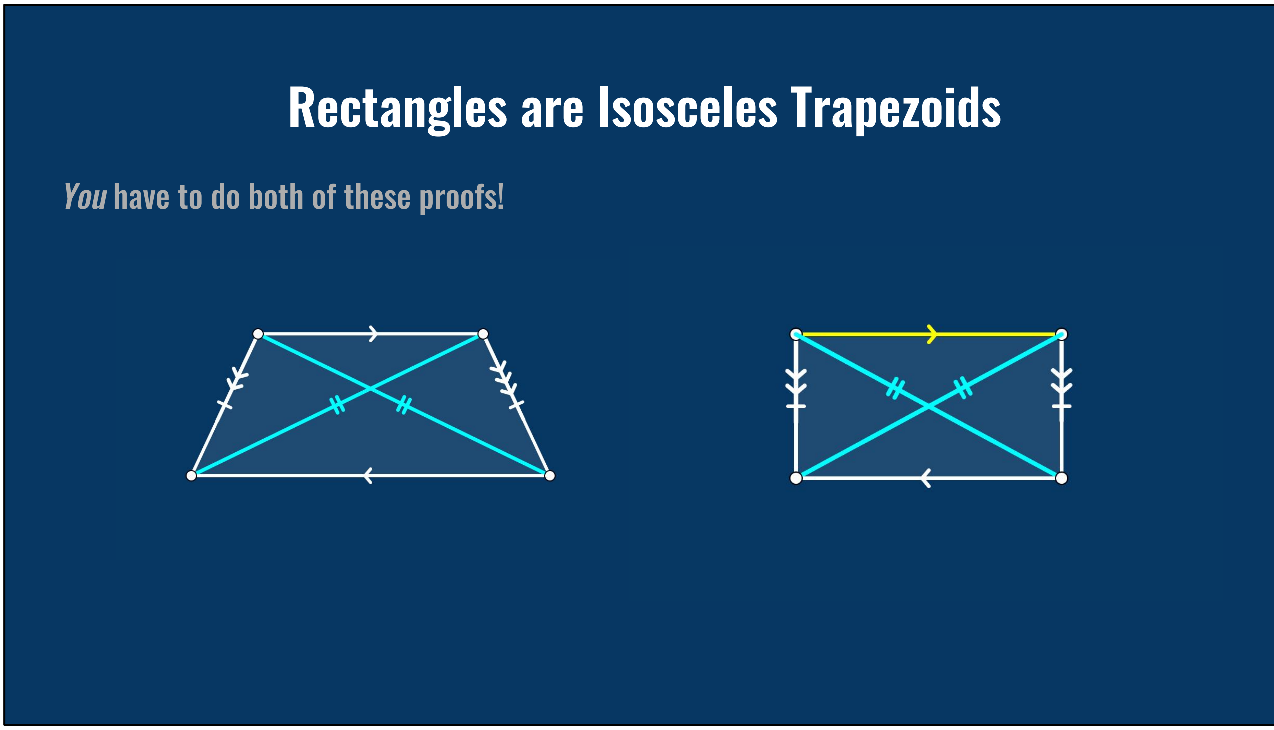
Construct the perpendicular bisector of the bottom base, forming two quadrilaterals.

This is also perpendicular to the other base because they are parallel.

The two quadrilaterals are congruent by ASASA.

Thus the top base has been bisected, so we have the constructed line as the perpendicular bisector of both bases.

This is the midline symmetry we seek, and so the trapezoid is isosceles.



[John]

The full weight of this approach is evident when you consider isosceles trapezoids, since these are the trapezoids we really care about anyway.

Every property that an isosceles trapezoid has, a rectangle has also -- congruent base angles, congruent diagonals, you name it.

[Will]

Hmm... that is true.

[John]

For example, rectangles have congruent diagonals, right?

[Will]

Yes

[John]

How do you prove that? Is it something like....

[Will]

Whoa, it’s the exact same proof.

[John]

Yes. That’s because this is reality. Join me in enlightenment.

[2:00] [Will]

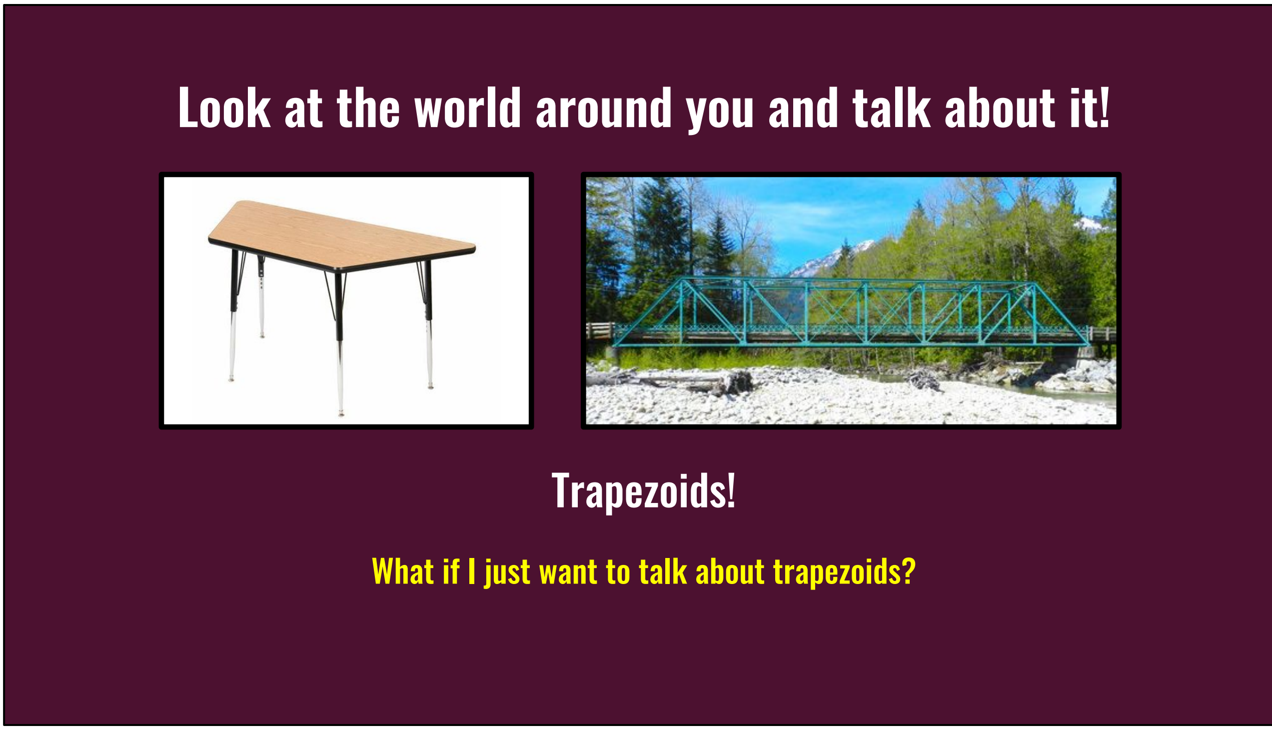
I accept that some things about the inclusive approach are superior:

- I agree it is more consistent with the way we classify the other quadrilaterals
- I agree that parallelograms have every property that trapezoids have, if you define trapezoids inclusively and that rectangles have every property of isosceles trapezoids
- I agree that the proofs are more efficient

But that doesn't mean I have to accept the inclusive definition.

It works in theory, but does in work in practice? What are the drawbacks? I think that there are a lot.

If all the theoretical arguments are on your side, why is the inclusive camp having so much trouble convincing the rest of us?



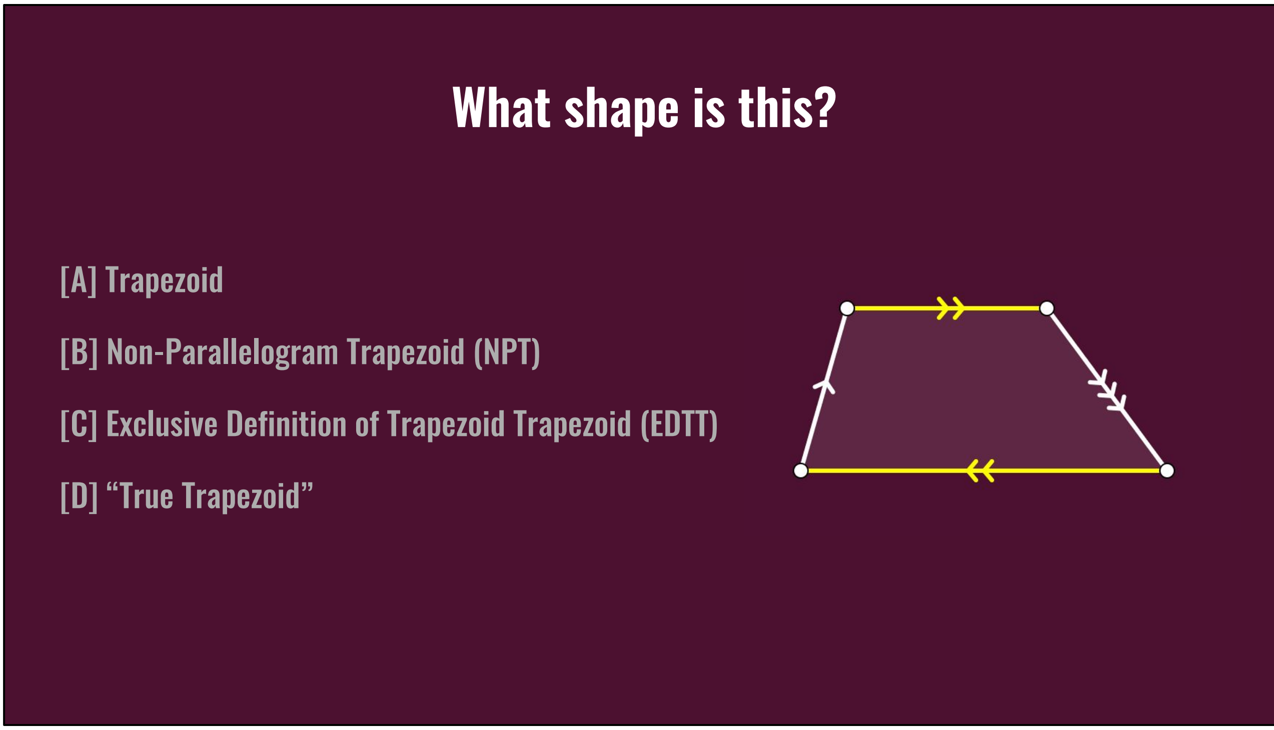
[Will]

What shape is this table? What shape is this bridge?

What if I just want to talk about trapezoids? Sometimes I just want to talk about trapezoids.

We live in a real world and that world is full of shapes. We want to be able to talk about those shapes, using terms that are descriptive.

You want to seize the word “trapezoid” and define it in a particular way to serve your theoretical purposes. But that’s not how words work! We invent words (mostly) for pre-existing concepts. Trapezoids are all around us! Let’s talk about them!



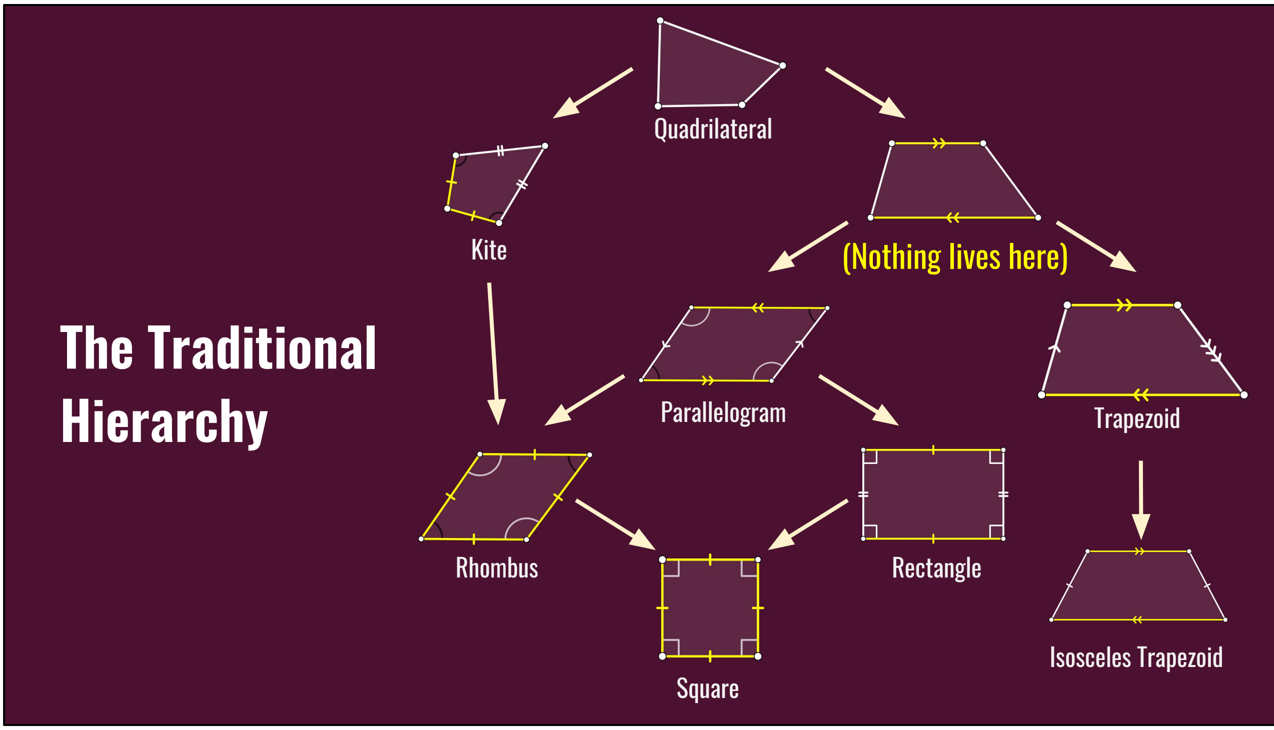
[Will]

[A] Not specific or descriptive enough. THIS SHAPE has non-parallel legs. If I just call it a trapezoid, according to your definition, I’m being vague and non-descriptive.

[B] Clunky and annoying

[C]
[D]

Man, if only there were some term for a quadrilateral with exactly one pair of parallel sides. Hmm, someone should invent a word for that. Oh, I don’t know, how about… TRAPEZOID!!!!



[Will]

This is the traditional classification.

It presents the shapes as they are. Sort of.

Every shape either has no pairs of parallel sides, exactly one pair of parallel sides, or exactly two.

We have a convenient word for quadrilaterals with two pairs of parallel sides. We call them parallelograms.

We need a convenient word for quadrilaterals with exactly one pair of parallel sides. I call them trapezoids.

Your classification avoids the issue by just refusing to talk about these objects at all. But you can't make shapes go away by just refusing to allow me a word to describe them.

You fixate on quadrilaterals with at least one pair of parallel sides, but there are no such objects! You obsess over a theoretical point on the quadrilateral family tree where nothing actually lives. I'm just out here trying to name the animals.

[John]

But now you're just arguing for exclusivity in general. You want the word "trapezoid" to refer to the special case in which the legs are NOT PARALLEL. But where does this exclusivity end? Are squares a kind of rectangle according to you? Do you want all the quadrilateral definitions to be exclusive?

[Will]

Well…

Another look at the benefits of exclusivity -- via Euclid

Of quadrilateral figures, a **square** is that which is both equilateral and right-angled; an **oblong** that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a **rhomboid** that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called **trapezia**.

Euclid's Elements Book 1 Definition 22

[Will]

I found this surprising.

Euclid [date] defined all the quadrilaterals with the exclusive definition [Euclid's Elements Book 1 Definition 22]. For example, a rhombus was defined by Euclid as a quadrilateral figure that was equilateral but not right-angled. Here's the full content from Euclid on this issue -- really it's not much, given how much ink he spilled on other parts of geometry:

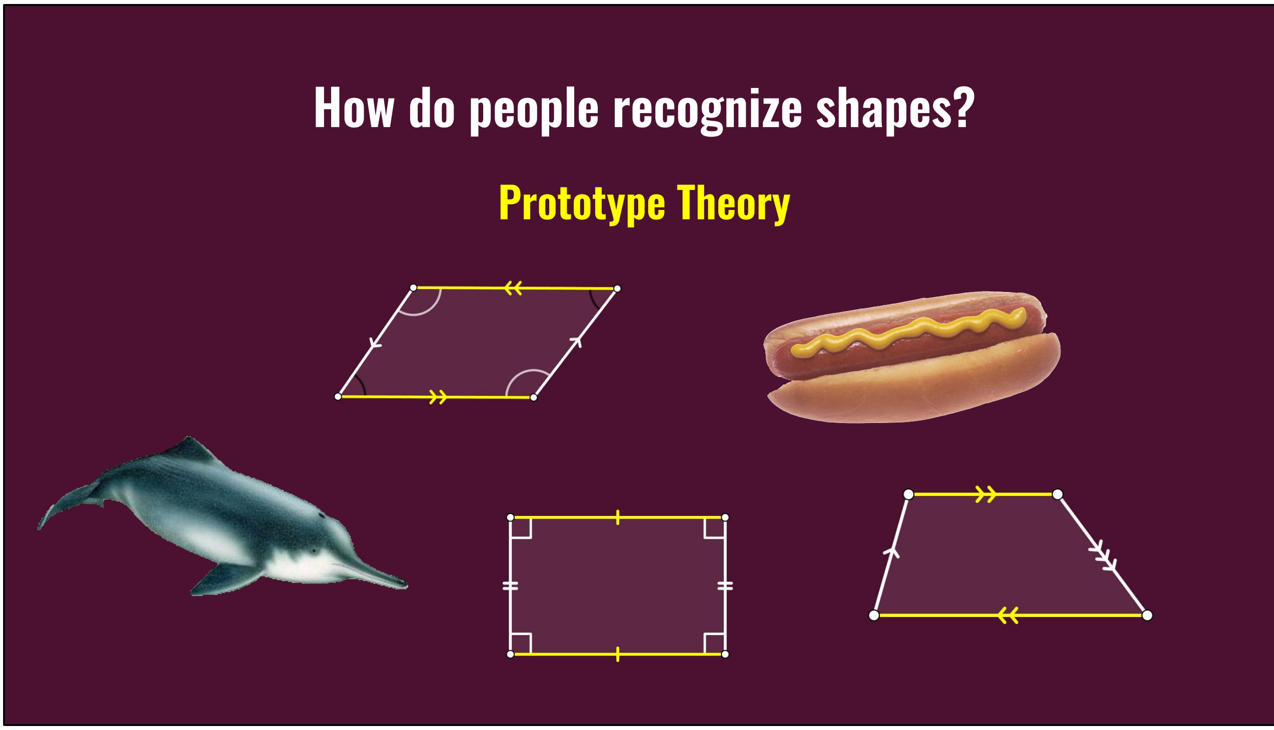
He defines an "oblong" (we might say rectangle) to explicitly exclude squares, a rhombus to explicitly exclude squares, and a "rhomboid" (we might say parallelogram) to explicitly exclude rhombi and rectangles.

[John]

Yes, I also found this surprising to learn. I've seen this before. But I think this is one of the few times that Euclid made bad choices. He also barely followed up on these definitions to prove properties of these figures. If he had bothered to, as we do now, I think he would have realized the absurdity of proving properties of a square and an oblong separately, when those properties depend merely on the figures having all right angles.

Mathematical educators and reformers in the 19th century corrected this theoretical flaw in Euclid around the time that people finally started using other geometry textbooks besides The Elements.

Now we can prove rectangle properties and since a square is a kind of rectangle, the square inherits all of those properties. In fact, the square already had all those properties to begin with! The fact that we can prove that it has all those properties by appealing to rectangles shows us that a square truly is a kind of rectangle. It would be crazy to define rectangle in a way that excludes squares.



[2:10] [Will]

Ok, I can see why exclusivity is awkward. But on the other hand, there is a huge amount of research in cognitive science about how people recognize categories. There is also a huge literature devoted to how students and teachers think and learn about quadrilaterals. Learning to classify quadrilaterals is hard!

There is evidence that humans recognize categories, not with a list of necessary and sufficient conditions, but instead by comparing objects to prototypical members of that category.

Under mammal, for example, people have a mental image of a moose or a fox and therefore have no trouble classifying a bear, for example, as a mammal. But people have a lot of trouble classifying a dolphin as a mammal because a dolphin is so visually distinct from the prototypical mammal.

This explains the crazy internet phenomenon of "Is a hotdog a sandwich?" People mostly don't have a formal definition of sandwich based on essential properties. They just compare a hotdog to their iconic mental image of sandwich and get confused.

[John]

Yes, but we really do need to precisely define our objects in mathematics. We can't do formal reasoning without it. It can’t be ambiguous whether a mathematical object fits into a category.

[Will]

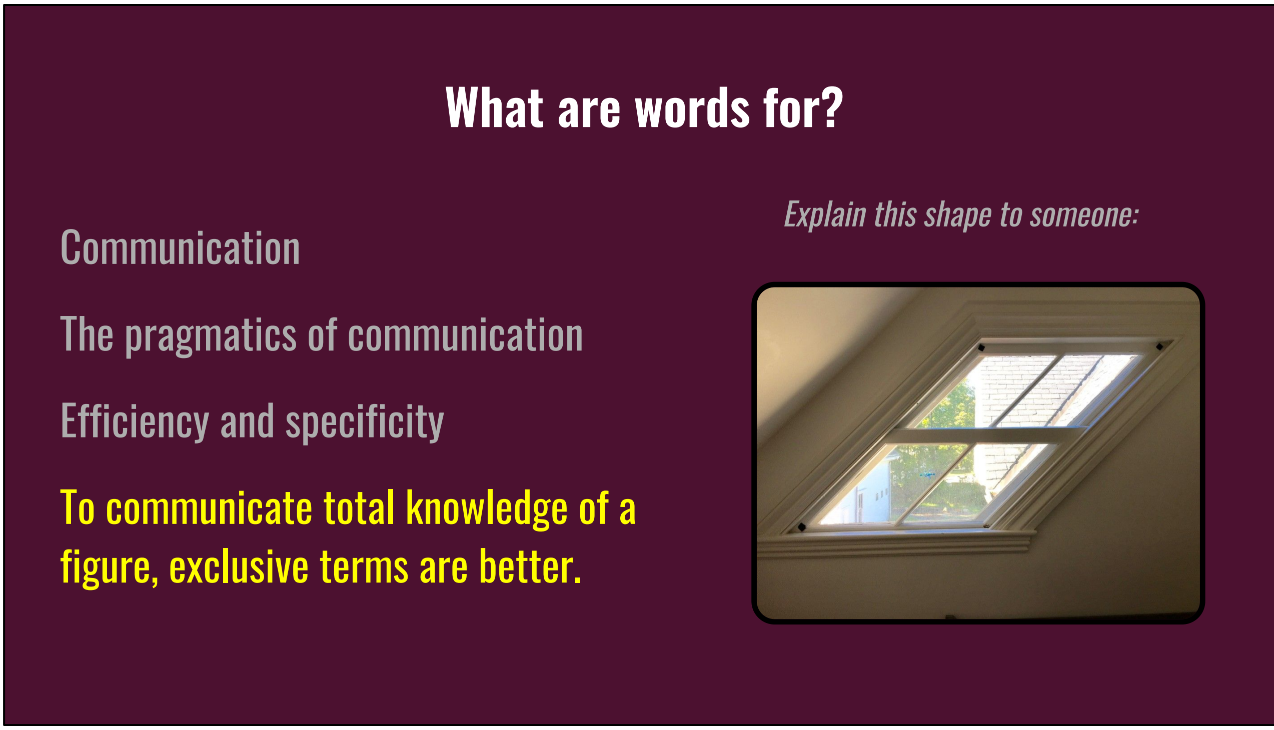
Yes, but I think we should at least respect how hard this is for students, since it’s so different from the normal way that they cognitively process categorization in their daily lives.

[John]

Part of the rationale for a unit on quadrilaterals is to train the students to adopt this formal framework of terms with precise definitions. The entire point is to learn how to define objects via their essential properties in a natural hierarchy. That’s why the inclusive approach is so superior.

Definitions in math are arbitrarily chosen, true, but we evaluate their strength based on what consequences those definitions have and how much power they give us.

Mathematicians are always in the business of hijacking English words and making them more precise. Words like continuous, regular, equal, series, or product all have very particular meanings in various mathematical contexts. Even the word “or” we make more precise. This is a huge and vital part of mathematics.



[Will]

But we can't completely ignore how language is used in normal life. Language is used for communication between a speaker and a listener.

Communication is by default cooperative. And a lot of communication is just one agent transferring information to another .

Unless you have reason to be suspicious, you assume as the listener that the speaker is being as helpful as possible at all times.

There is an entire subfield of linguistics called Pragmatics which analyzes how speakers and listeners navigate through a discourse making assumptions about intent.

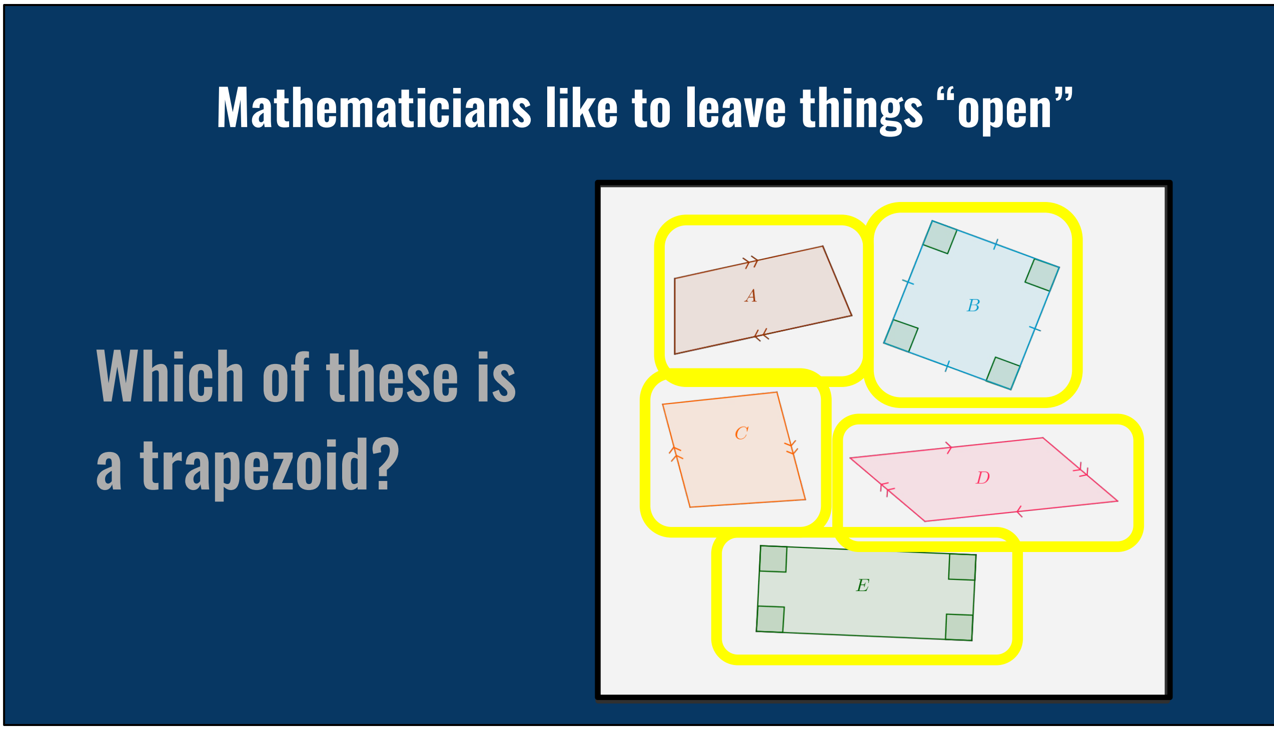
If I have "total knowledge" of a figure and want to communicate that information about the figure to another person, than exclusive terms are always better. If I have a square behind my back and I tell you I have a rectangle, I'm holding out on you.

If we accept the standard inclusive hierarchy, then there's no word for a parallelogram that's not a rectangle or a rhombus. But what if I have one of those shapes? Take a look at this window. Since everyone here in the audience is looking at this photo, we now all have “total knowledge”. How would we describe this to a listener? Well, we can safely call it a parallelogram and not really worry about misinterpretation.

The listener will assume by default that I am being as specific as possible because to give less information in a discourse than is available is a pragmatic violation. If my window were a square, then I would have said “square”, the listener can reasonably assume. And if my window were a rectangle, I would have just said “rectangle”. It would be a violation of pragmatics to call my window a parallelogram if it were in fact in a rectangle, since rectangle is more specific. So by choosing to use the word “parallelogram” in my discourse, even though a rectangle is a type of parallelogram, the listener can assume that my window is not a rectangle. And so, in practice, the word “parallelogram” is interpreted exclusively!

There is thus constant pressure from everyday experience to interpret terms exclusively, even if the terms have been precisely defined inclusively.

It’s probably part of why students resist the inclusive definitions and why adults who don’t do formal mathematical reasoning lapse back into the exclusive definitions.



[2:15] [John]

Ok, you’ve made some valid points. You say that “To communicate total knowledge of a figure, exclusive terms are better.” But in mathematics, and in life I would argue, we are operating in situations where we do not have total knowledge of a figure. In fact, in math, we intentionally put ourselves in this position.

Here’s a simple test question: *Which of the following quadrilaterals are trapezoids?*

You guys have been listening closely enough to say “stop, wait, under *which definition?*” Good. Very good.

Let’s start with the inclusive definition -- which ones have at least two parallel sides? -- ALL OF THEM.

If you are using the the exclusive definition, then you can’t be sure that *any* of them are trapezoids. In order for (A) and (C) to be trapezoids, under the exclusive definition, you must prove that two sides are parallel AND the two remaining sides are not parallel (and you can’t assume that from the picture…especially for (C)!). A and C live on the tree where you said nothing lives!

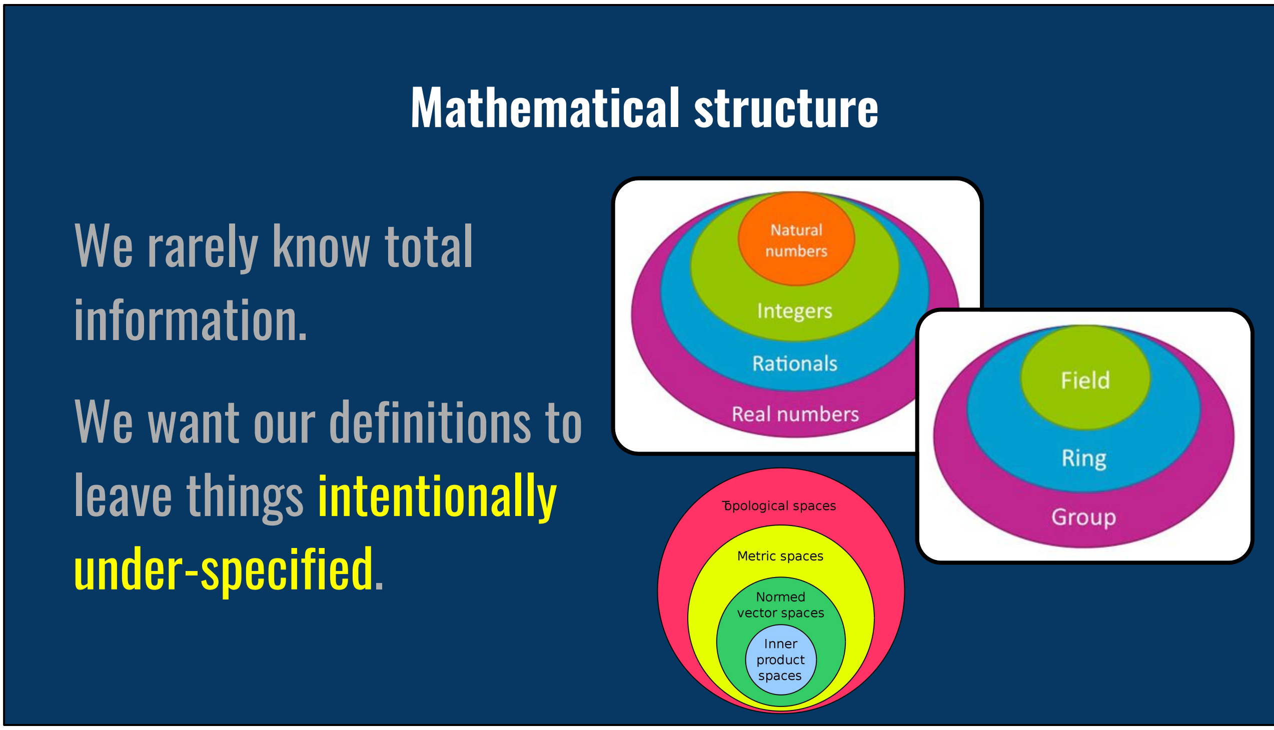
When we choose to have definitions that leave certain facts "open," we're also saying something deep about the way we do mathematics. This inductive journey is what if feels like to do mathematics, to prove something. In order to "show" a quadrilateral is a trapezoid, one has to only prove two sides parallel. But under your definition, you would have to not only prove one pair of sides parallel but also prove the other pair of sides is *non-parallel!*

Can you see the absurdity of the exclusive definition now?

These marks we make on figures refer not to the truth about the figure, but only what we know so far about the truth of the figure. In geometric reasoning, we often don't have total knowledge at the beginning.

We are making a statement about what things we are certain about and which parts we are leaving open. I admit that this is an abstract idea that is of mostly of interest just to mathematicians. But it arises in any field where terms must be precisely defined, like science or even law.

Can we think of other non-mathematical areas of life where we desire to talk about objects with partial information? Biology and taxonomy (identifying a new species), law (we definitely know this is a tort issue…but we don't know much more, or consider when federal and state law are in conflict…federal law is sometimes intentionally vague..)



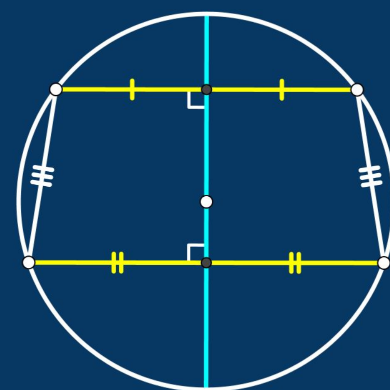
[John]

The nature of geometric proof is that we often don’t know total information. We want our definitions to be open and flexible. Think of how we define rationals so they include integers, for example. Or think of the most general algebraic object, the group. Under specified objects bring us great power!

Reasoning with partial information

Given a cyclic quadrilateral with opposite sides parallel, prove that the other pair of sides are congruent.

- Construct a diameter perpendicular to one of the bases. Because the bases are parallel, this is perpendicular to the other base too.
- A diameter perpendicular to a chord bisects the chord, so the constructed diameter is the perpendicular bisector of both bases.
- Because we have midline symmetry, the trapezoid is isosceles, and the desired result follows.



[John]

Here's an example of a statement we might want to prove. It's neutral with respect to your definition of trapezoid. Let's take a look at how the proof might go using the different definitions of trapezoid.

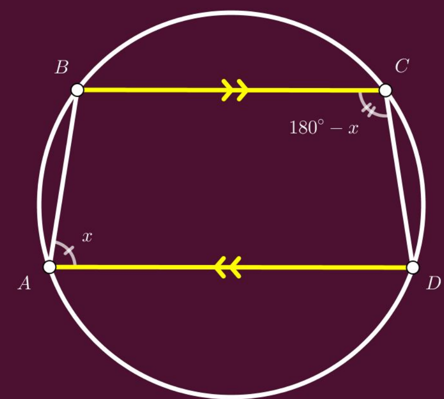
If you take the exclusive definition, you can't say this is a trapezoid or use anything you know about trapezoids to prove the statement. But under the inclusive definition, you can immediately say this is a trapezoid without worrying about the other sides being parallel. And once you can say this is a trapezoid, you can bring to bear all of the facts and theorems you know about trapezoids to this current task.

Proof under the inclusive definition: Construct diameter through one base. This bisects the base since the base is a chord. Since the bases are parallel, the previously constructed diameter is also perpendicular to the other base, and thus bisects it as well. Since the diameter constructed serves as a perpendicular bisector for both bases, we have midline symmetry, thus the quadrilateral is an isosceles trapezoid.

Proof with exclusive definition

Given a cyclic quadrilateral with opposite sides parallel, prove that the other pair of sides are congruent.

- If it's a parallelogram, we're done
- Otherwise, it's a trapezoid
- A property of cyclic quadrilaterals gives us that *A* and *C* are supplementary
- But *C* and *D* are also supplementary
- So *A* = *D* and the trapezoid is therefore isosceles



[Will]

Well, that looks like a trapezoid.

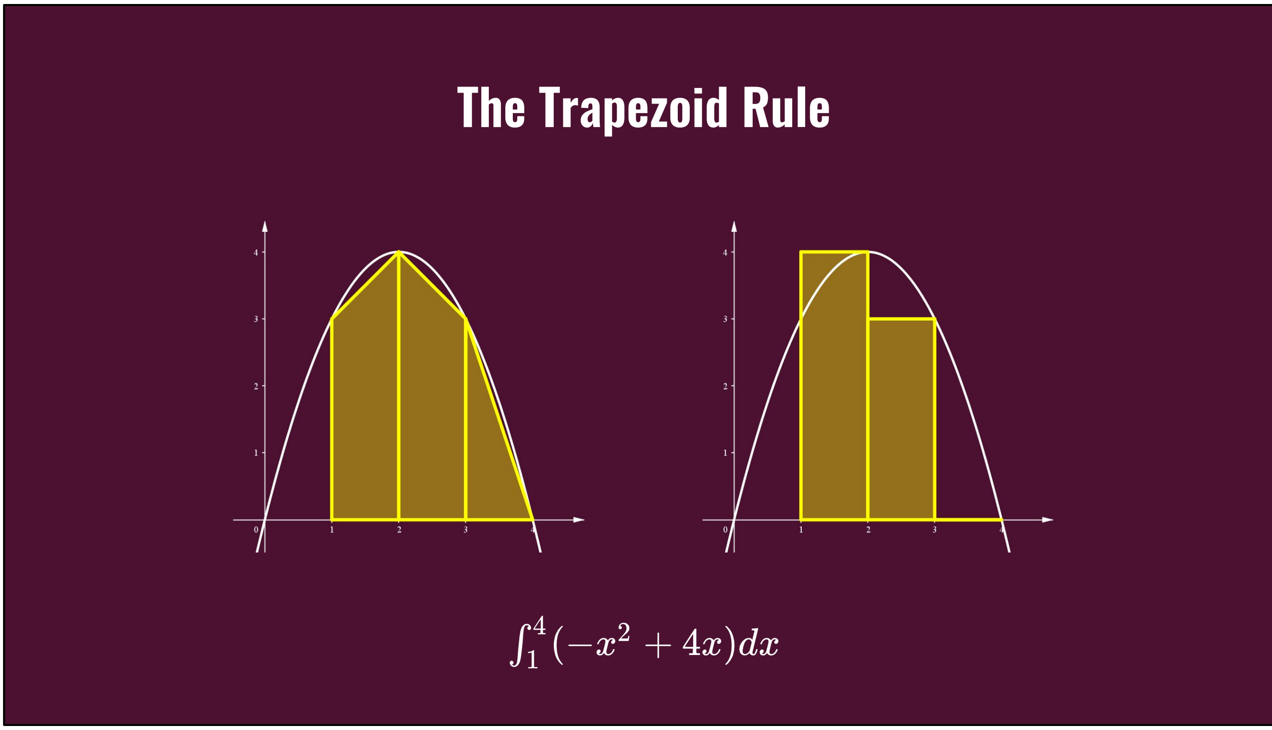
[John]

But according to you, is it a trapezoid or not?

[Will]

According to me, I can't say if it's a trapezoid or not because I don't know anything about the other sides yet. I admit that this is a bit awkward.

But, inelegant as it is, I am not stuck!



[2:20] [Will]

The Trapezoid Rule is a way of estimating the value of an integral. You can prove that in the limit it gives you the correct value for the integral and you can prove various properties of the rule, like upper bounds on the error.

But all of this is totally independent of how you define a trapezoid and what you choose to name this rule.

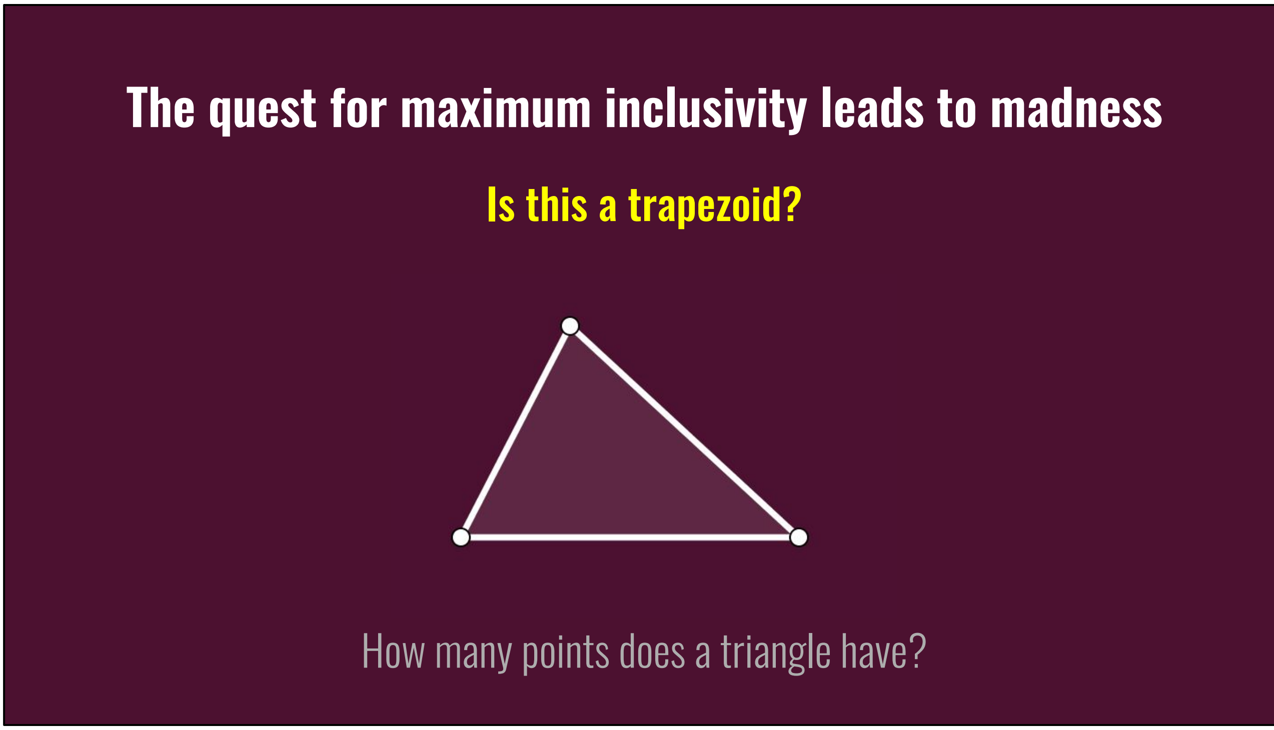
We chose to name it “Trapezoid Rule” because if you draw a picture of what’s going on, most of the shapes are trapezoids. But you can’t then turn around and claim that since the rule is called the Trapezoid Rule, that every shape formed by the rule must be a trapezoid.

In fact, consider using the trapezoid rule with n=3 on the integral from x=1 to x=4 of f(x) = -x^2+4x. The final “trapezoid” is a triangle!

Is a triangle a trapezoid?!?

It gets even worse. What if we try to approximate the integral using the right rectangular approximation method, n=3. Well then that final “rectangle” is just a line segment.

Is a line segment a rectangle?!?



[Will]

Is a triangle a trapezoid? It's a serious question.

You say that, for example, the area formula that works for the trapezoid also works for the parallelogram. I agree. But does it not *also* work for a triangle where b_1 is "zero."

(Drag a trapezoid around until it's a triangle.)
(Drag a triangle around until it's a segment.)
(Drag a triangle around until it's a point!)

Built into the definition of quadrilateral, is that we have four DISTINCT points. Thus we have some exclusivity lying at the heart of our quadrilateral definitions and we can't define our way out of things unless we also allow for quadrilaterals to be triangles, segments, even points.

[John]

Word. That's a good point. I hadn't really thought about that. Let's slow down a bit.

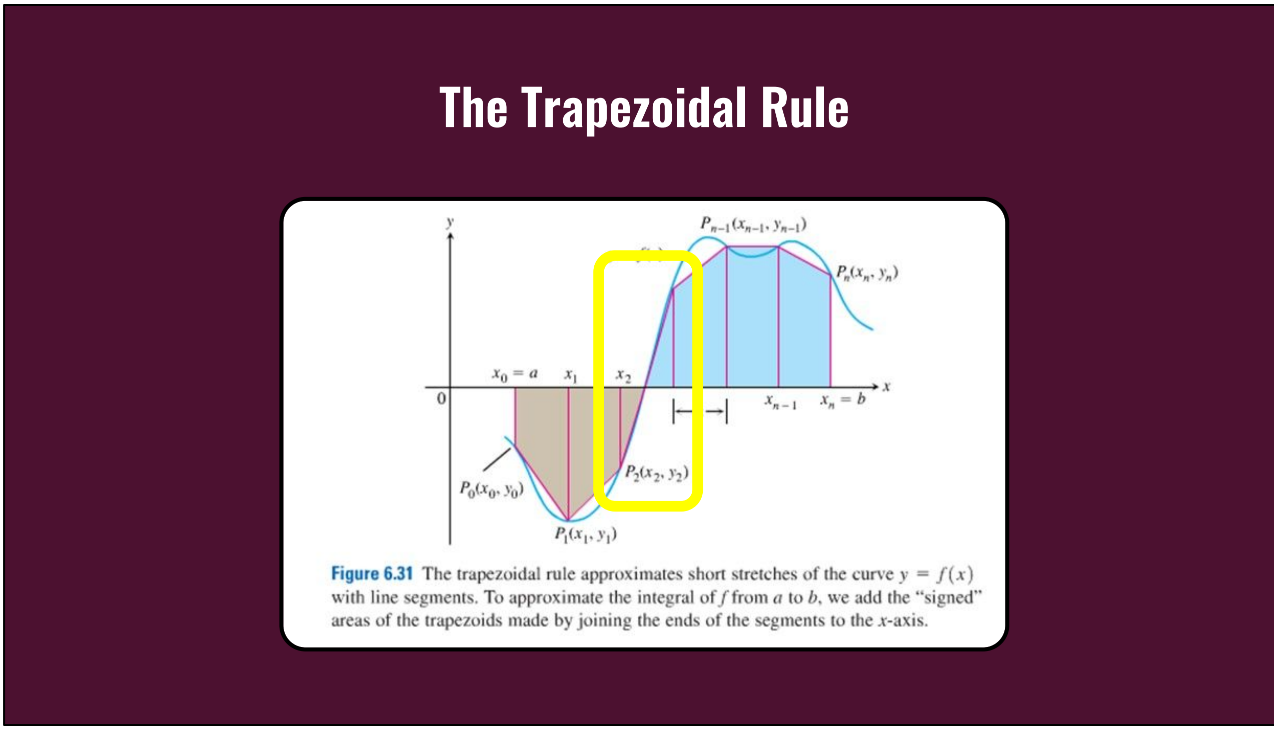


Figure 6.31 The trapezoidal rule approximates short stretches of the curve $y = f(x)$ with line segments. To approximate the integral of f from a to b, we add the "signed" areas of the trapezoids made by joining the ends of the segments to the x-axis.

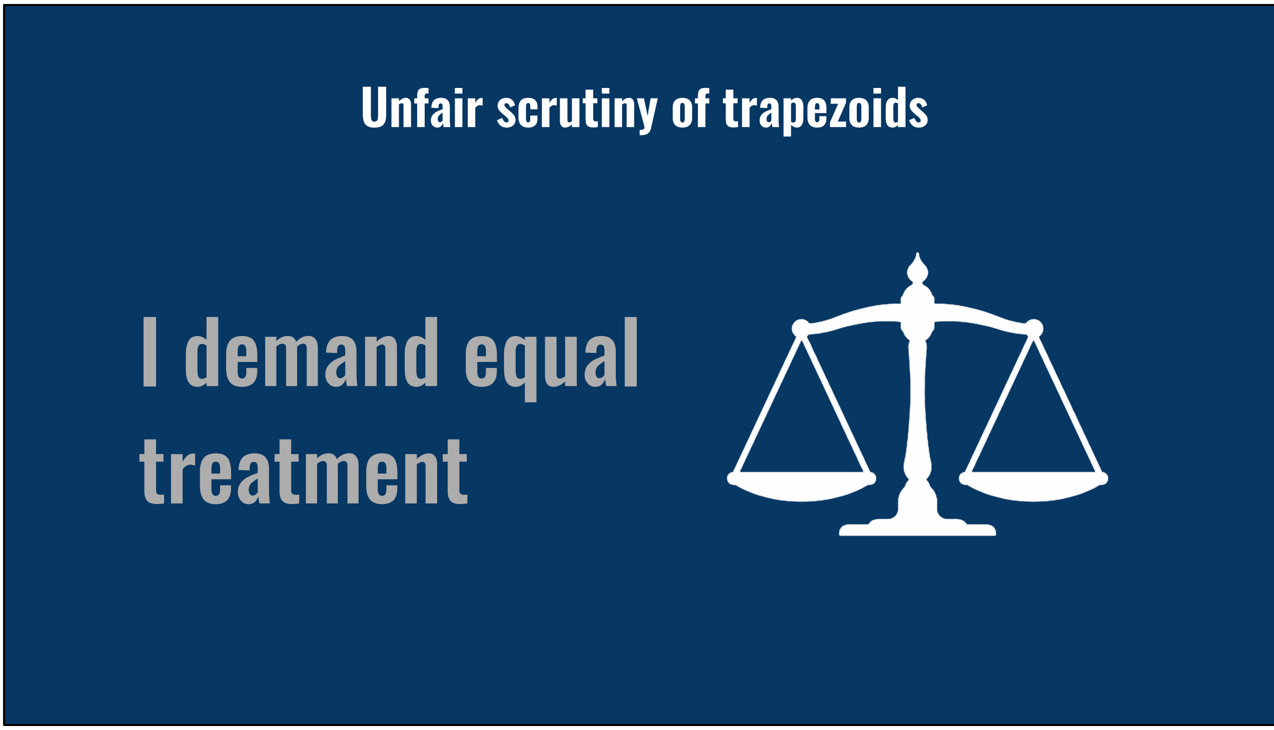
[Will]

Oh, and I think it's interesting that when you showed this slide earlier that you didn't highlight *this* "trapezoid." VERY INTERESTING.

Is *this* a trapezoid, John? Really?

[John]

Okay, I concede on this point. The trapezoidal rule really doesn't give either of us ammunition.



[John]

I’m not arguing for inclusivity in everything and I never said that. I guess I only ask for the kind of charity that you might give the other quadrilaterals.

Insisting that trapezoids be subjected to the "is it a triangle?" test or the "is it a bow tie?" test is unfair because it goes too far. We don't subject the other quadrilaterals to this kind of scrutiny. I just ask for the exact same charity with trapezoids.

Triangles are not quadrilaterals. We all agree on this.

We all have to draw the line somewhere. I think the line I’m proposing seems most natural, but you want to draw the line right through the middle of the quadrilateral hierarchy!

Coming Together

Summary of arguments

Exclusive Definition	Inclusive Definition
Tradition: everything works fine; why fix what isn't broken?	Consistency: Equal treatment with all quadrilaterals
Simplicity: Just a truncated triangle; isn't this what we all mean?	Elegance & Inheritance: Everything true about trapezoids is true of parallelograms
Ease: Cognitively easier to process	Efficiency: Proof without redundancy
Honest and clear: Communicate maximum information at all times	Open: Can reason with partial information (awkward with exclusive definition)

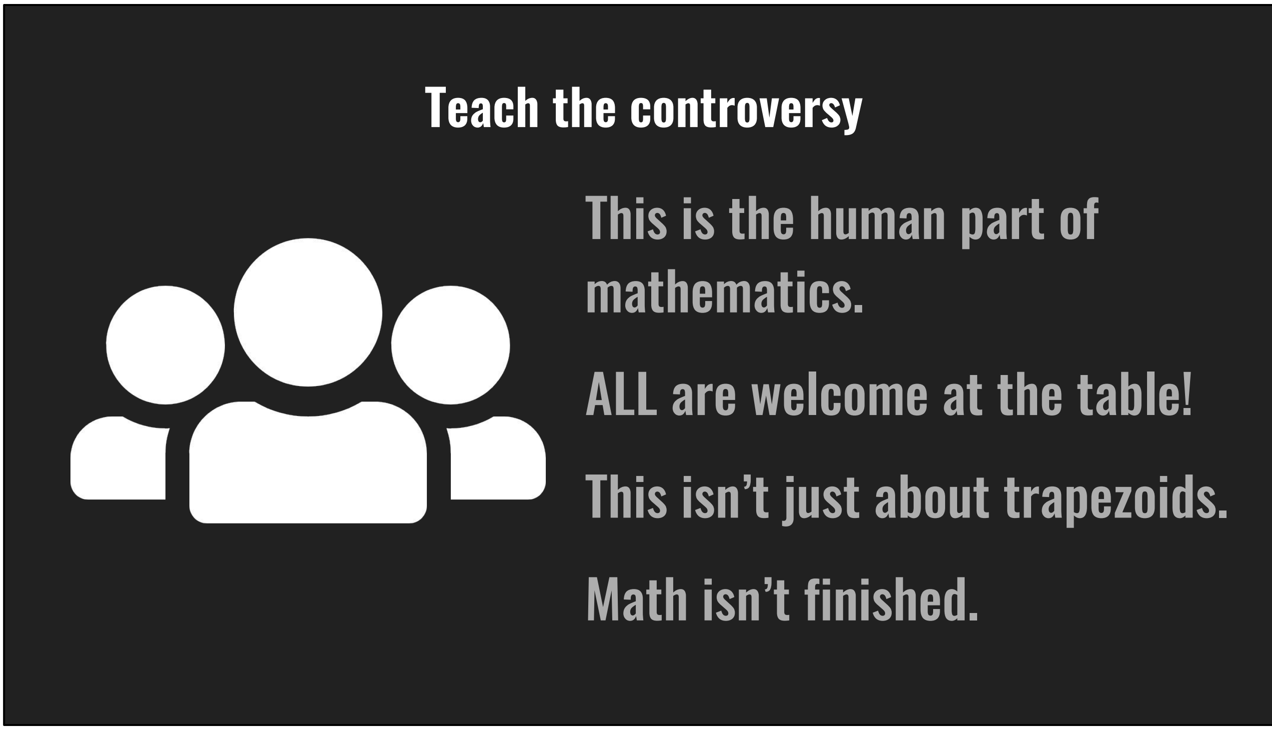
[John]

As we come together, just for fun, we will now summarize the strengths of the *our opponents* arguments. We do this in the spirit of brotherhood!

The exclusive definition has the advantage of being trustworthy -- it always worked well, so why fix it? It is simple, and many people just want to think of a trapezoid as the bottom of a triangle. Why increase the cognitive demand with extra levels of abstraction? And Will also made a great point about the fact that when we communicate, in general, we ought to communicate maximum information. If you *know* a trapezoid has nonparallel sides, why wouldn't you say so?

[Will]

And the strong arguments for the inclusive side are consistency with all the other quadrilateral definitions, the elegance gained through "inheritance," the added efficiency of not having to prove things again for parallelograms, rectangles, etc, and the ability to reason with partial information, especially in the context of proof. Leaving mathematical objects "open" actually allows for more deductive power.



[John] [WIll can chime in]

Why are quadrilaterals important at all? Why is this topic important? The goal is to get us thinking about definitions. This isn’t just about trapezoids.

This is how mathematics works. Definitions change over time. This is the human part of mathematics.

So what do we teach our students?

It might be best to “teach the controversy,” since you would be involving your students in a rich discussion about the nature of definitions in mathematics. Definitions are arbitrarily chosen, but the definitions we choose have different consequences–some of which are powerful and some of which are useless, some of which are beautiful and some of which are ugly. This would let your students see that “doing mathematics” is a creative human endeavor, not a formulaic endeavor.

Teach the controversy. Involve students in the discussion! Show them that math isn’t finished -- it is a conversation and they, regardless of their age or experience, can enter into that conversation.

Further Resources

Inclusive Definition

- John’s blog where the debate began: mrchasemath.wordpress.com
- Richard Rusczyk from AoPS on YouTube: youtu.be/xoXLbOWRBMA
- CCCSS Progressions for 5th Grade: http://commoncoretools.me/wp-content/uploads/2014/12/ccss_progression_gk6_2014_12_27.pdf
- “Draggable” heirarchy of quadrilaterals organized by symmetry by Michael de Villiers: http://dynamicmathematicslearning.com/quad-tree-new-web.html
- A whole book about definitions that has a sophisticated discussion about trapezoids: *The Classification of Quadrilaterals: A Study in Definition*

Thanks for coming! Stick around to discuss!

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Poll the audience

Let’s Check out the results

bit.ly/2Khz4YK

Comments from the poll

Parallelograms deserve to be more than just a special case!

A rectangle and a trapezoid shouldn't be considered the same thing in some cases. It just isn't right. When somebody says trapezoid everybody thinks of something with exactly two parallel lines and two non parallel lines and the definition should reflect that.

It's all about the base(s)🎶🎧

if a trapezoid had at least two parallel sides then wouldn't a parallelogram be a trapezoid? *sounds fake*

Comments from the poll

I think they are very interesting shapes and no matter what this debate may try to say about them, they are all beautiful. :-)

They shouldn’t be this divisive.

It's a trapezoid or it's a parallelogram. There's no need to confuse them. Silly purists with their silly ways of defining things.

If it has more than 2 parallel sides, it's not a trapezoid, it's another shape for which we already have a name!!! For example you wouldn't (or at least shouldn't) call a square a trapezoid - *that would be a crime against mathematics.*

Comments from the poll

I used to use the exclusive definition because that was all I knew. Now that I know more, I prefer the inclusive definition as it allows for more connections to other quadrilaterals.

make trapezoids great again

Thinking about a parallelogram as a special kind of trapezoid makes me feel icky.