THE GAP TRAP AND OTHER RISKY REASONING:
THE KRYPTONITE OF FRACTION FLUENCY
Agenda

- Comparing Fractions
- Valid/Invalid Strategies
- Developing Residual Reasoning Language
- Super Power Strength of Residual Reasoning
1. $\frac{2}{3}$  $\frac{1}{2}$

2. $\frac{1}{3}$  $\frac{3}{8}$

3. $\frac{5}{6}$  $\frac{2}{5}$

4. $\frac{3}{4}$  $\frac{1}{2}$

5. $\frac{7}{8}$  $\frac{9}{10}$

6. $\frac{2}{5}$  $\frac{5}{12}$

7. $\frac{4}{5}$  $\frac{1}{3}$

8. $\frac{2}{3}$  $\frac{5}{8}$

9. $\frac{3}{4}$  $\frac{1}{3}$

10. $\frac{5}{8}$  $\frac{4}{6}$

11. $\frac{1}{6}$  $\frac{2}{5}$

12. $\frac{7}{10}$  $\frac{8}{12}$

13. $\frac{4}{5}$  $\frac{7}{10}$

14. $\frac{2}{3}$  $\frac{3}{5}$

15. $\frac{7}{12}$  $\frac{6}{10}$

16. $\frac{3}{8}$  $\frac{4}{10}$
1. $\frac{2}{3}$ $\frac{1}{2}$
2. $\frac{1}{3}$ $\frac{3}{8}$
3. $\frac{5}{6}$ $\frac{2}{5}$
4. $\frac{3}{4}$ $\frac{1}{2}$
5. $\frac{7}{8}$ $\frac{9}{10}$
6. $\frac{2}{5}$ $\frac{5}{12}$
7. $\frac{4}{5}$ $\frac{1}{3}$
8. $\frac{2}{3}$ $\frac{5}{8}$
9. $\frac{3}{4}$ $\frac{1}{3}$
10. $\frac{5}{8}$ $\frac{4}{6}$
11. $\frac{1}{6}$ $\frac{2}{5}$
12. $\frac{7}{10}$ $\frac{8}{12}$
13. $\frac{4}{5}$ $\frac{7}{10}$
14. $\frac{2}{3}$ $\frac{3}{5}$
15. $\frac{7}{12}$ $\frac{6}{10}$
16. $\frac{3}{8}$ $\frac{4}{10}$
\begin{align*}
\frac{2}{3} & \quad \frac{5}{8}
\end{align*}
\[
\frac{2}{3} \quad \frac{5}{8}
\]

5 is greater than 2

8 is greater than 3
\[ \frac{2}{3} \quad \frac{5}{8} \]
\[
\frac{2}{3} \quad \frac{5}{8} \quad \frac{16}{24} \quad \frac{15}{24}
\]
\[ \frac{2}{3} > \frac{5}{8} \]

\[ \frac{16}{24} \quad \frac{15}{24} \]
It worked on all the others!

\[
\frac{2}{3} \quad \frac{5}{8}
\]

5 is greater than 2

8 is greater than 3
Comparing just the numerators and/or just the denominators
9. $\frac{3}{4}$  $\frac{1}{3}$

10. $\frac{5}{8}$  $\frac{4}{6}$

11. $\frac{1}{6}$  $\frac{2}{5}$

12. $\frac{7}{10}$  $\frac{8}{12}$

13. $\frac{4}{5}$  $\frac{7}{10}$

14. $\frac{2}{3}$  $\frac{3}{5}$

15. $\frac{7}{12}$  $\frac{6}{10}$

16. $\frac{3}{8}$  $\frac{4}{10}$
15. \( \frac{7}{12} \) \( \frac{6}{10} \) + 4

16. \( \frac{3}{8} \) \( \frac{4}{10} \) + 6

Gap reasoner would choose \( \frac{3}{8} \) as the greater fraction.
Gap reasoning is a combination of whole number reasoning and partial residual reasoning. Here, the student considers only the absolute difference between the numerator and denominator, with no consideration of the size of the denominator and therefore the size of the relevant parts (Clarke, 2009).
How would the gap reasoner compare these two fractions?

\[
\frac{4}{9} \quad \frac{6}{11}
\]
How would the gap reasoner compare these two fractions?

- Gap reasoner would say they are equal because:
  - 4 is 5 away from 9
  - 6 is 5 away from 11
Comparing $\frac{4}{5}$ to Other Common Fractions

Using Only Gap Reasoning

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Comparing $\frac{4}{5}$ to Other Common Fractions

Using Only Gap Reasoning

1/2  1/3  1/4  1/5  1/6  1/7  1/8  1/9  1/10  1/11  1/12

2/12 3/12 4/12 5/12 6/12 7/12 8/12 9/12 10/12 11/12

3/12 4/12 5/12 6/12 7/12 8/12 9/12

4/12 5/12 6/12 7/12 8/12 9/12 10/12

5/12 6/12 7/12 8/12 9/12 10/12 11/12

6/12 7/12 8/12 9/12 10/12 11/12 12/12

7/12 8/12 9/12 10/12 11/12 12/12

8/12 9/12 10/12 11/12 12/12

9/12 10/12 11/12 12/12

10/12 11/12 12/12

11/12 12/12
Comparing $\frac{4}{5}$ to Other Common Fractions
Using Only Gap Reasoning

\[
\begin{align*}
\frac{1}{2} & \quad \frac{1}{12} & \quad \frac{2}{12} & \quad \frac{3}{12} & \quad \frac{4}{12} & \quad \frac{5}{12} & \quad \frac{6}{12} & \quad \frac{7}{12} & \quad \frac{8}{12} & \quad \frac{9}{12} & \quad \frac{10}{12} & \quad \frac{11}{12} \\
\frac{1}{3} & \quad \frac{2}{3} & \quad \frac{1}{10} & \quad \frac{2}{10} & \quad \frac{3}{10} & \quad \frac{4}{10} & \quad \frac{5}{10} & \quad \frac{6}{10} & \quad \frac{7}{10} & \quad \frac{8}{10} & \quad \frac{9}{10} \\
\frac{1}{4} & \quad \frac{2}{4} & \quad \frac{3}{4} & \quad \frac{1}{8} & \quad \frac{2}{8} & \quad \frac{3}{8} & \quad \frac{4}{8} & \quad \frac{5}{8} & \quad \frac{6}{8} & \quad \frac{7}{8} \\
\frac{1}{5} & \quad \frac{2}{5} & \quad \frac{3}{5} & \quad \frac{4}{5} & \quad \frac{1}{6} & \quad \frac{2}{6} & \quad \frac{3}{6} & \quad \frac{4}{6} & \quad \frac{5}{6} & \quad \frac{6}{6}
\end{align*}
\]
Comparing $\frac{7}{8}$ to Other Common Fractions

Using Only Gap Reasoning

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<th>$\frac{1}{2}$</th>
<th>$\frac{2}{3}$</th>
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<th>$\frac{5}{6}$</th>
<th>$\frac{6}{7}$</th>
<th>$\frac{7}{8}$</th>
<th>$\frac{8}{9}$</th>
<th>$\frac{9}{10}$</th>
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<tr>
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<td>$\frac{9}{12}$</td>
<td>$\frac{10}{12}$</td>
<td>$\frac{11}{12}$</td>
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</tbody>
</table>
Comparing $\frac{7}{8}$ to Other Common Fractions

Using Only Gap Reasoning

\[
\begin{array}{cccccccccccc}
\times & \frac{1}{2} & 12 & 2 & 12 & 3 & 12 & 4 & 12 & 5 & 12 & 6 & 12 & 7 & 12 & 8 & 12 & 9 & 12 & 10 & 12 & 11 & 12 \\
\times & \frac{1}{3} & 3 & 10 & 2 & 10 & 3 & 10 & 4 & 10 & 5 & 10 & 6 & 10 & 7 & 10 & 8 & 10 & 9 & 10 & \times \\
\times & \frac{1}{4} & 4 & 8 & 2 & 8 & 3 & 8 & 4 & 8 & 5 & 8 & 6 & 8 & \times & 7 & 8 \\
\times & \frac{1}{5} & 5 & 6 & 2 & 6 & 3 & 6 & 4 & 6 & 5 & 6 & \times
\end{array}
\]
Reasoning: Invalid and Valid

Invalid:
Whole Number Reasoning
Gap Reasoning

Valid:
Residual Reasoning
Fraction Towers
Residual Reasoning is Valid!!!!!
Two Problems

- Jack ate \( \frac{1}{6} \) of a whole Twizzler. Stella ate \( \frac{1}{8} \) of a whole Twizzler. Who ate more?

- Jack ate \( \frac{5}{6} \) of a whole Twizzler. Stella ate \( \frac{7}{8} \) of a whole Twizzler. Who ate more?

- How did you reason through these problems?
- How might students reason through these problems?
Jack ate \( \frac{5}{6} \) of a Twizzler. Stella ate \( \frac{7}{8} \) of a Twizzler. Who ate more?

- Correct with valid reasoning
- Correct with invalid reasoning
- Incorrect, but consider the reasoning
<table>
<thead>
<tr>
<th><strong>Lydia’s response:</strong></th>
<th><strong>A.J.’s response:</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>They both ate the same amount because $\frac{5}{6}$ and $\frac{7}{8}$ are equal. They are equal because they are both one piece less than the whole.</td>
<td>Jack ate more because sixths are bigger pieces than eighths.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Ben’s response:</strong></th>
<th><strong>Jasmine’s response:</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Stella ate more because $\frac{7}{8}$ is only $\frac{1}{8}$ less than the whole, but $\frac{5}{6}$ is $\frac{1}{6}$ less than the whole. $\frac{5}{6}$ is farther away from the whole.</td>
<td>Stella ate more because 7 is greater than 5.</td>
</tr>
</tbody>
</table>

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<thead>
<tr>
<th><strong>Kaniah’s response:</strong></th>
<th><strong>Daniel’s response:</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Stella ate more because more squares are shaded in $\frac{7}{8}$ than in $\frac{5}{6}$.</td>
<td>Stella ate more because there's more circles in $\frac{7}{8}$ than in $\frac{5}{6}$.</td>
</tr>
</tbody>
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<tr>
<th><strong>Sitara’s response:</strong></th>
<th><strong>Manuel’s response:</strong></th>
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<tbody>
<tr>
<td>Jack ate more because $\frac{5}{6}$ has 5 pieces shaded and $\frac{7}{8}$ has 7 pieces shaded. With fractions, the smaller the number, the bigger it is.</td>
<td>They both ate the same amount because the same amount is shaded.</td>
</tr>
</tbody>
</table>
Correct with Invalid Reasoning

Jasmine’s response:
Stella ate more because 7 is greater than 5.

Kaniah’s response:
Stella ate more because more squares are shaded in $\frac{7}{8}$ than in $\frac{5}{6}$.

Daniel’s response:
Stella ate more because there’s more circles in $\frac{7}{8}$ than in $\frac{5}{6}$.
A.J.’s response:
Jack ate more because sixths are bigger pieces than eighths.

Manuel’s response:
They both ate the same amount because the same amount is shaded.

Lydia’s response:
They both ate the same amount because \( \frac{5}{6} \) and \( \frac{7}{8} \) are equal. They are equal because they are both one piece less than the whole.

Sitara’s response:
Jack ate more because \( \frac{5}{6} \) has 5 pieces shaded and \( \frac{7}{8} \) has 7 pieces shaded. With fractions, the smaller the number, the bigger it is.
Ben’s response:

Stella ate more. $\frac{7}{8}$ is $\frac{1}{8}$ less than the whole. $\frac{5}{6}$ is $\frac{1}{6}$ less than the whole so it’s further to the left because sixths are greater than eighths.
Strategy Development

- Whole Number Reasoning
- Build Concept of the Whole
- Gap Reasoning
- ?
- Residual Reasoning and Benchmarking
Students have moved away from the unreliable notions of gap reasoning towards the reliable concepts of residual reasoning when students do the following:

- They recognize there is an equal number of missing pieces from the whole (or some other benchmark).
- Then they consider the size of the missing pieces.
- The use the size of the missing pieces to determine which fraction is farther away from the benchmark.
How Fifth Graders Solved This

- Jack ate $\frac{1}{6}$ of a whole Twizzler. Stella ate $\frac{1}{8}$ of a whole Twizzler. Who ate more?

- Jack ate $\frac{5}{6}$ of a whole Twizzler. Stella ate $\frac{7}{8}$ of a whole Twizzler. Who ate more?
Fraction Strips

- Glue
- Scissors
- Number lines
Each person gets two yellow strips and two blue strips.

Fold each yellow strip into eighths.

Fold each blue strip into sixthls.
Glue a whole unit onto each double number line... sixths and eighths
Glue 1 sixth and 1 eighth
Glue 5 sixths and 7 eighths
1 Whole
Mark the Benchmarks
\[ \frac{7}{8} \quad \frac{5}{6} \]
4/8  3/6
3/8

2/6

0

1

1/2

1
$\frac{5}{8}$  $\frac{4}{6}$
More students experience success with mathematics if the introduction of procedures is delayed for the purpose of developing conceptual knowledge of a fraction’s magnitude (Charlambous & Pitta-Pantazi, 2007).
To win the battle....

- We must identify the enemy:
  - Moving too quickly to procedures
  - Overcoming the power of Gap Reasoning
  - Stopping Gap Reasoning before it becomes the go-to comparison strategy
That’s Good

- because of our emphasis on the whole
- and considering what part of the whole is not filled
- and using the denominators articulated by our state standards

That’s Bad

- A teacher’s typical instruction actually perpetuates gap reasoning
- students take that information in and it evolves into the fraction comparison strategy of gap reasoning
Missing from our curriculum is initial purposeful fraction instruction that helps students confront the faultiness of gap reasoning and allows students to avoid the gap trap.